

The Brownian graph is not round

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The Open University, Milton Keynes, 16.4.2013

joint work with **Jonathan Fraser** and **Tuomas Orponen**



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...so if for $\varepsilon > 0$ we have

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-(s+\varepsilon)/2}, \xi \in \mathbb{R}^2 \implies I_s(\mu) < \infty \implies \dim_{\text{H}} K \geq s.$$

Fourier analysis and Hausdorff dimension

This motivates...

Definition (Fourier dimension)

$$\dim_{\mathbb{F}} K := \sup\{s \leq 2 : \exists \mu \in \mathcal{P}(K) \text{ with } |\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}, \xi \in \mathbb{R}^2\}$$

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Definition (Round sets)

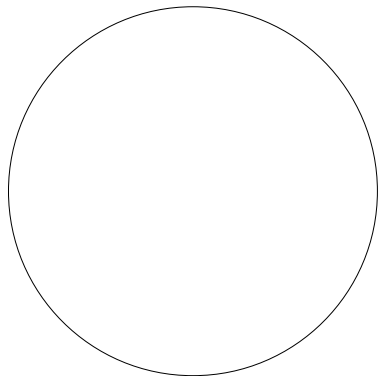
If $\dim_{\mathbb{F}} K = \dim_{\mathbb{H}} K$, we say that K is **round**.

- Round sets are also known as **Salem sets**.

Finding round sets

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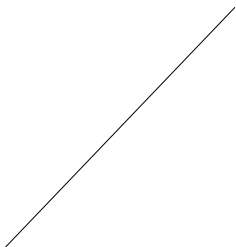
Unit circle S^1 is round...



...since $\widehat{\mathcal{H}^1 \llcorner S^1}$ decays like $|\xi|^{-1/2}$.

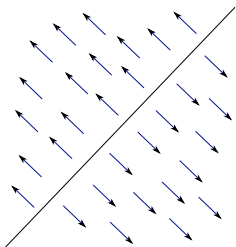
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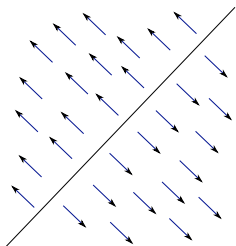


...since $\hat{\mu} \equiv 1$ on L^\perp for any $\mu \in \mathcal{P}(L)$!

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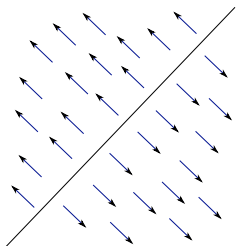
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- Punchline: $\dim_{\mathbb{H}}$ measures size, but $\dim_{\mathbb{F}}$ also contains information on **curvature**.

'Non-trivial' round sets are hard to construct deterministically; but

- there are some examples by Kahane and Kaufman; in particular some sets arising from **Diophantine approximation** are round.

Finding round sets III

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Many **random sets** are round... first such constructions are due to Salem, but the following key result is by Kahane.

Theorem (Kahane 1986)

Let $\omega: [0, \infty) \rightarrow \mathbb{R}$ be 1-dimensional Brownian motion, and let $K \subset [0, \infty)$ be compact. Then the **image** $\omega(K) \subset \mathbb{R}$ is a.s. round, with

$$\dim_{\mathbb{F}} \omega(K) = \dim_{\mathbb{H}} \omega(K) = \min\{1, 2 \dim_{\mathbb{H}} K\}.$$

Analogous result also holds for fractional Brownian motion.

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Kahane writes (1993):

“...proving almost sure roundedness for specific random sets is never easy and it remains an open program for most natural random sets: level sets and graphs of random functions in particular.”

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Kahane writes (1993):

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Later (2006), Shieh and Xiao explicitly ask:

“Are the graph and level sets of a stochastic process such as fractional Brownian motion Salem sets?”

Finding round sets V

The conjecture is partially confirmed for level sets:

Theorem (Fouché and Mukeru 2013)

Let $\omega : [0, \infty) \rightarrow \mathbb{R}$ be 1-dimensional (fractional) Brownian motion.
Then for $a \in \mathbb{R}$, the **level set**

$$L_\omega(a) = \{0 \leq t \leq 1 : \omega(t) = a\}$$

is round with positive probability and

$$\dim_{\text{F}} L_\omega(a) = \dim_{\text{H}} L_\omega(a) = \frac{1}{2}.$$

Graphs of Brownian motion

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The Hausdorff dimension part is classical:

Theorem (Taylor 1953, Adler 1977)

*Let $\omega : [0, \infty) \rightarrow \mathbb{R}$ be the 1-dimensional fractional Brownian motion with Hurst exponent $0 < H < 1$. Then, the **graph***

$$G_\omega := \{(t, \omega(t)) : t \in [0, \infty)\} \subset \mathbb{R}^2$$

a.s. satisfies

$$\dim_{\text{H}} G_\omega = 2 - H.$$

- In particular, $\dim_{\text{H}} G_\omega > 1$ a.s.

Fourier dimension of graphs

We proved:

Theorem (Fraser, Orponen, S. 2013)

Let $E \subset \mathbb{R}$ be a set, and let $f: E \rightarrow \mathbb{R}$ be a function. Then

$$\dim_{\mathbb{F}} G_f \leq 1.$$

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Combining this with Taylor's and Adler's results answers Kahane's, Shieh's and Xiao's questions on random graphs in the negative:

Corollary

The Brownian graph G_{ω} is a.s. **not** round/Salem.

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- **Marstrand's projection theorem:** if a planar set K has Fourier dimension $s \in [0, 1]$, then all projections of K onto lines have Fourier dimension $\geq s$ (folklore).
- **Falconer's distance set conjecture:** if a planar set K has Fourier dimension $s > 1$, then the distance set of K , namely

$$\Delta(K) = \{|x - y| : x, y \in K\},$$

has positive length (P. Mattila).

Proof II

For us, the key example is

- **Marstrand's slicing theorem:**¹ if a planar set K has $\dim_{\text{F}} K > 1$, then in every direction there are Leb positively many lines ℓ with

$$\dim_{\text{H}}[K \cap \ell] > 0.$$

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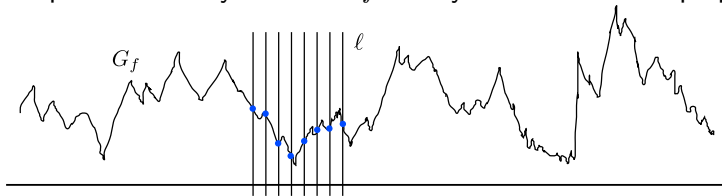
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- Graphs of arbitrary functions f clearly do not have this property:



Hence, they can have Fourier dimension *at most one!*

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- **The enemy:** classical proofs do not use Fourier analysis (i.e. relation with Fourier transforms and dimension is not clear).
- **The solution:** invent a Fourier analytic proof for Marstrand's slicing theorem.

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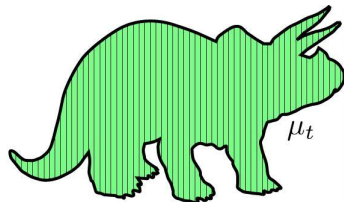
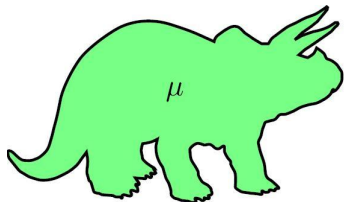
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- **Slice** the measure μ with vertical lines $L_t = \{(t, y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$ to obtain 'sliced measures' μ_t , supported on $K \cap L_t$.



Easy but important: $\mu_t \neq 0$ for Leb positively many t .
(this requires the decay assumption of $\widehat{\mu}$ and Plancherel's formula)

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- Consider the $(s - 1)$ -energies

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$$\implies I_{s-1}(\mu_t) < \infty \quad \text{for Leb a.e. } t \in \mathbb{R}.$$

$$\implies \dim_{\text{H}}[K \cap L_t] \geq s - 1 > 0 \quad \text{for Leb pos. many } t.$$

Q.E.D.

Open questions

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We only proved that $\dim_{\mathbb{F}} G_\omega \leq 1$.
- Is Fourier dimension countably (or even finitely) stable? I.e.

$$\dim_{\mathbb{F}} \left(\bigcup_{i \in \mathbb{N}} A_i \right) = \sup_i \dim_{\mathbb{F}} A_i?$$

Further results

Not being able to solve the first open question, we considered the following variant:

Question

What is the Fourier dimension of the graph of a typical function $f \in C[0, 1]$?

- Here we mean *typical* in the sense of Baire category.

Further results II

We proved:

Theorem (Fraser, Orponen, S. 2013)

The typical function $f \in C[0, 1]$ has the following property. If $\mu \in \mathcal{P}(G_f)$, then

$$\limsup_{|\xi| \rightarrow \infty} |\widehat{\mu}(\xi)| \geq \frac{1}{5}.$$

In particular, $\dim_{\mathbb{F}} G_f = 0$.

- The constant $1/5$ is not sharp (optimal constant unknown).
- Hausdorff dimension $\dim_{\mathbb{H}} G_f \geq 1$ for any $f \in C[0, 1]$, so our result implies that the graph of a typical function is *not* round/Salem!

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