Magnification dynamics and non-conformality

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joint work with **Andrew Ferguson** and **Jonathan Fraser**

University of BRISTOL × 2

THE UNIVERSITY OF WARWICK

Spirit of the talk

"*You take an obvious concept of a limit, and then, by the power of analysis, you can go to the limit many times, which creates structures that you have not seen before. You think you have not done anything but, amazingly, you have achieved something.*"

–Mikhail Gromov, March 2010

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- Hochman-Shmerkin (2012): Applications to projection theorems, which yielded a solution to a conjecture of Furstenberg (not the famous $\times 2, \times 3$ one however!)
- Hochman-Shmerkin (2013): Applications to equidistribution problems in metric number theory
- ...more in progress ☺

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- The **magnification** μ^D of a measure $\mu \in \mathcal{P}([0,1)^d)$ to D with $\mu(D) > 0$ is
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- If $k \in \mathbb{N}$, let $D_k(x) \in \mathcal{D}_k$ be the cube with $x \in D_k(x)$. Write
$$\Xi = \{(x,\mu) : \mu \in \mathcal{P}([0,1)^d) \text{ and } \mu(D_k(x)) > 0 \text{ for any } k \in \mathbb{N}\}.$$

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$$\Xi = \{(x,\mu) : \mu \in \mathcal{P}([0,1)^d) \text{ and } \mu(D_k(x)) > 0 \text{ for any } k \in \mathbb{N}\}.$$
- Define the **magnification operator** $M : \Xi \to \Xi$ by
$$M(x,\mu) = (T_{D_1(x)}(x), \mu^{D_1(x)})$$
on the **phase space** Ξ.

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- The measure component a micromeasure distribution is supported on **micromeasures** of μ at x (i.e. accumulation points of the '**minimeasures**' $\mu^{D_k(x)}$, $k \in \mathbb{N}$)
- Any micromeasure distribution Q is an M **invariant measure** i.e. $M_*Q = Q$.

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- It is also possible to use more general (regular) filtrations than dyadic. Then the dynamics is described by a Markov process (a **CP chain**) and the **CP distribution** is the stationary measure for this chain.

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(1) Q is the **only** micromeasure distribution of μ at μ almost every x;

(2) and at μ almost every x also the q-sparse scenery distributions

$$\frac{1}{N}\sum_{k=0}^{N-1}\delta_{M^{qk}(x,\mu)} \in \mathcal{P}(\Xi)$$

converge to some, possibly other, distribution Q_q for any $q \in \mathbb{N}$.

The condition (2) is technical, but is essential to carry geometric information from the micromeasure level back to μ.
Often $Q_q = Q$ for any $q \in \mathbb{N}$.

Example: Distortion of dimension under projections

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Theorem (Hochman-Shmerkin 2012)

Suppose μ generates an ergodic CP distribution Q. Then

(1) $\dim \pi_* \mu \geq E(\pi)$ for any $\pi \in \Pi_{d,k}$;

(2) for any C^1 map $f : \mathbb{R}^d \to \mathbb{R}^k$ without singular points

$$\dim f\mu \geq \operatorname*{ess\,inf}_{x \sim \mu} E(D_x f).$$

Moreover, for a.e. $\pi \in \Pi_{d,k}$ the expectation $E(\pi) = \min\{k, \dim \mu\}$.

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- **Example**: Fix integers $m < n$ and define $T_{m,n} : [0,1]^2 \to [0,1]^2$,

$$T_{m,n}(x,y) = (T_m(x), T_n(x)),$$

where $T_m : [0,1] \to [0,1]$ is given by

$$T_m(x) = mx \bmod 1.$$

Here $[0,1]$ is thought as a unit circle $\mathbb{T}$ and $[0,1]^2$ as the 2-torus $\mathbb{T}^2$.

Non-conformality II: $T_{m,n}$ invariant sets

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A subset $K \subset [0,1]^2$ is a **Bedford-McMullen carpet**, if $K = \bigcup_{i,j} f_{i,j}(K)$, where

$$f_{i,j} = \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix},$$

for some indices $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Then $K = T_{m,n}^{-1}(K)$.

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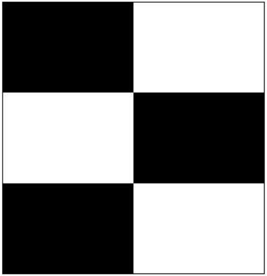

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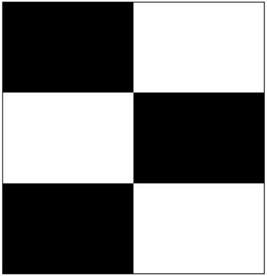

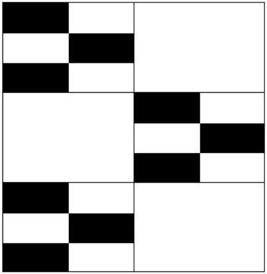

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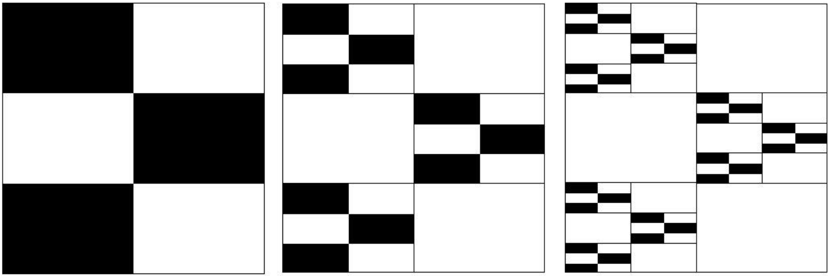

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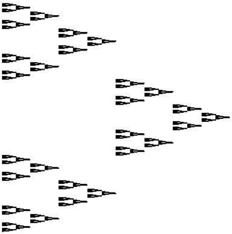

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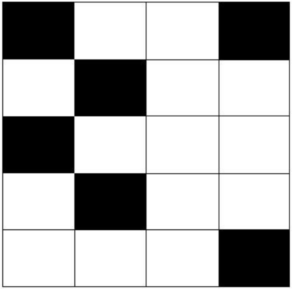

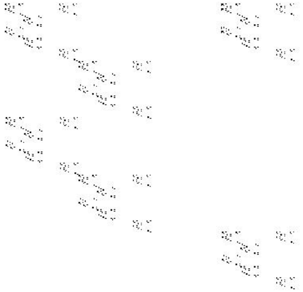

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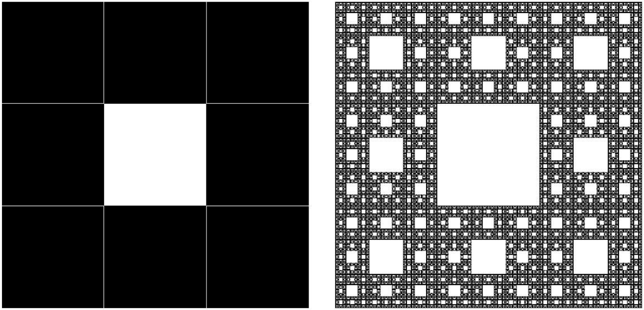

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- A $T_{m,n}$ invariant measure μ on $[0,1]^2$ is **Bernoulli** if all 'construction rectangles' are mutually μ independent. I.e.

$$\mu[(I \times J) \cap (I' \times J')] = \mu(I \times J)\mu(I' \times J')$$

for all m-adic intervals $I, I' \subset [0,1]$ and n-adic intervals $J, J' \subset [0,1]$.

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for all m-adic intervals $I, I' \subset [0,1]$ and n-adic intervals $J, J' \subset [0,1]$.
- **Example:** The **McMullen measure** of a Bedford-McMullen carpet is a $T_{m,n}$ invariant Bernoulli measure of maximal Hausdorff dimension.

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- It was proved by Käenmäki and Bandt (2011) that under mild assumptions the 'tangent sets' of Bedford-McMullen carpets (wrt. Hausdorff distance) are of the form

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- The product form of the tangents was exploited by Mackay (2011) when computing the conformal Assouad dimension of Bedford-McMullen carpets.

Non-conformality III: Magnification, cont.

Theorem

Any $T_{m,n}$ Bernoulli measure μ generates an ergodic CP distribution Q.

Non-conformality III: Magnification, cont.

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- Measure component $\tilde{Q}$ is the distribution of the random measure

$$\pi_1\mu \times \mu_x,$$

  where $x \sim \pi_1\mu$ and $\mu_x \in \mathcal{P}([0,1])$ is the conditional measure of μ with respect to the fibre $\pi_1^{-1}\{x\}$.
- Here we do not use dyadic cubes to magnify but more general but still **regular** partitions obtained from 'construction rectangles' $I \times J$, where $I, J \subset [0,1]$ are m-adic and n-adic intervals respectively. Hence the CP distribution Q is a stationary measure for an **ergodic CP chain**.

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Furstenberg's Conjecture (one of them, from 1960's)

If $X, Y \subset [0,1]$ are closed and T_2 and T_3 invariant respectively. Then

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Solved:

Theorem (Hochman-Shmerkin 2012)

If $\mu, \nu \in \mathcal{P}([0,1])$ are T_m and T_n invariant respectively and $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\dim \pi_*(\mu \times \nu) = \min\{1, \dim(\mu \times \nu)\}, \quad \pi \in \Pi_{2,1} \setminus \{\pi_1, \pi_2\}.$$

Obtained by constructing an ergodic CP distribution out of $\mu \times \nu$.

Application I: Projections, cont.

Conjecture

Suppose μ is a $T_{m,n}$ invariant measure and $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$, then

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Suppose K is a Bedford-McMullen carpet with $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$. Then

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Theorem

The conjecture above holds for $T_{m,n}$ invariant *Bernoulli* measures.

Theorem (Hochman-Shmerkin 2012)

Suppose μ generates an ergodic CP distribution Q. Then

(1) $\dim \pi_* \mu \geq E(\pi)$ for any $\pi \in \Pi_{d,k}$;

(2) for any C^1 map $f : \mathbb{R}^d \to \mathbb{R}^k$ without singular points

$$\dim f\mu \geq \operatorname*{ess\,inf}_{x \sim \mu} E(D_x f).$$

Moreover, for a.e. $\pi \in \Pi_{d,k}$ the value $E(\pi) = \min\{k, \dim \mu\}$.

- After suitable reparametrisation of $\Pi_{2,1}$, the map E is invariant under the **irrational** $\frac{\log m}{\log n}$ **rotation** of the circle, so E is constant as a lower semicontinuous function.

Application II: Distance sets

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The **distance set** of $K \subset \mathbb{R}^d$ is

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Distance set conjecture (Falconer, 1980's)

Suppose $K \subset \mathbb{R}^d$ is Borel and $\dim K \geq d/2$. Then $\dim D(K) = 1$.
Moreover, if $\dim K > d/2$, then $\mathcal{L}^1(D(K)) > 0$.

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Many people involved, current state:

- Bourgain (2003) found a small constant $\varepsilon > 0$ with

$$\dim D(K) \geq \frac{1}{2} + \varepsilon$$

  whenever $K \subset \mathbb{R}^2$ with $\dim K \geq 1$.
- Erdogan (2006) proved $\dim K > d/2 + 1/3$ in $\mathbb{R}^d$ yields positive measure for $D(K)$.
- Orponen (2011) proved $\dim D(K) = 1$ if K is a planar self-similar set with $\mathcal{H}^1(K) > 0$.

Application II: Distance sets, cont.

Theorem

If μ on $\mathbb{R}^2$ generates an ergodic CP distribution and $\mathcal{H}^1(\operatorname{spt}\mu) > 0$, then

$$\dim D(\operatorname{spt}\mu) \geq \min\{1, \dim\mu\}.$$

Application II: Distance sets, cont.

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If μ is **any** measure on $\mathbb{R}^2$ with $\dim\mu > 1$, then after a random translation of μ, there are micromeasures ν of μ at μ a.e. x with $\dim D(\operatorname{spt}\nu) = 1$.

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Corollary

If K is a Bedford-McMullen carpet with $\dim K \geq 1$, then $\dim D(K) = 1$.

- Using standard dimension approximation theorems via Bedford-McMullen 'type' carpets, this yields results for other **Lalley-Gatzouras** and **Barański** type self-affine carpets as well.

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- Conformal Hausdorff dimension of self-affine carpets (???)

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Kiitos!