Aero III/IV Calculus of Variations

1 Historical Development

The calculus of variations originated in problems to maximize or minimize certain integrals depending on functions. It is developed from several examples through history:

a) *Isoperimetric problem*: Find the closed curve (or surface) with maximal area (or volume) with a given total length (or surface area). In two dimension, it becomes the minimal surface (like soap bubble)

b) *Geodesics*: Find the shortest distance between two points on a general surface (like the sphere)

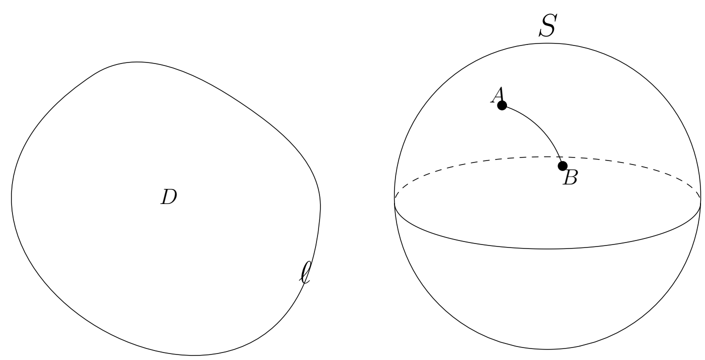


Figure 1: The isoperimetric problem and the geodesic on a sphere.

c) *Optical Path*: the path from one point to another has the least *time* (not the least distance if the media is not homogeneous). This fact can give another derivation of the law of reflection and refraction.

d) *Brachistochrone*: the curve that carries a bead from one place to another in the least amount of time. This problem was solved by several mathematicians, including the inventors of Calculus, Newton and Leibniz. The general formulation was proposed by Euler and Lagrange, leading to the celebrated Euler-Lagrange equation below.

e) *The inspired further development in mechanics*: The Lagrangian mechanics and Hamiltonian mechanics, Hamilton's principle, virtual work ...

Exercise: Name some examples of optimizing in your daily life or course work?

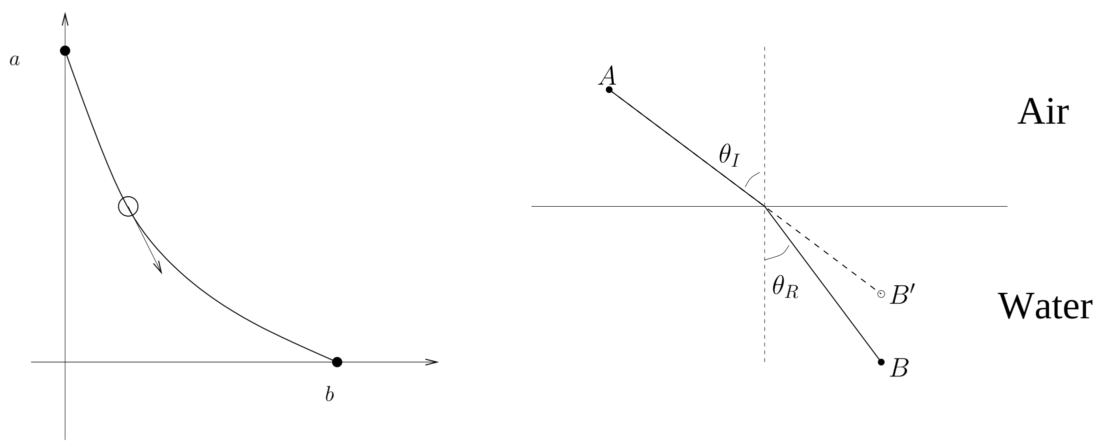


Figure 2: The Brachistochrone and the Fermat's problem of least time.

2 Formulation of the problem and derivation of the governing equation

The fundamental problem in the calculus of variations: Given a functional $I = I[y(x)]$, find the function $y(x)$ for which $I[y(x)]$ is maximum or minimum.

Here a functional takes a function $y(x)$ into a real number (function of function), for example:

$$I[y] = \int_a^b y(x)dx, \quad I[y] = \int_a^b \big(y(x)\big)^2 dx, \quad I[y] = \int_a^b \big(y'(x)\big)^2 dx.$$

Motivation for the characterizing equation: In calculus, to find the maximum or minimum of a function $f(x)$, for x in n-dimensional space, the usual approach is to find all the points x^* such that $\nabla f(x^*) = 0$. But $I[y]$ is defined for $y(x)$ in an infinite dimensional space, and we can not define *the gradient or partial derivatives* of $I[y]$ with respect to y.

The motivation comes from the equivalent condition to $\nabla f(x^*)$: *for any* $\tilde{x}$, if $\varphi(\epsilon) = f(x^* + \epsilon\tilde{x})$ has a maximum or minimum point at $\epsilon = 0$, then $f(x)$ has a maximum or minimum point at x^*. In fact,

$$0 = \varphi'(0) = \frac{d}{d\epsilon}\varphi(\epsilon)|_{\epsilon=0} = \frac{d}{d\epsilon} f(x^* + \epsilon\tilde{x})\Big|_{\epsilon=0} = \tilde{x} \cdot \nabla f(x^* + \epsilon\tilde{x})\Big|_{\epsilon=0} = \tilde{x} \cdot \nabla f(x^*).$$

Since $\tilde{x}$ is arbitrary, we can choose $\tilde{x} = \nabla f(x^*)$ to get $0 = \nabla f(x^*) \cdot \nabla f(x^*)$ or $\nabla f(x^*) = 0$. In this way, all the calculations are for the *scalar* function φ, regardless the dimension of the space of x. We are going to derive the governing equation for the maximizer or minimizer of $I[y]$, by first reducing it to a scalar point with a "test" function η.

Derivation of the Euler-Lagrange equation: Find the curve $y(x)$ that extremizes the integral

$$I[y] = \int_{x_0}^{x_1} f(x, y(x), y'(x))dx$$

subject to the end conditions $y(x_0) = y_0, y(x_1) = y_1$. Similarly, the extreme curve $y(x)$ is characterized by the fact that for any other function η, $\varphi(\epsilon) = I[y + \epsilon\eta]$, we have

$$0 = \varphi'(0) = \lim_{\epsilon\to 0} \frac{I[y+\epsilon\eta] - I[y]}{\epsilon} \equiv \frac{d}{d\epsilon} I[y+\epsilon\eta]\bigg|_{\epsilon=0}. \tag{1}$$

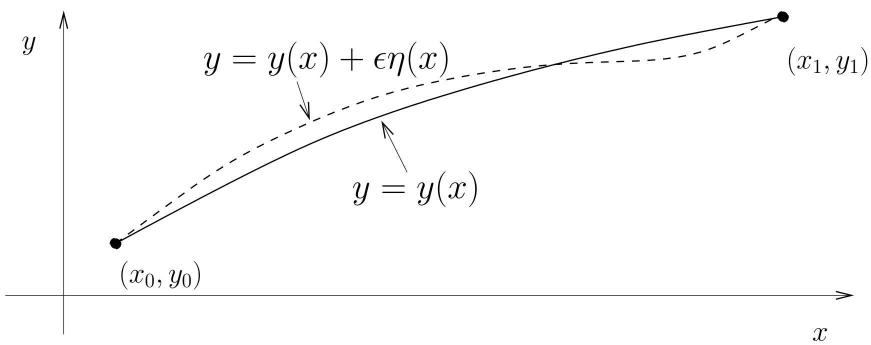


Figure 3: If $y = y(x)$ is a minimizer of $I[y]$, then $I[y + \epsilon]$ should be larger than $I[y]$ for any variation $y + \epsilon\eta$.

Remark (The boundary conditions). Here $y + \epsilon\eta$ should satisfies the same end condition as y, $y(x_0) + \epsilon\eta(x_0) = y_0, y(x_1) + \epsilon\eta(x_1) = y_1$ or $\eta(x_0) = \eta(x_1) = 0$. We need this boundary condition later to eliminate certain terms to get the final Euler-Lagrange equation.

For simple cases, we can evaluate the limit in (1) directly. For example when $I[y] = \int_{x_0}^{x_1} (y'(x)^2 + y(x)^2)dx$, then

$$\begin{aligned}
0 = \lim_{\epsilon\to 0} \frac{I[y+\epsilon\eta] - I[y]}{\epsilon} &= \int_{x_0}^{x_1} \left[\lim_{\epsilon\to 0} \frac{\big((y'(x)+\epsilon\eta'(x)\big)^2 + \big(y(x)+\epsilon\eta(x)\big)^2 - y'(x)^2 - y(x)^2}{\epsilon} \right] dx \\
&= \int_{x_0}^{x_1} \lim_{\epsilon\to 0} \left[2y'(x)\eta'(x) + 2y(x)\eta(x) + \epsilon(\eta'(x)^2 + \eta(x)^2)\right] dx \\
&= \int_{x_0}^{x_1} \big(2y'(x)\eta'(x) + 2y(x)\eta(x)\big) dx
\end{aligned}$$

For more complicated integrand $f(x, y, y')$, we can use the definition and partial derivatives to find the Euler-Lagrange equation characterizing extremal curves of the integrals.

The **Euler-Lagrange equation** for the stationary value of the integral $I[y] = \int_a^b f(x, y, y')dx$ can be derived using the definition (1):

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{I[y+\epsilon\eta] - I[y]}{\epsilon} &= \int_{x_0}^{x_1} \lim_{\epsilon\to 0} \frac{f(x, y+\epsilon\eta, y'+\epsilon\eta) - f(x,y,y')}{\epsilon} dx \\
&= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right] dx = 0. \qquad (2)
\end{aligned}$$

It is not immediately clear the conditions such that the last expression in (2) is identically zero, for any η. We have to convert η' to η, using integration by parts:

$$\begin{aligned} 0 = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right] dx &= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \eta \ dx + \frac{\partial f}{\partial y'}\eta \Big|_{x_0}^{x_1} \\ &= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \eta \ dx. \end{aligned} \tag{3}$$

During the last step, the end condition for η is used such that

$$\frac{\partial f}{\partial y'}\eta \Big|_{x_0}^{x_1} = \frac{\partial f}{\partial y'}(x_1, y(x_1), y'(x_1))\eta(x_1) - \frac{\partial f}{\partial y'}(x_0, y(x_0), y'(x_0))\eta(x_0) = 0.$$

Since η is arbitrary, we get the *Euler-Lagrange equation*:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0. \tag{4}$$

The derivative $\dfrac{d}{dx}$ is the total derivative and if we expand the second term

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = \frac{\partial^2 f}{\partial x \partial y'}\frac{dx}{dx} + \frac{\partial^2 f}{\partial y \partial y'}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y' \partial y'}\frac{dy'}{dx} = \frac{\partial^2 f}{\partial x \partial y'} + y'\frac{\partial^2 f}{\partial y \partial y'} + y''\frac{\partial^2 f}{\partial y' \partial y'}. \tag{5}$$

The Euler-Lagrange equation is usually a second order differential equation, because of the last term $y''\frac{\partial^2 f}{\partial y' \partial y}$ in (5). Moreover, the expansion $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'} \right)$ is in general very complicated, we expand it only when f is relatively simple in y' (for example, f is quadratic and we get a linear differential equation).

In this class, you can use the Euler-Lagrange equation (4) directly in your calculation (without the derivation). The main focus should be on how to solve (4), according the the precision form of f given.

How do you remember the Euler-Lagrange equation (4)**?** At least two ways:

(1) From the derivative above. Using δ to denote the *variation* (commonly used in math and engineering), then (forget about the bounary conditions)

$$\begin{aligned} \delta \int f(x, y, y')dx &= \int \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right] dx \\ &= \int \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}\delta y \right] dx = \int \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y dx. \end{aligned}$$

(2) If you vaguely remember the two terms in the Euler-Lagrange equation, then you can get the right form by notting (a) there is no terms like $\frac{\partial f}{\partial x}$ because the *variation* is with respect to y; (b) using dimensional analysis to determine the derivative $\frac{d}{dx}$ is on $\frac{\partial f}{\partial y}$ or $\frac{\partial f}{\partial y'}$. The equation (4) is the only one has right dimension of $\dfrac{\text{[the integral } f]}{\text{[the function } y]}$. The dimension of the equation $\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y} \right) = 0$ does not match ($\frac{[x][f]}{[y]} \neq \frac{[f]}{[x][y]}$).

3 Solution of the Euler-Lagrange equation

In general, the solutions can be classified into two classes:

(i) The Euler-Lagrange equation is **linear** (f is at most quadratic in y and y') and we can find the general solution. For example:

1) $\displaystyle\int_0^1 (y'^2 + y^2)dx, \quad y(0) = 0, y(1) = 1.$

2) $\displaystyle\int_0^1 \left[x^2\left(\frac{dy}{dx}\right)^2 + 2y^2\right] dx, \qquad y(1) = 0, y(2) = 1$

See the Appendix about general solutions of the second order ODEs.

(ii) The Euler-Lagrange equation has certain "symmetry" in the sense that x or y is missing in f, then we can get a first integral (reducing the equation to an ODE), and the final problem is reduced into integration w.r.t y or x only:

a) If x **is missing** (or $f = f(y, y')$), then $f - y'\dfrac{\partial f}{\partial y'} = c_1$ is a constant. In fact,

$$\begin{aligned}\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\frac{\partial f}{\partial y'} \\ &= y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \\ &= 0 \qquad\qquad \text{(by Euler-Lagrange equation)}\end{aligned}$$

Since x is missing, this first order equation can be rewritten as $y' = F(y, c_1)$ for some function F and can be integrated by separation of variable:

$$\int^y \frac{dy}{F(y, c_1)} = x + c_2.$$

Example. For the hanging chain, the shape of the chain is determined by the extremal curve of the integral $\int y\sqrt{1 + y'^2}dy$. Therefore

$$c_1 = f - y'\frac{\partial f}{\partial y'} = \frac{y}{\sqrt{1 + y'^2}},$$

or

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{c_1^2} - 1}.$$

The general solution is given by

$$x + c_2 = \int \frac{dy}{\sqrt{\frac{y^2}{c_1^2} - 1}} = c_1 \ln\frac{y + \sqrt{y^2 - c_1^2}}{c_1} + c_2.$$

b) If y **is missing** (or $f = f(x, y')$), then by the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} = 0.$$

This implies that $\partial f/\partial y' = c_1$ is a constant. We can also solve for y' to get $y' = G(x, c_1)$ and the general solution is given by

$$y = \int G(x, c_1)dx + c_2.$$

Example (Problem 3 in tutorial sheet)**.** Show that the extremal curves of the integral

$$I[\theta] = \int r^2 \left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]^{1/2} dr$$

Here ϕ is the dependent variable and is missing in the integrand. Therefore,

$$\frac{\partial f}{\partial \theta'} = r^4 \frac{\theta'}{\sqrt{1 + r^2\theta'^2}} = c_1,$$

or

$$\frac{d\theta}{dr} = \frac{c_1}{r\sqrt{r^6 - c_1^2}}.$$

Using the substitution $r^3 = c_1 \sec u$, we get

$$\frac{dr}{r} = \frac{1}{3}\tan u du, \qquad \sqrt{r^6 - c_1^2} = c_1 \tan u$$

and

$$\theta = \int \frac{c_1}{r\sqrt{r^6 - c_1^2}} dr = \int \frac{1}{3} du = \frac{u}{3} + \text{constant}$$

Therefore, the solution is given by $r^3 = c_1 \sec(3\theta + c_2)$.

Remark. In general, the integration in a) and b) are complicated, and you should look at the solution (if given) to find the right change of variable.

4 More examples

Example (Brachistochrone and cycloid) The speed of the bead at (x, y) can be obtained from energy conservation, $\frac{1}{2}mv^2 - mgy = 0$ or $v = (2gy)^{1/2}$. Therefore,

$$dt = \frac{ds}{v} = \left(\frac{1 + y'^2}{2gy}\right)^{1/2},$$

and the total time from $(0, 0)$ to (x_1, y_1) is

$$T = I[y] = \int_0^{x_1} \left(\frac{1 + y'^2}{2gy}\right)^{1/2} dx.$$

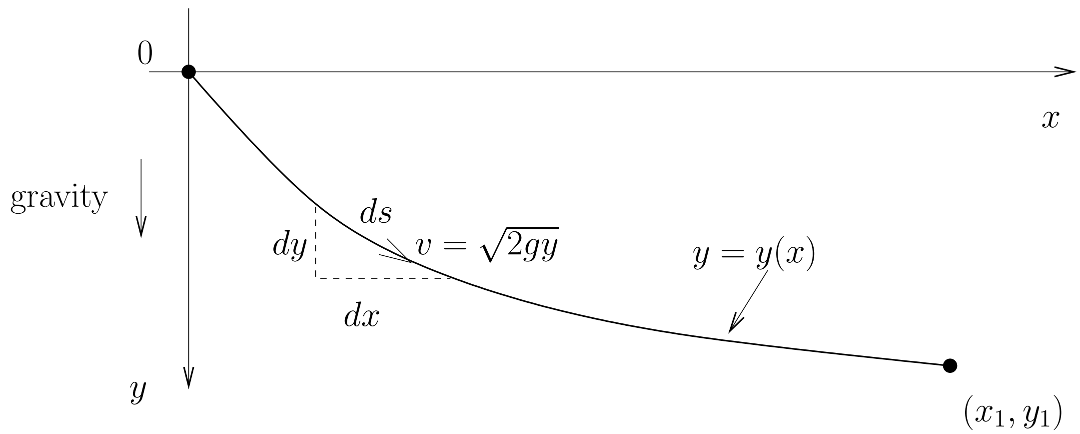


Figure 4: The brachistochrone: the curve that carries a bead from one place to another in the least amount of time.

Solution: Since $f = \left(\frac{1+y'^2}{2gy}\right)^{1/2}$ is independent of x,

$$f - y'\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy(1+y'^2)}}$$

is a constant, or $y(1+y'^2) = D^2$ for some constant D. The derivative y' can be obtained

$$y' = \sqrt{\frac{c_1^2}{y} - 1},$$

where the fact $y' \geq 0$ (with y increasing downwards) is used to get rid of the other solution $y' = -\sqrt{\frac{c_1^2}{y} - 1}$. The solution can be defined implicitly by

$$x = \int_y \frac{dy}{\sqrt{\frac{c_1^2}{y} - 1}} + c_2$$

or more conveniently in parametric form as follows. Let $y = c_1^2 \sin^2\theta$, then $dy = 2c_1^2 \sin\theta\cos\theta d\theta$ and

$$dx = \frac{dy}{\sqrt{\frac{c_1^2}{y} - 1}} = \frac{2c_1^2 \sin\theta\cos\theta}{\sqrt{\frac{1}{\sin^2\theta} - 1}} d\theta = 2c_1^2 \sin^2\theta d\theta.$$

Therefore,

$$x = 2c_1^2 \int \sin^2\theta d\theta = c_1^2 \int \big(1 - \cos 2\theta\big) d\theta = c_1^2\big(\theta - \frac{1}{2}\sin 2\theta\big) + c_2.$$

Using the boundary condition at the origin ($y = 0$ when $x = 0$), we get $c_2 = 0$ with $\theta_0 = 0$. Using the boundary at the end point,

$$c_1^2 \sin^2\theta_1 = y_1, \qquad c_1^2\big(\theta_1 - \frac{1}{2}\sin\theta_1\big) = x_1,$$

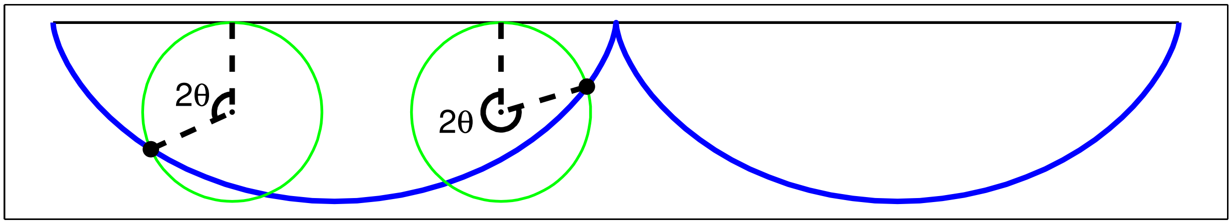


Figure 5: The solution to the brachistochrone is part of the cycloid.

from which we can find c_1 and θ_1, in terms of (x_1, y_1). Finally, the solution is given by

$$\begin{cases} x &= c_1^2\big(\theta - \frac{1}{2}\sin 2\theta\big), \\ y &= c_1^2 \sin^2\theta, \end{cases} \qquad 0 \le \theta \le \theta_1.$$

The resulting curve is called a cycloid. We can get the trajectory, which is a fixed point on a disk with radius c_1^2 rotating with angle 2θ. See Figure 5.

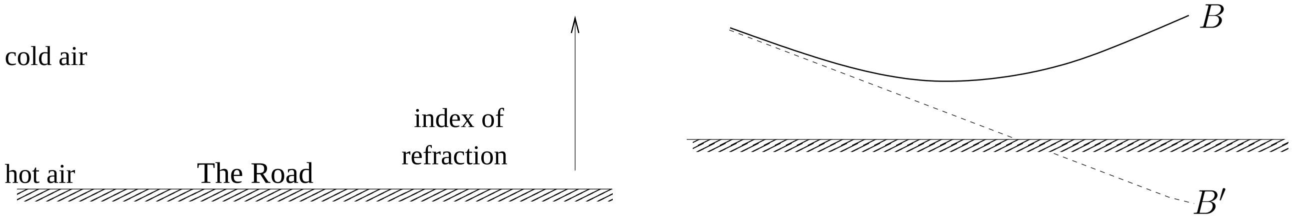


Figure 6: The creation of a mirage.

When light travels through a media, it is the time (instead of length) that is minimized. In a homogeneous media, where the refraction index n is constant, the path is a straight line. When light travels from one media to another, the law of refraction can be obtained by minimizing the traveling time. Even in one media, the index of refraction can differ significantly, for instance the air near the ground is very hot and the corresponding index of refraction is smaller. This may lead to mirage.

Example (The mirage) Find the extremal curve to

$$\int_{x_0}^{x_1} (1+\kappa y) n_0 \sqrt{1+y'^2}\, dx,$$

there the index of refraction is $(1+\kappa y)n_0$.

Solution: Since $f = (1+\kappa y)n_0\sqrt{1+y'^2}$ is independent of x, constant, or

$$f - y'\frac{\partial f}{\partial y'} = (1+\kappa y)n_0\sqrt{1+y'^2} - y'(1+\kappa y)n_0\sqrt{\frac{y'}{1+y'^2}} = \frac{(1+\kappa y)n_0}{\sqrt{1+y'^2}}$$

is a constant, denoted as c_1. We can solve y' from the equation $\frac{(1+\kappa y)n_0}{\sqrt{1+y'^2}} = c_1$, that is

$$y' = \pm\sqrt{\frac{(1+\kappa y)^2 n_0^2}{c_1^2} - 1}.$$

The solution is

$$x = \pm \int \frac{dy}{\sqrt{\frac{(1+\kappa y)^2 n_0^2}{c_1^2} - 1}} + c_2.$$

We can find the integral by the change of variable

$$\frac{(1+\kappa y)n_0}{c_1} = \cosh\theta. \tag{6}$$

Therefore

$$\sinh\theta d\theta = \frac{\kappa n_0}{c_1}dy, \qquad \sqrt{\frac{(1+\kappa y)^2 n_0^2}{c_1^2} - 1} = \sqrt{\cosh^2\theta - 1} = \sinh\theta.$$

The solution can be written as

$$x = \pm\frac{c_1}{\kappa n_0}\int d\theta + c_2 = \pm\frac{c_1}{\kappa n_0}\theta + c_2,$$

or $\theta = \pm\frac{\kappa n_0}{c_1}(x - x_1)$. Substituting θ into (6), we get

$$y = \frac{c_1}{\kappa n_0}\cosh\frac{\kappa n_0}{c_1}(x - x_0) - \frac{1}{\kappa}.$$

The most important information from the solution is that y is always *convex* (or *concave up*).

Exercise (Straight lines are the paths with shortest distance) Show that the shortest path between two points on a plane is a straight line. In other words, show that the extremal curve of

$$I[y] = \int_{x_0}^{x_1} \sqrt{1+y'^2}dx,$$

with the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$ is a straight line.

Exercise (Which method to choose?) Find the extremal curve of the integral $\int(y^2 - y'^2)dx$ (a) by solving the *linear* Euler-Lagrange equation (b) by realizing the fact that $f = y^2 - y'^2$ is independent of x. Which way is easier and faster?

5 Extremal curves with integral constraint

In practice, many problems involves constraints: the total length of a suspension bridge or a hanging chain, the perimeter or total area in isoperimetric problem. The general problem can be formulated as:

Find the extremal curve of

$$I[y] = \int_{x_0}^{x_1} f(x, y, y')dx, \qquad y(x_0) = y_0, \quad y(x_1) = y_1$$

with constraint

$$\int_{x_0}^{x_1} g(x, y, y')dx = J_0.$$

Similar to optimization of a scalar function with constraints, we have to introduce the *Lagrange multiplier* λ and replace f by

$$\tilde{f}(x, y, y') = f(x, y, y') - \lambda g(x, y, y').$$

The solution from the **Constrained-Euler-Lagrange** equation $\dfrac{\partial \tilde{f}}{\partial y} = \dfrac{d}{dx}\dfrac{\partial \tilde{f}}{\partial y'}$ has three parameters c_1, c_2 and λ, which are determined by the two end conditions and the constraint.

Remark. We can also use $\tilde{f}(x, y, y') = f(x, y, y') + \lambda g(x, y, y')$ in the definition, everything is the same except the opposite sign. But do keep the sign in front of λ consistent in one problem.

Example (Eigenfrequency). Find the extremal curve $y(x)$ of the functional

$$\int_0^1 \frac{1}{2}y'^2 dx, \qquad y(0) = 0, \; y(1) = 0$$

subject to the constraint

$$\int_0^1 \frac{1}{2}y^2 dx = 1.$$

Solution: Since $f = y'^2/2$ and $g = y^2/2$,

$$\tilde{f} = f - \lambda g = \frac{1}{2}y'^2 - \frac{\lambda}{2}y^2.$$

The corresponding Constrained-Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial \tilde{f}}{\partial y'}\right) - \frac{\partial \tilde{f}}{\partial y} = \frac{d}{dx}y' + \lambda y = y'' + \lambda y = 0.$$

The solution depends on the sign of λ:

i) When $\lambda < 0$, $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{\sqrt{\lambda}x}$. Using the boundary conditions $y(0) = 0, y(1) = 0$, we get $c_1 = c_2 = 0$. But the solution $y(x) = 0$ does not satisfy the constraint above.

ii) When $\lambda = 0$, $y = c_1 + c_2 x$. From the boundary conditions, we still get the same trivial solution $y(x) = 0$, which does not satisfy the constraint.

iii) When $\lambda > 0$, $y = c_1 \sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x$. The boundary condition $y(0) = 0$ implies that $c_2 = 0$, and the other boundary condition leads to $y(1) = c_1 \sin\sqrt{\lambda} = 0$. To get a non-trivial solution (or $c_1 \neq 0$), we must have $\sin\sqrt{\lambda} = 0$ or $\sqrt{\lambda} = k\pi$ for $k = 1, 2, \cdots$. Consequently,

$$\lambda_k = k^2\pi^2$$

and the coefficient c_1 can be obtained from the constraint

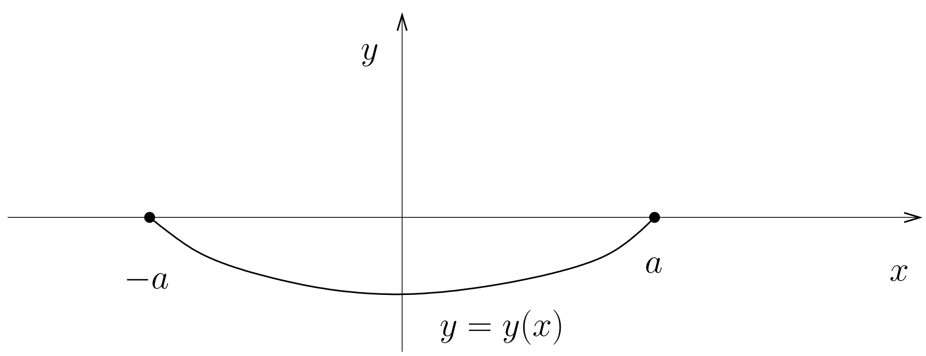
$$1 = \int_0^1 \frac{1}{2} y^2 dx = \frac{c_1^2}{2}\int_0^1 \big(\sin(k\pi x)\big)^2 dx = \frac{c_1^2}{4},$$

or $c_1 = \pm 2$.

Therefore, the extremal curves are $y_k = 2\sin k\pi x$, with the corresponding Lagrange multiplier $\lambda_k = k^2\pi^2$.

Remark. In the previous example, the Lagrange multiplier can be interpreted as the eigenfrequency. Here y_1 is the minimizer of the constrained problem, but all other y_ks with $k > 1$ are not minimizers.

Example (Suspension bridge and catenary). Find the curve that minimize the total (rescaled) gravitational potential $\int_{-a}^{a} y\sqrt{1+y'^2}dx$ with $y(a) = y(-a) = 0$, subject to the total length of the curve $\int_{-a}^{a}\sqrt{1+y'^2}dx = L > 2a$.



Solution: We have $\tilde{f} = f - \lambda g = (y-\lambda)\sqrt{1+y'^2}$. Since $\tilde{f}$ does not depend on x,

$$\tilde{f} - y'\frac{\partial f}{\partial y'} = \frac{y-\lambda}{\sqrt{1+y'^2}} = c_1$$

is a constant. Then derivative y' can be solved from this equation

$$y' = \pm\sqrt{\frac{(y-\lambda)^2}{c_1^2} - 1}. \tag{7}$$

Using the change of variable $y = \lambda + c_1\cosh u$, we have

$$dy = c_1 \sinh u du, \qquad \sqrt{\frac{(y-\lambda)^2}{c_1^2} - 1} = \sinh u.$$

The differential equation (7) is reduced to (the sign can be absorbed by c_1).

$$dx = \frac{dy}{\sqrt{\frac{(y-\lambda)^2}{c_1^2} - 1}} = c_1 du,$$

or $u = x/c_1 + c_2$. Therefore the solution is given by $y = \lambda + c_1 \cosh(c_2 + x/c_1)$. The boundary conditions $y(a) = y(-a) = 0$ become

$$0 = \lambda + c_1 \cosh\left(\frac{a}{c_1} + c_2\right), \quad 0 = \lambda + c_1 \cosh\left(-\frac{a}{c_1} + c_2\right),$$

which can be solved as $c_2 = 0, \lambda = -c_1 \cosh a/c_1$. Substituting the solution $y = c_1\big(\cosh(x/c_1) - \cosh(a/c_1)\big)$ into the constraint

$$L = \int_{-a}^{a} \sqrt{1 + y'^2}dx = \int_{-a}^{a} \cosh\frac{x}{c_1}dx = 2c_1 \sinh\frac{a}{c_1}.$$

Solve $c_1(> 0)$ from this algebraic equation, then we get λ and the curve $y(x)$.

6 Some Extensions (not required for the exam)

There are many general extensions to the Euler-Lagrange equation, higher order derivatives, multiple independent variables and several dependent functions:

(I) If f depends on higher order derivatives, say $f = f(x, y, y', y'')$, then the Euler-Lagrange equation becomes fourth order

$$f - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0.$$

(II) If f depends on multiple independent variables, say x_1 and x_2, with the corresponding partial derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ (that is $f = f(x_1, x_2, u, u_{x_1}, u_{x_2})$), then we get a partial differential equation:

$$0 = \frac{\partial f}{\partial u} - \frac{d}{dx_1}\frac{\partial f}{\partial u_{x_1}} - \frac{d}{dx_2}\frac{\partial f}{\partial u_{x_2}} = \frac{\partial f}{\partial u} - \text{div } \mathbf{F},$$

where $\mathbf{F} = (u_{x_1}, u_{x_2})$.

(III) If f depends on several function, say $f = f(x, u, v, u_x, v_x)$, then we get a system of equation

$$0 = \frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u_x}, \qquad 0 = \frac{\partial f}{\partial v} - \frac{d}{dx}\frac{\partial f}{\partial v_x}.$$

The equations that are Euler-Lagrange equations of special f:

	Equation	The functional
The beam equation	$EI\dfrac{d^4}{dx^4}w(x) = q(x)$	$\displaystyle\int_0^L \left(\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - q(x)w(x)\right) dx$
Euler equation	$\frac{d^2}{dt^2}X(\alpha, t) = -\nabla p$	$\displaystyle\frac{1}{2}\int \left\lvert \frac{d}{dt}X(\alpha, t)\right\rvert^2 d\alpha$ + incompressibility
Kepler's problem	$\dfrac{d}{dt}(mr^2\dot{\theta}) = 0,\ m\ddot{r} = mr\dot{\theta}^2 - \dfrac{GMm}{r^2}$	$\displaystyle\int_{t_0}^{t_1} \left(\frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{GMm}{r}\right) dt$
Wave equation	$u_{tt} = \Delta u$	$\displaystyle\int\int \left(\frac{1}{2}u_t^2 - \frac{1}{2}\lvert\nabla u\rvert^2\right) dtdx$

Two major types problems arising in calculus of variations:

(1) Steady state that minimize the energy, for example the catenary minimizes the total gravitational energy or the soap bubble minimizes the total surface area. You get a **boundary value problem**.

(2) Mechanic or time-dependent systems corresponds to an *action* $\mathcal{A}$, where the action is the integral (w.r.t time) of the *Lagrangian* $\mathcal{L}$ defined to be the **difference** of kinetic energy and potential energy. For example, the trajectory of a particle $\mathbf{x}(t)$ with mass m under the force $-\nabla V(\mathbf{x})$, then the equation can be obtained from $\int_{t_0}^{t_1}\left(\frac{m}{2}|\dot{\mathbf{x}}|^2 - V(\mathbf{x})\right)dt$, or the Newton's equation of the second law. You get a **initial value problem**. In this case, the total energy $\frac{m}{2}|\dot{\mathbf{x}}|^2 + V(\mathbf{x})$ (the **sum** of kinetic and potential energy) is conserved.

A General solutions of linear second order ODEs

Linear second order ODEs have the form

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{8}$$

The equation is called *homogeneous* if $f(x) \equiv 0$. In general, the coefficients $a(x), b(x), c(x)$ are either constants or monomials (for example $a(x) = x^2, b(x) = x, c(x) = 1$).

A.1 Homogeneous linear equations with constant coefficients

The equation is $ay'' + by' + cy = 0$ where a, b, c are constants and a is nonzero. We can find the general solution with the characteristic equation (using the ansatz $y(x) = e^{\mu x}$)

$$a\mu^2 + b\mu + c = 0.$$

(1) If μ_1 and μ_2 are distinct real numbers (when $b^2 - 4ac > 0$), then the general solution is

$$y(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}.$$

(2) If $\mu_1 = \mu_2$ (when $b^2 - 4ac = 0$), then the general solution is

$$y(x) = c_1 e^{\mu_1 x} + c_2 x e^{\mu_1 x}.$$

(3) If $\mu_1 = \alpha + i\beta$ and $\mu_2 = \alpha - i\beta$ are complex numbers (when $b^2 - 4ac < 0$), then the general solution is
$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x.$$

Example.

(1) $y'' + y = 0$

(2) $y'' - y = 0$

(3) $y'' + 2y' + y = 0$

(4) $y'' + 2y' + 2y = 0$

A.2 Homogeneous linear equations with monomials

These linear ODEs have the form

$$ax^2y'' + bxy' + cy = 0.$$

We are looking for solutions of the form $y(x) = x^\ell$, whose characteristic equation (by putting $y(x) = x^\ell$ into above equation)

$$a\ell(\ell - 1) + b\ell + c = 0.$$

There are two solutions ℓ_1 and ℓ_2 for the quadratic equation and the general solution of the differential equation is
$$y(x) = c_1 x^{\ell_1} + c_2 x^{\ell_2}.$$

A.3 Solutions of equations with inhomogeneous terms

The general solution of the equation

$$a(x)y'' + b(x)y' + c(x)y' = f(x)$$

has the structure $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$, where c_1 and c_2 are two constants (determined by end conditions), y_1 and y_2 are the solutions of the homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$, and $y_p(x)$ is **any** solution of the inhomogeneous equation. The particular solution can be obtained by inspection:

Example. Find the general solution of the following equations

(1) $y'' - y = x, y(0) = 1, y(1) = e^1 - 1$

(2) $x^2y'' + xy' = x^4, y(1) = \frac{1}{16}, y(2) = \frac{1}{4}$