# Aero III/IV Complex Variables

Many of the basic properties of complex variables are a *generalization* of those of real variables. The most striking difference is the concept of (complex) *differentiable functions* or *analytic functions*. It requires more than ordinary differentiability of the real and imaginary parts, the so called *Cauchy-Riemann* condition.

In this section, the basics of complex numbers, elementary complex functions and analytic functions will be briefly reviewed first. We will focus on some applications of the complex variables, and then move to complex line integrals, the most important topics in the remaining course.

## 1 Basic concepts and arithmetics of Complex numbers

1) The fundamental relation: $i^2 = -1$

2) Two representations:

   (i) Cartesian representation: $z = x + iy$

   (ii) Polar representation: $z = re^{i\theta}$

   In general, the Polar representation is preferred for certain manipulations like $n$-th roots $z^{1/n}$, the logarithm $\log z$ and other calculations related to trigonometric functions.

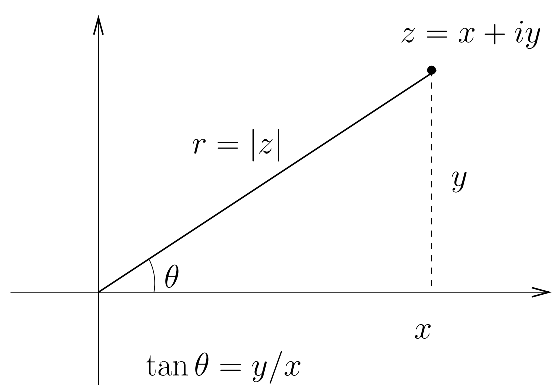
3) Related concepts for $z = x + iy = re^{i\theta}$:

   (i) Real part $x = \text{Re}z$

   (ii) Imaginary part $y = \text{Im}z$

   (iii) Modulus $r = |z| = \sqrt{x^2 + y^2}$

   (iv) Argument $\theta = \text{Arg}z = \arctan \frac{y}{x}$




4) These two representations are related by the *Euler equation*:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

   where $\theta$ is measured in *radian* like $\pi/3, \pi/2$ (instead of degree).

5) Basic operations between complex numbers:

(i) (Addition) If $z = x + iy$ and $w = u + iv$,

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v).$$

(ii) (Multiplication)

$$zw = (x + iy)(u + iv) = xu - yv + i(xv + yv).$$

(iii) (Division)

$$\frac{z}{w} = \frac{x + iy}{u + iv} = \frac{(x + iy)(u - iv)}{u^2 + v^2} = \frac{ux + vy + i(uy - vx)}{u^2 + v^2}.$$

6) Another important concepts is *complex conjugate* $\bar{z} = x - iy$

(i) $\bar{\bar{z}} = z$

(ii) $\mathrm{Re}z = \dfrac{z + \bar{z}}{2}$, $\mathrm{Im}z = \dfrac{z - \bar{z}}{2i}$

(iii) $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$

(iv) $|\bar{z}| = |z|$, $|z|^2 = z\bar{z}$, $|zw| = |z||w|$

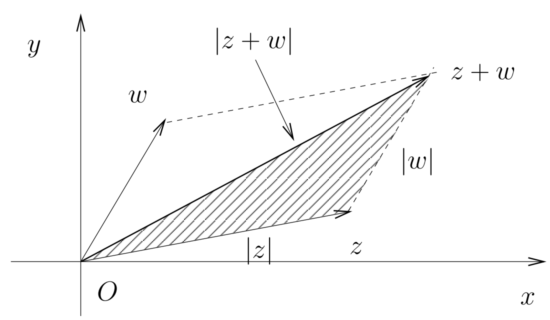
7) There are also important inequalities that can be used later (to estimate some of the terms in contour integration)

(i) $|\mathrm{Re}z| \le |z|$, $|\mathrm{Im}z| \le |z|$

(ii) $|z + w| \le |z| + |w|$

(iii) $\big||z| - |w|\big| \le |z - w|$

The equality in (ii) is achieved when $\mathrm{Arg}z = \mathrm{Arg}w$.




8) De Moirre's formula

$$\big(\cos\theta + i\sin\theta\big)^n = \cos n\theta + i\sin n\theta.$$

9) Special curves on the complex plane (in terms of equation or parametrization):

(i) General straight line: $z = (1 - t)z_0 + tz_1$. If we restrict $t \in [0, 1]$, then this is a line segment connecting $z_1$ and $z_2$

(ii) The real axis: $\mathrm{Im}z = 0$

(iii) The imaginary axis: $\mathrm{Re}z = 0$

(iv) The straight line going through the middle point of $z_1$, $z_2$, and perpendicular to $z_1 - z_2$: $|z - z_1| = |z - z_2|$

(v) The circle with centre $z_0$ and radius $r$: $|z - z_0| = r$. We can also parametrized by $z = z_0 + re^{i\theta}$ with $\theta \in [0, 2\pi)$.

(vi) Annular region: $r_1 \leq |z - z_0| \leq r_2$

10) ***n*-the roots:** Given $z$, find $w$ such that $w^n = z$ (and $w$ is denoted by $z^{\frac{1}{n}}$).

Let $z = r\big(\cos\theta + i\sin\theta\big), w = R\big(\cos\phi + i\sin\phi\big)$. By Demoirre's formula,

$$R^n\big(\cos n\phi + i\sin n\phi\big) = r\big(\cos\theta + i\sin\theta\big).$$

Therefore, $R^n = r$ and $n\phi = \theta + 2k\pi$ with $k = 0, \pm 1, \pm 2, \cdots$. This implies that

$$w = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \qquad k = 0, 1, \cdots, n-1,$$

which are $n$ distinct points lying equally-spaced on a circle of radius $r^{1/n}$.

**Exercise 1.1.** Find the modulus of $\frac{3+i}{1+i}$.

**Exercise 1.2** (The Circles of Apollonius)**.** The equation $|z - z_1|/|z - z_2| = \lambda(> 0)$ for $\lambda \neq 1$ is a circle, called the Circles of Apollonius.

(i) What is the curve corresponding to $\lambda = 1$?

(ii) If $\lambda \neq 1$, substituting $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ into the equation $|z - z_1|/|z - z_2| = \lambda$, we get (with $z = x + iy$)

$$\lambda = \frac{|z - z_1|}{|z - z_2|} = \frac{|(x - x_1) + i(y - y_1)|}{|(x - x_2) + i(y - y_2)|} = \frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\sqrt{(x - x_1)^2 + (y - y_1)^2}}. \tag{1}$$

Confirm that this is a circle (the expressions for the center and the radius are very complicated).

(iii) Once we are sure it is circle, we can find the center and the radius in other alternative ways, using the geometry.

Let $z_3$ and $z_4$ be the intersection points of the circle with the straight line connecting $z_1$ and $z_2$. Without loss of generality, we can take $\lambda > 1$ (the case $0 < \lambda < 1$ works similarly), which looks like in Figure 1. The advantage using this geometric information is that we can get rid of the modulus from the governing equations $|z - z_1|/|z - z_2| = \lambda$. That is

$$\lambda = \frac{z_1 - z_3}{z_3 - z_2} = \frac{z_1 - z_4}{z_2 - z_4},$$

which gives

$$z_3 = \frac{\lambda z_2 + z_1}{\lambda + 1}, \qquad z_4 = \frac{\lambda z_2 - z_1}{\lambda - 1}.$$

By the symmetry, the centre of the radius should be on this straight line, and $z_3$, $z_4$ are on the opposite sides of a diameter. Therefore, the centre of the circle is

$$z_0 = \frac{1}{2}(z_3 + z_4) = \frac{\lambda^2 z_2 - z_1}{\lambda^2 - 1}, \tag{2}$$

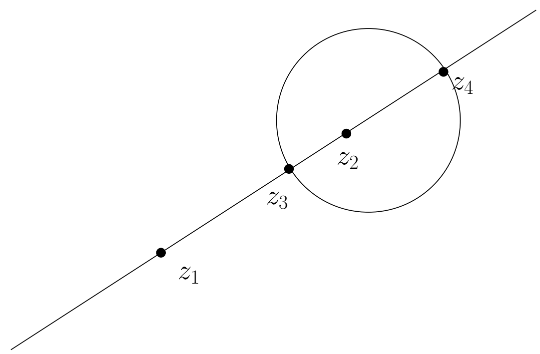



Figure 1: The circle of Apollonius of all the points $|z - z_1|/|z - z_2| = \lambda > 1$, which intersects the straight line (connecting $z_1$ and $z_2$) at $z_3$ and $z_4$.

and the radius is

$$R = \frac{|z_3 - z_4|}{2} = \frac{\lambda}{|\lambda^2 - 1|}|z_1 - z_2|, \tag{3}$$

which should be compared with the centre and radius calculated from the algebraic equation (1).

(iv) Check that when $0 < \lambda < 1$, the centre and the radius are still given by (2) and (3), respectively.

(v) Find the curve on the $w$-plane, which is transformed from the circle $|z| = 1$ by the map $w = (z+2)/(z-2)$.

*The following exercises show some applications of complex numbers, but are less related to the rest of the course.*

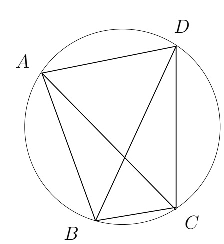
**Exercise 1.3.** From the identity $e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}$ and the arithmetics of multiplication of complex numbers, derive the following trigonometric identities:

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$
$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

**Exercise 1.4.** Shown that

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin\theta - \sin(n+1)\theta - \sin\theta\cos(n+1)\theta + \cos\theta\sin(n+1)\theta}{2 - 2\cos\theta},$$

using the relation between Euler equation (The right hand side can be simplified further).




**Exercise**. Show that the four points $A, B, C, D$ are on the same circle if and only if the distances satisfies the condition

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

Hint: Notice the condition that $A, B, C, D$ are on the same circle can be characterized by

$\angle ADC + \angle ABC = \pi$ (or equivalently $\angle BAD + \angle BCD = \pi$) and the identity $(z_A - z_C)(z_B - z_D) = (z_A - z_B)(z_C - z_D) + (z_A - z_D)(z_B - z_C)$, where $z_A, z_B, z_C$ and $z_D$ are the complex numbers corresponding to the four points.

**Exercise 1.5.** Shown from the property $|z_1 + z_2| \leq |z_1| + |z_2|$, we can get

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|.$$

Moreover, for any real number $a_1, \cdots, a_n, b_1, \cdots, b_n$, we have the inequality

$$\sqrt{a_1^2 + b_1^2} + \cdots + \sqrt{a_n^2 + b_n^2} \leq \sqrt{(a_1 + \cdots + a_n)^2 + (b_1 + \cdots + b_n)^2}.$$

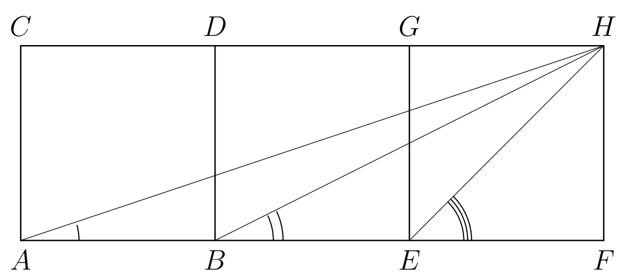



Figure 2: Sum of three angles

**Exercise 1.6.** Three squares placed side by side as shown in Figure 2. Prove that the sum of $\angle HAF$, $\angle HBF$ and $\angle HEF$ is a right angle. Hint: Let the vectors $\overrightarrow{AH}, \overrightarrow{BH}$ and $\overrightarrow{CH}$ be the complex number $3+i, 2+i$ and $1+i$ respectively, the the sum of the three angles is the argument of the product $(3+i)(2+i)(1+i)$.

# 2 Elementary complex functions

**Exponential function $e^z = e^x\big(\cos y + i \sin y\big)$:** Basic algebraic properties:

(i) When $z = x$ (i.e. $y = 0$), familiar exponential of real variable; when $z = iy$, $e^{iy} = \cos y + i \sin y$, for instance $e^{i\pi} = -1$.

(ii) $e^{z_1 + z_2} = e^{z_1} e^{z_2}$; $(e^z)^n = e^{nz}$.

**Trigonometric functions:** From the Euler's formula

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x$$

for real variable $x$, we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These definitions can be generalized from real variable $x$ to complex variable $z$, i.e.,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Other trigonometric functions can be defined similarly,

$$\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z},\ \sec z = \frac{1}{\cos z},\ \csc z = \frac{1}{\sin z}.$$

It can be verified that all trigonometric identities of real variables are valid, when the real variables are replaced by complex variables. For example (try it!),

$$\begin{aligned}\sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2,\\ \cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2,\\ \sin^2 z + \cos^2 z &= 1, \cdots\end{aligned}$$

**Hyperbolic functions:** The hyperbolic functions of complex variable are defined analogously as for real variable:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^z}{2},$$

and

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \mathrm{sech} z = \frac{1}{\cosh z}, \quad \mathrm{csch} z = \frac{1}{\sinh z}.$$

The identities of hyperbolic functions of real variables hold also for complex variables, like

$$\cosh^2 z - \sinh^2 z = 1.$$

Finally we have the following relations between trigonometric functions and their hyperbolic counterparts:

$$\sinh iz = i \sin z, \quad \sin iz = i \sinh z, \quad \cosh iz = \cos z, \quad \cos iz = \cosh z.$$

# 3 Functions of a complex variable, limit and continuity

Definition of **a complex function** $f(z) : z \to w = f(z)$. If for each $z$ (in a suitable region $R$ in $z$-plane), there exists a unique complex $w(z)$, then we say that $w(z)$ is a function of $z$, written as $w = f(z)$.

*Remark.* We isolate this definition here, because of **multi-valued function** we encounter later, like $z^\alpha, \ln z, \cdots$.

The concept of **limit** and **continuity** of a complex function is similar to those of a real function. The sequence $\{z_n\}$ **converges to the limit** $z_0 \in \mathbb{C}$ if $\lim_{n\to\infty} z_n = z_0$, or equivalently $\lim_{n\to\infty} |z_n - z_0| = 0$.

Suppose $f(z)$ is defined at all points in some neighbourhood of $z_0$ (except possibly at $z_0$ itself). We say $f(z)$ has limit $w_0$ if as $z$ approaches $z_0$ (along any path), $w$ approaches $w_0$. This is written as

$$\lim_{z\to z_0} f(z) = w_0.$$

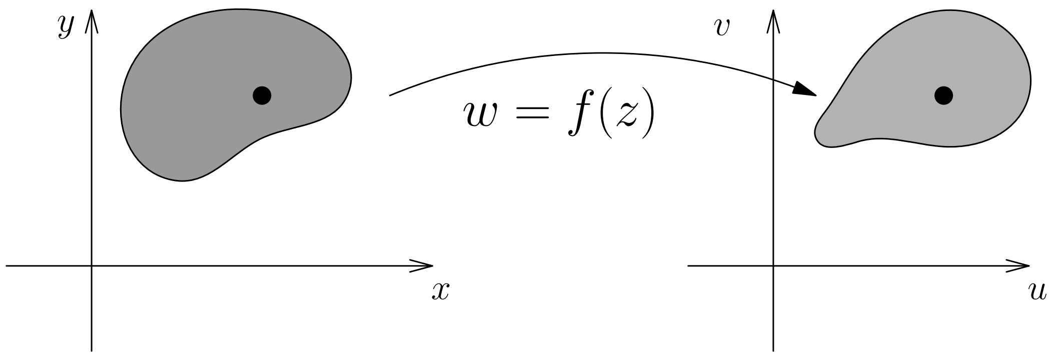



Figure 3: Two separate planes to represent the complex function $w = f(z)$.

**Exercise 3.1.** Find the limit of $f(z) = \dfrac{z^2 + iz + 2}{z - i}$ as $z \to i$.

A complex function $f(z)$ is **continuous** at $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0),$$

regardless of the manner in which $z \to z_0$.

The combination complex conjugate, modulus and their related (triangle inequalities) can be used to show the continuity of many simple functions effectively.

**Example 3.1.** Show that the following functions are continuous on the finite complex plane: (1) $f(z) = \bar{z}$; (2) $f(z) = \mathrm{Re}z$; (3) $f(z) = |z|$.
*Solution*: (1) $|f(z) - f(z_0)| = |\bar{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0|$, therefore

$$\lim_{z \to z_0} |f(z) - f(z_0)| = \lim_{z \to z_0} |z - z_0| = 0.$$

(2) $|f(z) - f(z_0)| = |\mathrm{Re}z - \mathrm{Re}z_0| = |\mathrm{Re}(z - z_0)| \leq |z - z_0|$, therefore

$$\lim_{z \to z_0} |f(z) - f(z_0)| \leq \lim_{z \to z_0} |z - z_0| = 0.$$

(3) We use the triangle inequality $\big||z| - |z_0|\big| \leq |z - z_0|$ and

$$\lim_{z \to z_0} |f(z) - f(z_0)| \leq \lim_{z \to z_0} |z - z_0| = 0.$$

For continuity of a function $f(z)$, we can just look at the real $u$ and imaginary part $v$ of $f(z)$, which is reduced to the continuity of $u$ and $v$ as a function of $x$ and $y$. To show some discontinuous functions, one way is to choose different sequence of $z_n$ approaching $z_0$ to get different limits.

**Example 3.2.** Show that

$$f(z) = \begin{cases} \mathrm{Im}z/\mathrm{Re}z, & \mathrm{Re}z \neq 0, \\ 0, & \mathrm{Re}z = 0. \end{cases}$$

is not continuous at $z_0 = 0$.
*Solution*: Choose $z = t + it \to 0$ as $t \to 0$, then we have $f(z) = f(t + it) = 1 \neq 0$. Therefore the limit $f(z)$ when $z \to 0$ does not exist.

# 4 Differentiability

To avoid the problem of defining differentiability of a function near the boundary, we only consider functions defined on *open set* like $\{z : |z| < 1\}$, instead of *closed set* like $\{z : |z| \leq 1\}$.

**Definition 4.1** (Differentiation)**.** A complex function $f$ is defined on an open subset $D$ of $\mathbb{C}$ is **differentiable** at $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, denoted by $f'(z_0)$.

Notice that the above limit is **independent** on the path. In other words, if you get different limits along different paths, then the function is *not differentiable* (this is exactly how we show that some functions are not differentiable).

**Example 4.1.** Show that $f(z) = z^n$ is differentiable for positive integral $n$.
*Solution:* Using binomial expansion,

$$\lim_{\Delta z \to 0} \frac{(z_0 + \Delta)^n - z_0^n}{\Delta z} = \lim_{\Delta z \to 0} \frac{\left(z_0^n + \binom{n}{1} z_0^n \Delta z + \cdots\right) - z_0^n}{\Delta z_0} = n z_0^{n-1}.$$

Therefore, $f(z) = z^n$ is differentiable for all $z$, and $f'(z) = nz^{n-1}$.

**Example 4.2.** Show that $f(z) = \bar{z} = x - iy$ is not differentiable.
*Solution:*

$$\lim_{\Delta z \to 0} \frac{(\overline{z_0} + \overline{\Delta z_0}) - \overline{\Delta z_0}}{(z_0 + \Delta z_0) - z_0} = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \begin{cases} 1, & \text{if } \Delta y = 0, \\ -1, & \text{if } \Delta x = 0. \end{cases}$$

Since the limits are different on different paths, $f(z) = \bar{z}$ is not differentiable anywhere.

*Remark.* Differentiability of a complex function is quite a severe requirement. Functions that are differentiable in the real line may NOT differentiable in the complex plane, because of the limit can be taken away from the real line. For example consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ x = 0, & x = 0, \end{cases}$$

then $f'(x)$ exists at $x = 0$ as a real function of $x$. Now if $f$ is extended to the whole complex plane

$$f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0, \\ x = 0, & z = 0, \end{cases}$$

then $f'(z)$ does not exist at $z = 0$. When the origin is approached from the real axis ($z = x \in \mathbb{R}$),

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0.$$

When the origin is approached from the imaginary axis $z = iy$, then

$$\lim_{y\to 0} \frac{e^{-1/(iy)^2}}{iy} = \lim_{y\to 0} \frac{e^{1/y^2}}{iy}$$

does not exist. Therefore, $f(z) = e^{-1/z^2}$ is not differentiable at $z = 0$.

A function $f(z)$ is called **analytic** in a region $R$ if $f(z)$ is defined and differentiable at all points of $R$. A function $f(z)$ is said to be analytic at a point $z_0$, if $f(z)$ is differentiable in a neighbourhood of $z_0$.

*Remark.* Analyticity of a function at $z_0$ is stronger than differentiability. Show that $f(z) = z\bar{z} = |z|^2$ is differentiable at $z = 0$, but it is not analytic at $z = 0$.

The differentiability of a complex function $f(z) = u(x,y) + iv(x,y)$ is characterized by the so-called **Cauchy-Riemann equations**. They are derived by taking $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ respectively:

$$f'(z) = \lim_{\Delta x\to 0} \left( \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right) = u_x(x,y) + iv_x(x,y),$$

$$f'(z) = \lim_{\Delta y\to 0} \left( \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+i\Delta y) - v(x,y)}{i\Delta y} \right) = -iu_y(x,y) + v_y(x,y),$$

or

$$\boxed{u_x = v_y, \qquad v_x = -u_y} \qquad \text{(The Cauchy-Riemann Equations).}$$

The Cauchy-Riemann equations are also sufficient condition for differentiability. In fact, by Taylor expansion, for $\Delta x$ and $\Delta y$ small,

$$\Delta u = u_x \Delta x + u_y \Delta y + o(|\Delta z|), \quad \Delta v = v_x \Delta x + v_y \Delta y + o(|\Delta z|).$$

Therefore,

$$\begin{aligned} \Delta f &= \Delta u + i\Delta v \\ &= (u_x \Delta x + u_y \Delta y) + i(v_x \Delta x + v_y \Delta y) + o(|\Delta z|) \\ &= (u_x + iu_y)(\Delta x + i\Delta y) + o(|\Delta z|). \end{aligned} \tag{4}$$

Therefore,

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = u_x + iv_x := f'(z).$$

**Theorem 4.1.** *A function $f(z) = u + iv$ is analytic in a region if and only if $u_x, u_y, v_x, v_y$ are continuous functions and satisfy the C-R equations.*

**Exercise 4.1.** Show that the Cauchy-Riemann equations is equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r},$$

when $u$ and $v$ are represented in polar coordinates.

A lot of common complex functions like $\bar{z}, |z|^2, \text{Re}z$ are NOT differentiable (show it). As a rule of thumb, if you write the function $f(z)$ in terms of $z$ and $\bar{z}$ and there is **non-trivial dependence on** $\bar{z}$ (the partial derivative w.r.t $\bar{z}$ is non-zero), then the function is **not differentiable**. For example,

$$|z|^2 = z\bar{z}, \qquad \text{Re}z = \frac{z+\bar{z}}{2},$$

both depend nontrivally on $\bar{z}$ and hence non-differentiable. Actually $\partial(z\bar{z})/\partial\bar{z} = z = 0$ if $z = 0$, we can show that $|z|^2$ is differentiable at the origin ($u_x = u_y = v_x = v_y = 0$). On the other hand, if $f$ is a "differentiable" function of $z$ alone, then we can proceed as in real variables.

**Exercise 4.2.** Use the Cauchy-Riemann equation to show that $f'(z) = e^z$ for $f(z) = e^z$.

**Special properties satisfied by** $u$ **and** $v$**:**

(a) Both $u$ and $v$ satisfied the Laplace equation, $\nabla^2 u = 0, \nabla^2 v = 0$, called *harmonic functions.*

(b) The level curves of $u =$ constant and $v =$ constant intersect at right angle, or equivalently $\nabla u \cdot \nabla v = 0$. Here the normals $n_1, n_2$ to the level curve $u(x, y) = c_1$ and $v(x, y) = c_2$ are

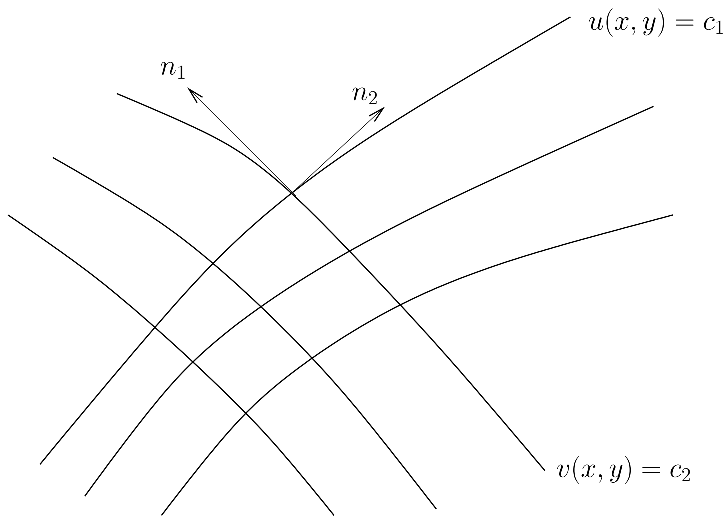



Figure 4: The level curves of $u$ and $v$ are orthogonal to each other.

$$\vec{n}_1 = \frac{(u_x, u_y)}{\sqrt{u_x^2 + u_y^2}}, \qquad \vec{n}_2 = \frac{(v_x, v_y)}{\sqrt{v_x^2 + v_y^2}}.$$

Therefore,

$$\vec{n}_1 \cdot \vec{n}_2 = \frac{u_x v_x + u_y v_y}{\sqrt{u_x^2 + u_y^2}\sqrt{v_x^2 + v_y^2}} = 0.$$

(c) Using the change of variable $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$ or $z = x + iy, \bar{z} = x - iy$, we get

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Therefore, the Laplace equation $\nabla^2 u = 4\frac{\partial^2}{\partial z \partial \bar{z}} u = 0$ can be integrated, to obtain $u = g(z) + h(\bar{z})$. The fact that $u$ is real (if it is true) puts some constraints on $g$ and $h$, leading finally to *the real or imaginary part of an analytic function.*

(d) Similarly, the biharmonic equation (or the beam equation in 2D) $\nabla^4 u = 0$ is equivalent to $16\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} u = 0$, whose general solution is given by

$$u = \bar{z} f(z) + g(z) + z h(\bar{z}) + j(\bar{z})$$

or $u = \mathrm{Re}\bar{z} f(z) + g(z)$ if $u$ is real.

# 5 Two applications

**Application in ideal flow:** A steady flow is called *ideal* if it is incompressible and irrotational. In two dimension, in terms of the velocity component $\vec{v} = (v_1, v_2)$, these conditions are

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \qquad \text{(incompressible)} \qquad (5a)$$

$$\frac{\partial v_1}{\partial y} - \frac{v_2}{\partial x} = 0. \qquad \text{(irrotational)} \qquad (5b)$$

These two conditions are exactly the Cauchy-Riemann equations, which motivate the introduction of a *complex velocity potential* $\Omega(z) = \phi(x, y) + i\psi(x, y)$. Here $\phi$ is the *velocity potential* ($(v_1, v_2) = \nabla\phi$ or $v_1 = \partial\phi/\partial x, v_2 = \partial\phi/\partial y$), and $\psi$ is the *stream function* ($v_1 = \partial\psi/\partial y, v_2 = -\partial\psi/\partial x$). Therefore, the velocity $\vec{v} = (v_1, v_2)$ can be obtained from the derivative of $\Omega$, i.e, $\Omega'(z) = v_1 - iv_2$.

For many flow patterns, we can get the velocity by a suitable choice of $\Omega(z)$. For example, for a flow around the cylinder of radius $a$, we have

$$\Omega(z) = v_0(z + \frac{a^2}{z}).$$

The velocity components in polar coordinates are

$$v_1 = v_0\left(1 - \frac{a^2 \cos 2\theta}{r^2}\right), \qquad v_2 = -v_0\frac{a^2 \sin 2\theta}{r^2}.$$

It is easy to see that these velocity has the right boundary condition on the cylinder $r = a$ and at infinity (only horizontal velocity $v_0$).

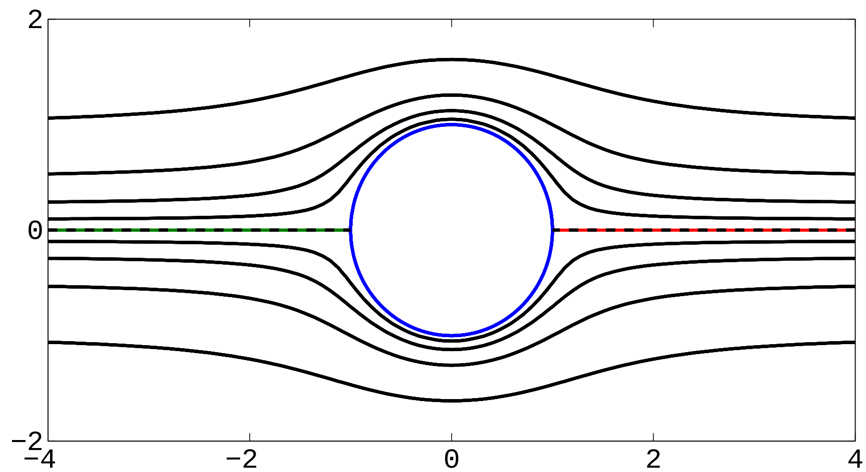



Figure 5: The ideal flow around an infinite cylinder.

**Application in Stokes flow:** The governing equation for $\mathbf{v} = (v_1, v_2)$:

$$\mu\nabla^2\mathbf{v} - \nabla p = 0, \quad \mathrm{div}\mathbf{v} = 0.$$

Similarly, introducing the *stream function* $\psi$, such that $v_1 = \partial\psi/\partial y, v_2 = -\partial\psi/\partial x$ and define the (scalar) vorticity by

$$\omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = -\nabla^2\psi.$$

Then

$$\frac{\partial \omega}{\partial x} = -\nabla^2\frac{\partial \psi}{\partial x} = \nabla^2 v_2 = \frac{1}{\mu}\frac{\partial p}{\partial y}, \quad \frac{\partial \omega}{\partial y} = -\nabla^2\frac{\partial \psi}{\partial y} = -\nabla^2 v_1 = -\frac{1}{\mu}\frac{\partial p}{\partial x},$$

which implies that

$$f(z) = \omega + \frac{i}{\mu}p$$

is analytic.

*Remark.* As we can see from the above a few examples, complex variable techniques can be conveniently employed to represent harmonic functions and solutions in many simplified problems. But the real difficulty lies in *connecting the boundary conditions*. We will get a partial answer in Section 8, using complex contour integration.

# 6 Analyticity and derivatives of elementary functions

- We start with the basic one

$$f(z) = e^z = e^x\big(\cos y + i\sin y\big)$$

with $u = e^x\cos y, v = e^x\sin y$. Then

$$u_x = e^x\cos y = v_y; \qquad v_x = e^x\sin y = -u_y.$$

Hence the Cauchy-Riemann equations are satisfied for all $z = x + iy$, and $e^z$ is analytic for all $z$. Furthermore,

$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.$$

- It follows from the definitions and rules of differentiation and limits that

$$\cos z, \quad \sin z, \quad \cosh z, \quad \sinh z$$

are all analytic. Their derivatives are the same as in real variables:

$$\frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\sinh z = \cosh z.$$

# 7 Multi-valued functions

Multi-valued functions arise naturally as the *inverse of single-valued* functions.

## 7.1 The logarithm function $\ln z$

For a real $x > 0$, there exists a unique real $y$ such that $e^y = x$. This defines $y$ as a function of $x$, $y = \ln x$.

**Definition:** Now for a complex $z$, there are *infinitely many* $w$ such that $e^w = z$. In analogy with real variable, we write

$$w = \ln z.$$

Let $w = u + iv, z = re^{i\theta}$ (or $\theta = \arg z, r = |z|$),

$$e^w = e^{u+iv} = e^u \cdot e^{iv} = re^{i\theta}.$$

This implies that $e^u = r, v = \theta$, or

$$w = \ln|z| + i\theta = \ln|z| + i\arg z.$$

Therefore,

$$\ln z = \ln|z| + i\arg z = \ln|z| + i(\text{Arg}z + 2n\pi), \qquad n = 0, \pm1, \cdots, -\pi < \text{Arg}z \le \pi.$$

There are infinite *branches*: they all have the same real part while the imaginary parts differ by a multiple of $2\pi$.

We can also define a single-valued function, or the **principal value** of $\ln z$ as

$$\text{Ln}z = \ln|z| + i\text{Arg}z,$$

which corresponds to $n = 0$.

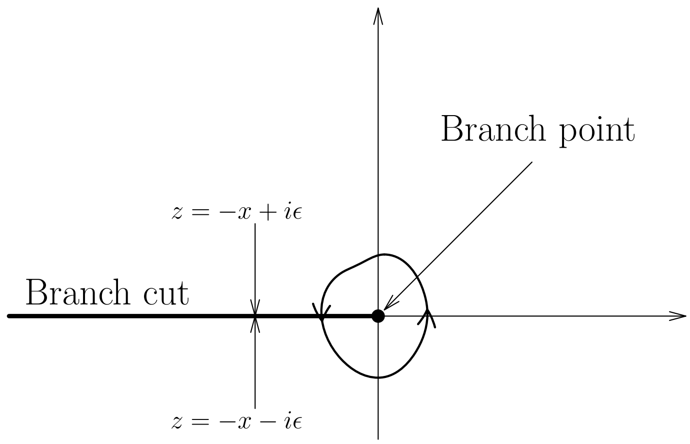



Figure 6: Branch point, branch cut and different limits when approaching from two sides of the branch cut.

**Branch point:** Consider how Ln$z$ change when $z$ transverses a small circuit around $z = 0$.

We consider the limit of the logarithm of $z = -x \pm \epsilon i$ for $x > 0$, as $\epsilon(> 0)$ goes to zero:

$$\lim_{\epsilon \to 0^+} \mathrm{Ln}(-x + \epsilon i) \to \log x + i\pi, \qquad \lim_{\epsilon \to 0^+} \mathrm{Ln}(-x - \epsilon i) \to \log x - i\pi.$$

Therefore, there is a jump in Ln$z$ by $2\pi i$ as $z$ crosses the negative x-axis.

A point $z_0$ is a **branch point** if a *multi-valued* function $f(z)$ does not return to its original values when $z$ transverses a small circuit around this point $z_0$.

The function $\ln z$ is not continuous in the whole plane, as jumps by $2\pi i$ as $z$ crosses the negative x-axis. The make $\ln z$ (including Ln$z$) continuous, we need to "cut out" the negative x-axis, which is called a **branch cut**. The resulting plane is called **cut plane**.

**Analyticity of Ln$z$:** From $\mathrm{Ln} z = \ln r + i\theta$, we have $u = \ln r = \ln(x^2 + y^2)^{1/2}, v = \theta = \arctan \frac{y}{x}$. The Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$ (or the equivalent ones $u_r = \frac{1}{r} v_\theta, v_r = -\frac{1}{r} u_\theta$ in polar coordinates) are satisfied except at $r = 0$.

Therefore, Ln$z$ is analytic in the cut plane, and $f'(z) = 1/z$.

## 7.2 Generalized power function $e^z$ with complex $\alpha$

**Definition:** $z^\alpha \equiv e^{\alpha \ln z}$.

In general, $z^\alpha$ is multi-valued. If $\alpha = m/l$ for integers $m$ and $l$ ($\alpha$ is rational),

$$z^\alpha = e^{\frac{m}{l}\left(\ln r + i\theta + 2n\pi i\right)} = r^{\frac{m}{l}} e^{i\frac{m}{l}\theta} e^{\frac{2n\pi i m}{l}}, \qquad -\pi < \theta \le \pi,\ n = 0, 1, \cdots, l-1.$$

**Principal value:** $e^{\alpha \mathrm{Ln} z}$

- Analytic in the same cut plane as for Ln$z$.
- $\dfrac{d}{dz} z^\alpha = \dfrac{d}{dz} e^{\alpha \mathrm{Ln} z} = \alpha e^{\alpha \mathrm{Ln} z} \dfrac{d}{dz} \mathrm{Ln} z = \alpha z^{\alpha - 1}$.

**Example 7.1.** Find $(1-i)^{(1+i)}$.
*Solution*: we have

$$(1-i)^{(1+i)} = \exp\left((1+i)\ln(1-i)\right) = \exp\left((1+i)[\ln\sqrt{2} + i(-\frac{\pi}{4} + 2n\pi)]\right),$$

since $1-i = \sqrt{2}\exp\left(i(-\frac{\pi}{4}+2n\pi)\right)$. Collecting the real and imaginary part in the exponents,

$$\begin{aligned}(1-i)^{(1+i)} &= \exp\left\{\left(\ln\sqrt{2} + \frac{\pi}{4} - 2n\pi\right) + i\left(\ln\sqrt{2} - \frac{\pi}{4} + 2n\pi\right)\right\} \\ &= \sqrt{2}e^{\frac{\pi}{4}-2n\pi}\left\{\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right\}, \qquad n = 0, \pm 1, \pm 2, \cdots\end{aligned}$$

The principle value is the one with $n = 0$

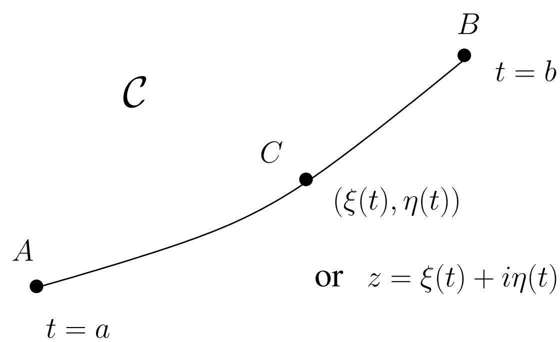
# 8 Complex line integral

A **curve** or **path** $\mathcal{C}$ in the plane can be described by *parametric equations*

$$x = \xi(t), y = \eta(t)$$

or $z = \gamma(t) = \xi(t) + i\eta(t)$, for $a \le t \le b$.

Examples of curves including the line segment $z(t) = 1 + it,\ t \in [0,1]$ connecting the points $z(0) = 1$ and $z(1) = 1 + i$, and the circle $z(\theta) = e^{i\theta},\ \theta \in [0, 2\pi]$.




Such a curve is *smooth* if $\gamma'(t)$ is continuous in $[a, b]$.

The **complex integration** of $f(z)$ along some curve $\mathcal{C}$ parametrized by $\gamma(t)$ for $t \in [a, b]$ is defined by

$$\int_{\mathcal{C}} f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If $\mathcal{C}$ is a closed curve, the integration direction is assumed to be anti-clockwise, unless otherwise stated.

**Example 8.1.** Integrate $z^n$ around the unit circle for any integer $n$.
*Solution*: The closed curve can be parametrized by $z(t) = e^{it}$ with $t \in [0, 2\pi)$.

$$\oint_{\mathcal{C}} z^n dz = \int_0^{2\pi} (e^{it})^n d(e^{it}) = i\int_0^{2\pi} e^{i(n+1)t}dt = \begin{cases} 0, & \text{if } n \ne -1, \\ 2\pi i, & \text{if } n = -1. \end{cases}$$

**Basic properties of path integrals** :

(1) **Fundamental theorem of calculus**: If $f'(z)$ exist ($f$ is analytic) and $\mathcal{C}$ is smooth, then

$$\int_{\mathcal{C}} f'(z)dz = f(z_2) - f(z_1)$$

where $\mathcal{C}$ joints $z_1$ and $z_2$.

(2) **Orientation Reversal** : $\int_{-\mathcal{C}} f(z)dz = -\int_{\mathcal{C}} f(z)dz$. Here $-\mathcal{C}$ means reverse the direction of the integration path.

(3) **Joining of two paths**: If $\mathcal{C}$ consists of two curves $\mathcal{C}_1$ and $\mathcal{C}_2$, then

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}_1} f(z)dz + \int_{\mathcal{C}_2} f(z)dz.$$

(4) **Linearity**: For any constants $\kappa_1$, $\kappa_2$ and any functions $f(z)$, $g(z)$,

$$\int_{\mathcal{C}} \Big[\kappa_1 f(z) + \kappa_2 g(z)\Big] dz = \kappa_1 \int_{\mathcal{C}} f(z)dz + \kappa_2 \int_{\mathcal{C}} g(z)dz.$$

(5) **Independence of the parametrization**: If $\gamma(t)$ and $\psi(s)$ are two parametrizations of the same path $\mathcal{C}$, then

$$\int_{\mathcal{C}} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\psi(s))\psi'(s)ds.$$

(6) **Estimation of integrals**

$$\left|\int_{\mathcal{C}} f(z)dz\right| \le \int_a^b |f(\gamma(t))||\gamma'(t)|dt, \qquad (a < b).$$

With these properties, we can decompose an integral on a complicated path into different pieces.

**Example 8.2.** Find the integral $\int_{\mathcal{C}} \bar{z}dz$, where $\mathcal{C}$ is the boundary of the unit square (counter-clockwise orientation) with corners $0, 1, 1+i, i$.

*Solution*: The path can be decomposed into four line segments:

$$\begin{aligned}
\gamma_1 &= z(t) = t, \quad t \in [0,1] \\
\gamma_2 &= z(t) = 1 + it, \quad t \in [0,1] \\
\gamma_3 &= z(t) = i + 1 - t, \quad t \in [0,1] \\
\gamma_4 &= z(t) = (1-t)i, \quad t \in [0,1].
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\mathcal{C}} \bar{z}dz &= \int_{\gamma_1} \bar{z}dz + \int_{\gamma_2} \bar{z}dz + \int_{\gamma_3} \bar{z}dz + \int_{\gamma_4} \bar{z}dz \\
&= \int_0^1 tdt + \int_0^1 (1-it)idt + \int_0^1 (-i+1-t)(-1)dt + \int_0^1 (t-1)i(-i)dt \\
&= \frac{1}{2} + \left(i + \frac{1}{2}\right) + \left(i - \frac{1}{2}\right) - \frac{1}{2} \\
&= 2i.
\end{aligned}$$

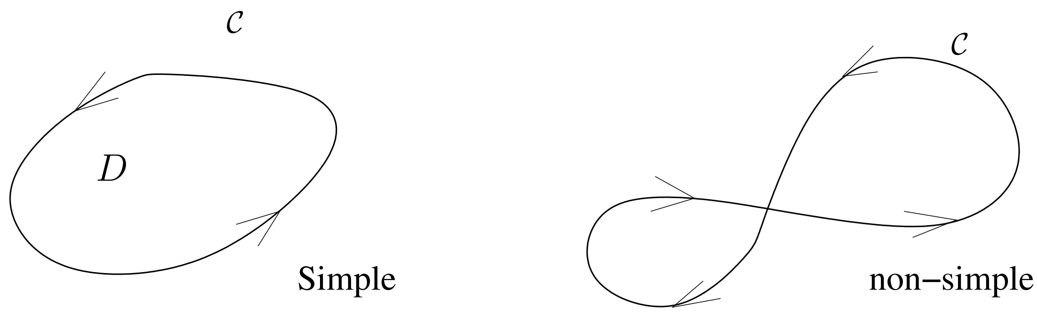



Figure 7: Simple vs non-simple curves.

# 9 Cauchy's integral theorem

**Theorem 9.1.** *If $f(z)$ is analytic on a simple closed curve $\mathcal{C}$ and in the region enclosed by $\mathcal{C}$, then*

$$\boxed{\oint_{\mathcal{C}} f(z)dz = 0.}$$

Here *simple curve* means a curve which does not intersect with itself.

*Proof.* Suppose $f(z) = u(x,y) + iv(x,y)$, then

$$\oint_{\mathcal{C}} f(z)dz = \oint_{\mathcal{C}} [u(x,y) + iv(x,y)](dx + idy) = \oint_{\mathcal{C}} (udx - vdy) + i\int_{\mathcal{C}} (vdx + udy).$$

The two integrals are now integrations of functions of real variables. By Green's theorem,

$$\oint_{\mathcal{C}} (udx - vdy) = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = 0,$$
$$\oint_{\mathcal{C}} (vdx + udy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0.$$

In the last step, the Cauchy-Riemann equations are used. □

*Remark.* **(a)** In order to use Green's Theorem, we require $f'(z)$ to be continuous in $D$. But Cauchy's theorem is still true without this — a more general proof was given by E. Goursat. **(b)** The theorem is also true for non-simple curves, which has a finite number of intersections (decomposing into simple curves).

**Consequences of Cauchy's Theorem** We have the following three equivalent statements about contour integrals, if $f$ is analytic on the paths and the region enclosed by the paths:

(I) **Vanishing integrals for closed path (Cauchy's theorem)**. If $\mathcal{C}$ is a closed path, then $\int_{\mathcal{C}} f(z)dz = 0$.

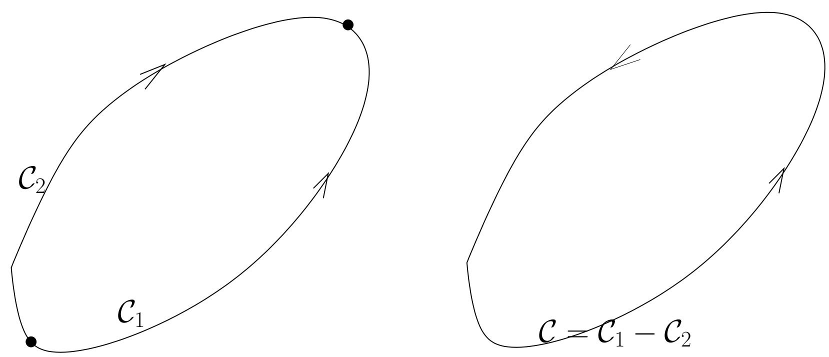



Figure 8: Two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ form a closed path $\mathcal{C}$, or $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$.

(II) **Path independent integral**. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two paths connecting $z_1$ and $z_2$, then $\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two paths connecting $z_1$ and $z_2$, then $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ is a closed path. Then using the properties of path integrals,

$$0 = \int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}_1 - \mathcal{C}_2} f(z)dz = \int_{\mathcal{C}_1} f(z)dz - \int_{\mathcal{C}_2} f(z)dz.$$

Therefore, $\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$.

(III) **Deformation of path**. If $\mathcal{C}_1$ can be deformed into $\mathcal{C}_2$, then $\int_{\mathcal{C}_1} f(z)dz = \int_{\mathcal{C}_2} f(z)dz$.

Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two closed curves as in Figure 9 (the left one) and $f$ is analytic on both and in the region between them. Introduce 'crosscut' $AB$ and $A'B'$, then $\mathcal{C} = \mathcal{C}_1 + AB - \mathcal{C}_2 + B'A'$ is closed. Then by Cauchy's theorem,

$$0 = \oint_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}_1} f(z)dz + \int_{AB} f(z)dz - \int_{\mathcal{C}_2} f(z)dz + \int_{B'A'} f(z)dz.$$

The 'crosscut' $B'A'$ can be chosen to be $-AB$ and $\int_{AB} f(z)dz = \int_{B'A'} f(z)dz$. Therefore,

$$\oint_{\mathcal{C}_1} f(z)dz = \oint_{\mathcal{C}_2} f(z)dz$$

*Remark.* Usually in calculation, the contour is deformed into a *circle* (easier to evaluate).

The result can be extended to deform one contour into multiple contours as in Figure (9) (the right one):

$$\oint_{\mathcal{C}} f(z)dz = \oint_{\mathcal{C}_1} f(z)dz + \oint_{\mathcal{C}_2} f(z)dz,$$

provided that $f$ is analytic on $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$, and in the shaded region.

*Remark.* For closed path, we can shrink the path into *one point*. When $f$ has no singular point, then the integral should vanish, so is the original integral on the undeformed path.

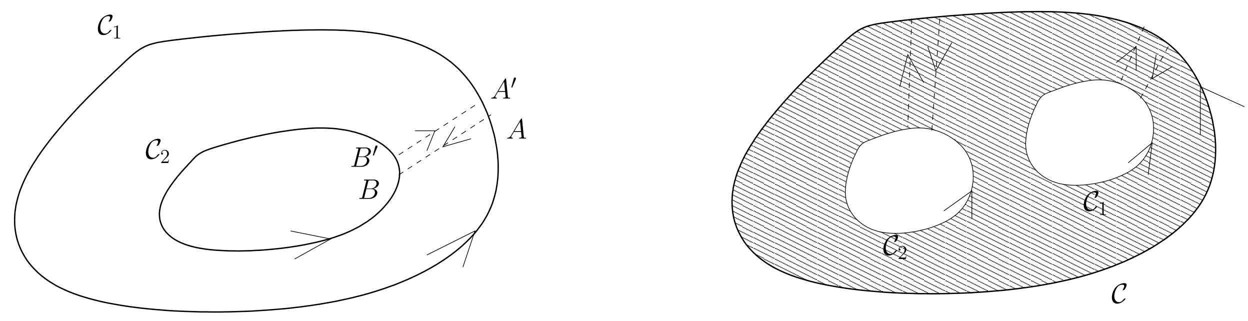



Figure 9: Deformation of one contour into other contour(s).

The function $f$ should be analytic on the region encircled by the paths, which is not true with singular points or "obstacles" (where the functions are not defined), as in Figure 10. If $f$ is not defined (or not analytic) on the shaded region, then $\mathcal{C}_1$ can not be deformed into $\mathcal{C}_2$ and in general

$$\oint_{\mathcal{C}_1} f(z)dz \neq \oint_{\mathcal{C}_2} f(z)dz.$$

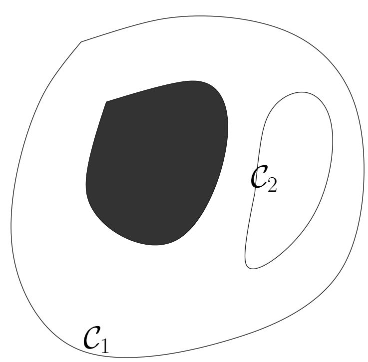



Figure 10: One example of non-deformable paths, if $f$ is not analytic on the shaded "obstacle".

**Theorem 9.2** (Fundamental theorem of calculus)**.** *If $\gamma$ is a path with parameter interval $[a, b]$, and $F$ is defined on a domain containing the path $\gamma$ and is analytic, then*

$$\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

*Especially, if the path is closed ($\gamma(a) = \gamma(b)$), then $\int_\gamma F'(z)dz = 0$.*

*Remark.* The antiderivative of any analytic function $f$ given by

$$F(z) = \int_{[a,z]} f(\xi)d\xi$$

is well-defined for any path from $a$ to $z$, since the integral is independent of the path. Moreover, we have $F$ is analytic and $F'(z) = f(z)$.

## 10 Cauchy integral formula

**Theorem 10.1** (Cauchy integral formula). *Let $f$ be analytic inside and on a simple closed curve $\mathcal{C}$, then for any $z$ inside $\mathcal{C}$,*

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi. \tag{6}$$

*Proof.* Since $f(\xi)/(\xi - z)$ is analytic inside $\mathcal{C}$ except at $z$, the contour $\mathcal{C}$ can be deformed into a circle of radius $\epsilon$, i.e.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi &= \frac{1}{2\pi i}\int_{|\xi - z| = \epsilon} \frac{f(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z + \epsilon e^{i\theta})}{\epsilon e^{i\theta}} \epsilon e^{i\theta} i d\theta \qquad (\xi = z + \epsilon e^{i\theta}) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(z + \epsilon e^{i\theta}) d\theta. \qquad (7)
\end{aligned}$$

Since $f$ is analytic at $z$, it is continuous at that point too. Therefore

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi = \lim_{\epsilon \to 0} \frac{1}{2\pi}\int_0^{2\pi} f(z + \epsilon e^{i\theta}) d\theta = f(z).$$

□

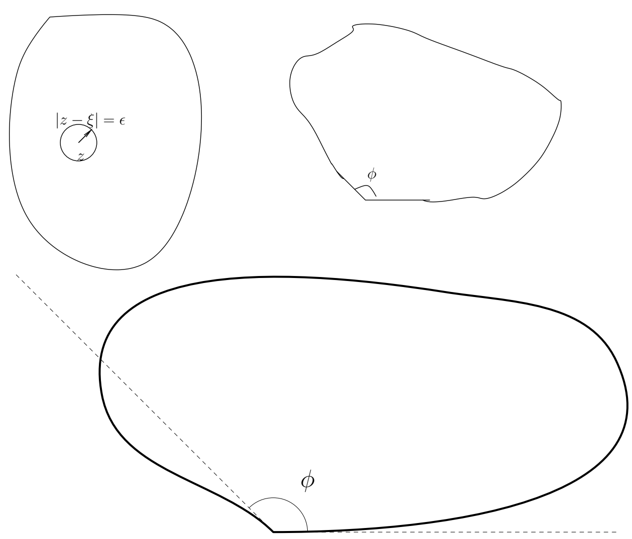



Figure 11: Left: the deformation of the contour $\mathcal{C}$ to a circle $|z - \xi| = \epsilon$; right: the interior angle $\phi$.

When $z$ is outside $\mathcal{C}$, then $f(\xi)/(\xi - z)$ is analytic on the domain bounded by $\mathcal{C}$ and the integral is zero. When $z$ is on the contour $\mathcal{C}$, the integral has to be defined with care (for example using *principle value*), because the integral becomes singular when $\xi \to z$. If the contour is smooth, then we get the average $f(z)/2$; otherwise it is $\frac{\phi}{2\pi} f(z)$, where $\phi$ is the interior angle in Figure 11. In summary,

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} f(z), & \text{if } z \text{ is inside } \mathcal{C}, \\ 0, & \text{if } z \text{ is outside } \mathcal{C}, \\ \frac{\phi}{2\pi} f(z), & \text{if } z \text{ is on } \mathcal{C}. \end{cases}$$

*Remark.* From the Cauchy integral formula (6), if $f$ is known on a contour $\mathcal{C}$ and $f$ is analytic inside, then the value of $f$ at any point inside is known (a representation formula in terms of *boundary values*).

*Remark.* If we look at the real and imaginary part of $f$ in (6) when the contour $\mathcal{C}$ is a circle $|\xi - z| = r$, we get the following **mean value formula**:

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta) d\theta,$$

$$v(x, y) = \frac{1}{2\pi} \int_0^{2\pi} v(x + r\cos\theta, y + r\sin\theta) d\theta.$$

In fact, the Cauchy integral formula (6) can be taken as the mean value formula of $f$ on the contour, with weight $\frac{1}{2\pi i}\frac{1}{\xi - z}$. The value of $z$ given in the weight determines the location of the value of $f$. Especially, the sum of the weight is unit, in the sense that

$$\int_{\mathcal{C}} \frac{1}{2\pi i} \frac{1}{\xi - z} d\xi = 1$$

for any $z$ inside the contour.

**Exercise.** Evaluate $\int_{\mathcal{C}} e^z/(z-1) dz$ where $\mathcal{C}$ is the circle $|z - 2| = 2$.

Since both sides of the Cauchy integral formula (6) are analytic inside $\mathcal{C}$, we can take derivative w.r.t $z$ $n$ times and get the **Cauchy's formula for derivatives**:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi. \tag{8}$$

**Example 10.1.** Find $\int_{\mathcal{C}} \frac{\cos z}{z^3} dz$ where $\mathcal{C}$ is the unit circle i) using (8); ii) using the series expansion $\cos z = 1 - z^2/2! + z^4/4! + \cdots$.

*Solution*: i) Write the integral in the form of (8),

$$\int_{\mathcal{C}} \frac{\cos z}{z^3} dz = \frac{2\pi i}{2!} \frac{2!}{2\pi i} \int_{\mathcal{C}} \frac{\cos z}{(z-0)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \cos z \Big|_{z=0} = -\pi i.$$

ii) Expand $\cos z$ at the origin,

$$\frac{\cos z}{z^3} = \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) = \frac{1}{z^3} - \mathbf{\frac{1}{2!}\frac{1}{z}} + \frac{1}{4!} z + \cdots$$

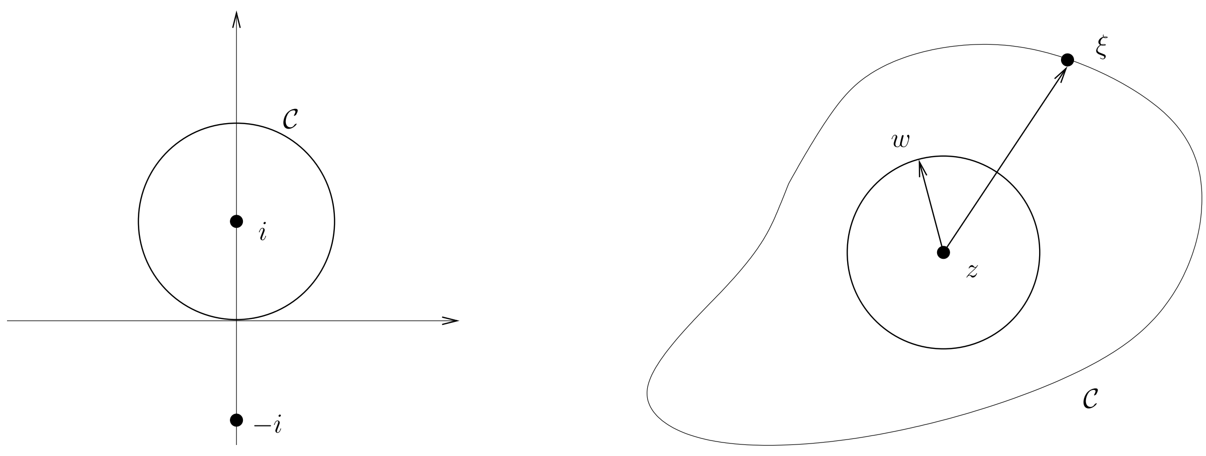
Only the $z^{-1}$ term matters in the contour integral, and

$$\int_{\mathcal{C}} \frac{\cos z}{z^3} dz = \int_{\mathcal{C}} \left(-\frac{1}{2!}\frac{1}{z}\right) dz = -\pi i.$$

*Remark.* When using the Cauchy integral formula, you should check which singular point is inside the contour and you can safely ignore those singular points outside the contour.

**Example 10.2** (Multiple singular points)**.** Find the contour integral $\int_{\mathcal{C}} \frac{z^2}{z^2+1} dz$, where $\mathcal{C}$ is the circle $|z-i| = 1$ as in Figure 12(a).

*Solution:* The integrand $z^2/(z^2+1)$ is not analytic inside $\mathcal{C}$, because of the singular point at $z = i$. We can write it as $\frac{z^2}{z^2+1} = \frac{z^2}{(z-i)(z+i)} = \frac{f(z)}{z-i}$, where $f(z) = z^2/(z+i)$ is analytic inside $\mathcal{C}$. Applying the Cauchy integral formula, $\int_{\mathcal{C}} \frac{z^2}{z^2+1} dz = \int_{\mathcal{C}} \frac{f(z)}{z-i} dz = 2\pi i f(i) = -\pi$.




(a) (b)

Figure 12: (a) Example 3.2; (b) Example 3.3

**Example 10.3** (Consistence between Taylor expansion and the Cauchy integral formula)**.** From the expression for the higher order derivatives in (8), we can express the Taylor expansion in terms of integrals:

$$\begin{aligned} &f(z) + (w-z)f'(z) + \frac{(w-z)^2}{2!} f''(z) + \cdots + \frac{(w-z)^n}{n!} f^{(n)}(z) + \cdots \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \left( \frac{1}{\xi - z} + \frac{w-z}{(\xi-z)^2} + \cdots + \frac{(w-z)^n}{(\xi-z)^{n+1}} + \cdots \right) f(\xi) d\xi \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{\xi - z} \left( 1 + \frac{w-z}{\xi - z} + \cdots + \left( \frac{w-z}{\xi-z} \right)^n + \cdots \right) f(\xi) d\xi. \end{aligned} \tag{9}$$

When $|w-z| < |\xi - z|$, what's inside the bracket is just a geometric series, and can be evaluated explicitly as

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{\xi - z} \frac{1}{1 - (w-z)/(\xi - z)} f(\xi) d\xi = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{\xi - w} f(\xi) d\xi.$$

This is exactly the Cauchy integral formula for $f(w)$. Therefore, we have

$$f(w) = f(z) + (w-z)f'(z) + \frac{(w-z)^2}{2!} f''(z) + \cdots + \frac{(w-z)^n}{n!} f^{(n)}(z) + \cdots .$$