Papers

Floor vibration

Floor vibration induced by dance-type loads: theory

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Synopsis
This paper is concerned with the response of floors to loading produced by dancing and aerobics, especially where the dancing involves jumping. Its purpose is to provide an analytical method for determining the response of floors to these loads. The characteristics of the load time history are dealt with initially, and, for calculation purposes, the load is expressed in terms of Fourier series. An analytical solution of the forced vibration of simply supported floors is developed, using plate theory and considering several modes of vibration. The number of Fourier terms that should be considered in the analysis is determined. The solution is then extended for other structures with different boundary conditions. It is predicted that significant accelerations may occur on relatively stiff floors induced by higher Fourier components of the load. (Verification of the method is provided in ref 6.)

Notation

a_n, b_n, r_n	are Fourier coefficients
a	is the acceleration (m/s²)
A_{ij}	is the generalised co-ordinate of the *ij* th mode (m)
B_{11}	is the structural factor corresponding to the fundamental mode
d_{11}	is the generalised displacement of the fundamental mode corresponding to static and uniform load G (m)
D	is the flexural rigidity (kg.m)
D_{ij}	is the dynamic magnification factor of the *ij* th mode for displacement
D^a_{ij}	is the dynamic magnification factor of the *ij* th mode for acceleration
D_{ijn}, D^a_{ijn}	are the *n* th component of D_{ij} and D^a_{ij}, respectively
D_n, D^a_n	are peak values of the *n* th component of the dynamic magnification factors
$f_{ij} = \omega_{ij}/2\pi$	is the *ij* th natural frequency (Hz)
$f_p = \omega_p/2\pi$	is the load frequency (Hz)
$F(t)$	is the load-time function (N/m²)
G	is the weight load of dancers per unit area (N/m²)
G_s	is the weight load of a single person (N)
k^*	is the modal stiffness
K_p	is the dynamic impact factor
L_x L_y	are the lengths of a floor along its x and y directions (m)
m	is the mass of a bare floor per unit area (kg/m²)
m_l	is the mass of people per unit area (kg/m²)
M	is the mass matrix
R	is the energy dissipated due to damping (N.m)
S_{ij}	is the mode participating factor of the *ij* th mode
t	is the time (s)
t_p	is the contact duration (s)
T_p	is the period of dance-type of load (s)
U	is the total strain energy (N.m)
V	is the total kinetic energy (N.m)
$W_{ij}(x,y)$	is the assumed *ij* th vibration mode
$w(x,y,t)$	is the displacement function
W	is the external work (N.m)
α	is the contact ratio
β_{ij}	is the ratio of excitation frequency to the *ij* th natural frequency
ζ	is the critical damping ratio
η	is the floor side ratio
θ_{ijn}	is the phase lag relating to the *ij* th mode and the *n* th load component
ϕ_n	is the phase lag of the *n* th sinusoidal term

Introduction
The dynamic behaviour of long-span lightweight floors is becoming increasingly important for two different forms of loading. First, vibrations produced by people walking on the floor can prove annoying to other users of the floor. Although this is a *serviceability problem*, it may be a limiting design criterion and hence can be significant. Secondly, loads from dancing or organised keep-fit exercises may be significant and, if resonance occurs, it may lead to excessive movements of the floor, hence it may become a *safety problem* as well as a *serviceability problem*. Either the designer can try to avoid problems of resonance by producing a floor with a sufficiently high fundamental frequency or he can calculate how the floor will respond under a given load and check that it is satisfactory. This paper provides a method of calculating the response of a floor to loads arising from dance-type activities.

Dance is movement with rhythmic steps and actions, usually to accompanying music. Similar movements include jumping, stamping, and aerobics. The loading is thus related to the dance frequency or the beat frequency of the music and is periodical. These loads are here termed dance-type loads. The maximum response produced by dancing occurs when jumping is involved, and it is this situation which is covered here.

The current recommendations concerning design loads for floors in the UK are given in BS 6399: *Part 1*[1]. For dance floors the equivalent static design load is given as 5 kN/m², but it is stated that this does not allow for dynamic loads due to crowds. Obviously, the loads produced by dancing and aerobics have a significant dynamic component, and, with functions like pop concerts which have large numbers of people crowded together and dancing to music, an extreme loading case can result. For these cases the current guidance does not make clear what loading to take. However, one thing that is clear is that a static analysis is not sufficient, simply because the dynamic nature of the load must be considered.

The main body of information on the response of floors to these loads is attributable to a group of research workers in Canada, and it is instructive to review the developments of the National Building Code of Canada for dealing with floor vibration. In 1970 the NBC recommended a minimum natural frequency of 5 Hz for floors subject to rhythmic activities. This recognises that people cannot dance at frequencies greater than 3.5 Hz and thus sets a minimum frequency of the floor to avoid resonance. Later, it became evident that energy was not restricted to the dance frequency but could be input at multiples of the dance frequency; hence, in 1975, NBC recommended a minimum of 10 Hz for floors subject to repetitive activities such as dancing. In 1985, NBC introduced a new clause requiring a dynamic analysis for floors with a fundamental frequency less than 6 Hz which may be subject to rhythmic activities[2,3], thus giving an alternative to the simplistic approach of avoiding resonance.

The supplement to the NBC[4] provides a method of calculating floor response to dance-type loads. The method is based on beam theory and models the floor as a single degree-of-freedom system (i.e. it considers only the fundamental mode). It is perhaps obvious that not all floors behave like simple beams, and it has also been suggested[5] that it is not sufficient to consider just the fundamental mode. In the following sections these ideas will be examined in some detail.

In order to explain the calculation of the response of floors to various dynamic human loads, it is useful to split the problem into its constituent parts:

(1) *Characterisation of the load.* This requires a knowledge of the number and weight of people, their distribution across the floor, the specific type of dance, the beat frequency, the crowd effect, and the load model.
(2) *Evaluation of the characteristics of the floor vibration.* This requires knowledge of the basic flooring system, including its stiffness, mass, support conditions, assumed behaviour, and the effect of the involvement of people.

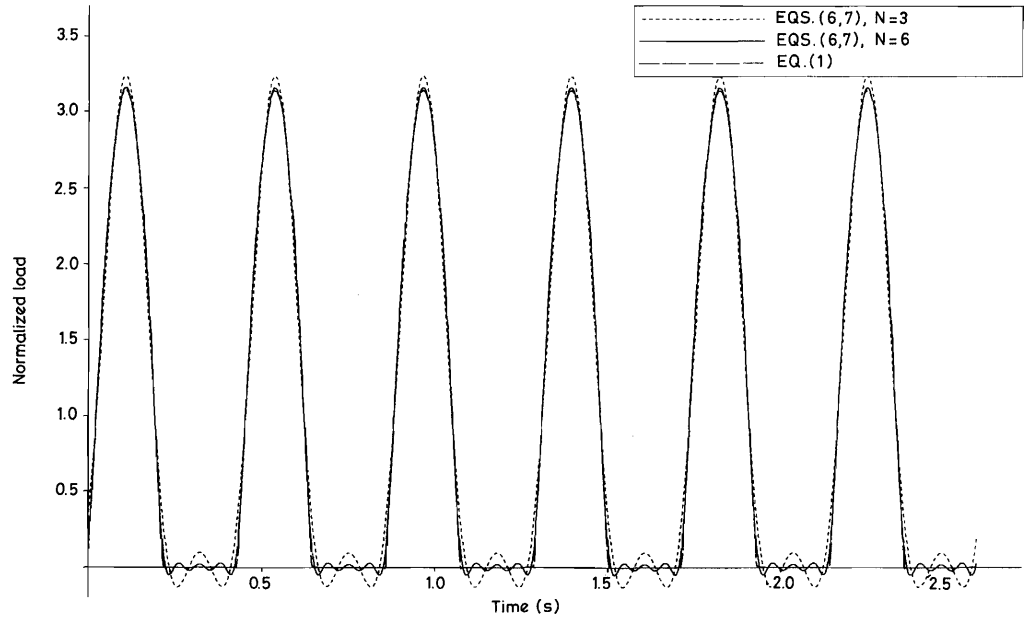


Fig 1. Load-time history for jumping

(3) *Calculation of the response of the floor to the dancing loads.* This calculation is based on the information from (1) and (2).

This paper deliberately does not define the number (or density) of people to be considered or the range of beat frequencies, as these will be defined by appropriate Codes of Practice when they are written. What the paper does is to provide an analytical model for dance-type loads and develops a method of calculating floor response induced by the loads. The procedure for calculating floor response is summarised in Appendix A. Experimental and numerical verification of the method is provided in a separate paper[6], together with a comparison with the NBC recommendations and other published data.

Characterisation of dance-type loads

In order to assess the response of a floor to a particular type of dancing in which jumping is involved, it is necessary to know what loads will be produced. This requires an estimation of the number and weight of people who will be dancing in the area of concern, or the load density. (Information on this topic can be found in refs 2 and 7). Also, there are many different types of dancing and a wide range of beat frequencies for music; however, dancing frequencies tend to be in the range 1.5–3.5 Hz.

There are a number of different dances but, for analytical purposes, it is convenient to split them into two categories. The first is when the dancer is always in contact with the floor and the second involves jumping when contact with the floor is not maintained. The first type of dancing is simple to model and is primarily a sinusoidal load at the dance frequency[4]. The second type of dancing is more complex and potentially much more severe because energy is input at the dance frequency and also at multiples of the dance frequency, which means that many more floors could be excited at resonance under this type of dancing. It is this type of loading which will be dealt with here. In fact, the first type of dancing is a special case of the second one when the duration of jumping reduces to zero and a single forcing frequency is encountered.

The load time history of the jump dancing can be described by a high contact force for a certain time t_p (contact duration) followed by zero force when the feet leave the floor. It has been proposed that the load time function for running can be expressed by a sequence of semi-sinusoidal pulses. The load time histories for other activities that involve jumping or aerobics are somewhat similar. The function within one period is given[7] by

$$F(t) = \begin{cases} K_p G \sin(\pi t/t_p) & 0 \le t \le t_p \\ 0 & t_p \le t \le T_p \end{cases} \qquad \text{....(1)}$$

where

K_p is F_{max}/G, impact factor
$F_{max.}$ is the peak dynamic load/unit area
G is the weight of dancers/unit area
t_p is the contact duration
T_p is the period of dancing load

The contact duration t_p can vary from 0 to T_p corresponding to different movements and activities. The contact ratio α is defined as follows:

$$\alpha = \frac{t_p}{T_p} \le 1.0 \qquad \text{....(2)}$$

Thus different contact ratios α characterise different rhythmic activities. For analysis purposes, it is useful to express eqn. (1) in terms of Fourier series

$$F(t) = G\left(a_o + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T_p}t + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi}{T_p}t \right)$$
$$= G\left[a_o + \sum_{n=1}^{\infty} r_n \sin\left(\frac{2n\pi}{T_p}t + \phi_n\right)\right] \qquad \text{....(3)}$$

where the Fourier coefficients and phase lags are determined as follows:

$$\alpha_0 = \frac{2K_p\alpha}{\pi} \qquad r_n = \sqrt{\alpha_n^2 + b_n^2} \qquad \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

when $2n\alpha = 1$, $n = 1,2,3,...$
then $a_n = 0$, $b_n = \alpha K_p$
otherwise

$$a_n = \frac{K_p\alpha}{\pi}\left[\frac{\cos(2n\alpha - 1)\pi - 1}{2n\alpha - 1} - \frac{\cos(2n\alpha + 1)\pi - 1}{2n\alpha + 1}\right]$$

$$b_n = \frac{K_p\alpha}{\pi}\left[\frac{\sin(2n\alpha - 1)\pi}{2n\alpha - 1} - \frac{\sin(2n\alpha + 1)\pi}{2n\alpha + 1}\right]$$

It had been observed experimentally[8] that the mean value of the time history of a vertical load corresponding to bouncing to music on toes or to rhythmic jumping was always equal to the weight of the performer. This was confirmed during the experiments undertaken for this investigation[6]. Expressing this observation analytically gives

$$\frac{1}{T_p}\int_0^{t_p} K_p G \sin\left(\frac{\pi t}{t_p}\right) dt = G \qquad \text{....(4)}$$

which reduces to

$$K_p = \frac{\pi}{2\alpha} \qquad \text{....(5)}$$

The impact factor K_p, corresponding to the different contact ratios or activities, could also be determined experimentally[9].
Substituting eqn. (5) into eqn. (3) yields

$$F(t) = G\left(1.0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T_p}t + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi}{T_p}t\right)$$
$$= G\left[1.0 + \sum_{n=1}^{\infty} r_n \sin\left(\frac{2n\pi}{T_p}t + \phi_n\right)\right] \quad ....(6)$$

$$r_n = \sqrt{\alpha_n^2 + b_n^2} \qquad \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

when $2n\alpha = 1$, $n = 1,2,3,...$
then $a_n = 0$, $b_n = \pi/2$
otherwise

$$a_n = 0.5\left[\frac{\cos(2n\alpha - 1)\pi - 1}{2n\alpha - 1} - \frac{\cos(2n\alpha + 1)\pi - 1}{2n\alpha + 1}\right]$$
$$b_n = 0.5\left[\frac{\sin(2n\alpha - 1)\pi}{2n\alpha - 1} - \frac{\sin(2n\alpha + 1)\pi}{2n\alpha + 1}\right] \quad ....(7)$$

Fig 1 displays the normalised load calculated using eqn. (1) with $\alpha = 0.5$ and $f_p = 1/T_p = 2.33$ Hz. For comparison, the load calculated using eqns (6) and (7) is also displayed, for examples adopting the first three and six terms of the Fourier series. In the figure the normalised load of 1.0 corresponds to the static weight of the dancer. It shows that using either three or six Fourier terms produces a reasonable representation of the load defined by eqn. (1).

The form of eqn. (6) is the same as that adopted in refs. 10 and 11, except that

(1) the Fourier coefficients are determined from measurements in refs. 10 and 11, while here they are calculated using eqn. (7);
(2) the Fourier coefficients were affected by the excitation frequency and different activities in refs. 10 and 11, but the effect from the frequency was ignored in their final presentation, while here they are a function of the contact ratio α corresponding to different activities and indirectly related to the excitation frequency;
(3) the analytical expression provides the phase lag ϕ_n for each sinusoidal component.

Further comparisons between refs. 12, 10 and 11 and the present work can be found in ref. 6.

Table 1 lists the first six Fourier coefficients and phase lags for different contact ratios. It can be seen that, to describe the load to a given accuracy, more terms are required as α decreases. The number of terms of Fourier series which should be included in analysis is determined in the next section, based on the induced response. The contact ratio α corresponding to different activities, can be evaluated from observations[12,10,11], and examples are given in ref. 6.

Another factor which should be considered is the dynamic crowd effect. When a crowd of people attempt the same repetitive movement, perfect synchronism is unlikely, thus the peak structural response produced by the group movement is smaller than it would be if the movement of all the individuals was perfectly synchronised, i.e. the load is attenuated due to the dynamic crowd effect. A method of determining this effect is presented in ref. 13 and a simple factor is suggested to modify the dynamic part of the proposed load characterisation (eqn. (6)).

Response of the floor to dance-type loads

Basic assumptions

For the initial analytical investigation, it is assumed that

(1) the floor is rectangular with uniform thickness and is simply supported along its four edges;
(2) the material works within its linear elastic region;
(3) the mass of the bare floor is uniformly distributed;
(4) the loads are uniformly distributed in the spatial domain but vary in the time domain as defined by eqn. (1);
(5) the human mass is not involved in the floor vibration for this type of loading[6].

Solutions for other floors and boundary conditions are developed later, based on the findings from the initial case.

TABLE 1 – Fourier coefficients and phase lags for different contact ratios

		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\alpha = 2/3$	r_n	1.28571	0.16364	0.13333	0.03643	0.02302	0.03175
	ϕ_n	$-\pi/6$	$\pi/6$	$-\pi/2$	$-\pi/6$	$\pi/6$	$-\pi/2$
$\alpha = 1/2$	r_n	1.57080	0.66667	0.00000	0.13333	0.00000	0.05714
	ϕ_n	0	$-\pi/2$	0	$-\pi/2$	0	$-\pi/2$
$\alpha = 1/3$	r_n	1.80000	1.28571	0.66667	0.16364	0.09890	0.01333
	ϕ_n	$\pi/6$	$-\pi/6$	$-\pi/2$	$\pi/6$	$-\pi/6$	$-\pi/2$
$\alpha = 1/4$	r_n	1.88562	1.57080	1.13137	0.66667	0.26937	0.00000
	ϕ_n	$\pi/4$	0	$-\pi/4$	$-\pi/2$	$\pi/4$	0

Plate equation of motion

The formulation of the equation of motion of a dynamic system can be established from the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{A}_{ij}}\right) + \frac{\partial U}{\partial A_{ij}} = \frac{\partial W}{\partial A_{ij}} - \frac{\partial R}{\partial \dot{A}_{ij}} \quad ....(8)$$

where

V is the kinetic energy of the system
U is the strain energy of the system
W is the external work done by the load
R is the energy dissipated due to the damping effect
A_{ij} is the generalised co-ordinate of the system
t is the time

The form of the solution depends on the distribution of the load and the boundary conditions, and here the displacement of the floor is assumed to be in the Navier form[14].

$$w(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}(t)\sin\frac{i\pi x}{L_x}\sin\frac{j\pi y}{L_y} \quad ....(9)$$

where L_x and L_y are the lengths of the sides of a rectangular floor and $A_{ij}(t)$ is the generalised co-ordinate of the i, j th mode.

The total kinetic energy of the system is

$$V = 0.5m\int_0^{L_x}\int_0^{L_y}\left(\frac{\partial w}{\partial t}\right)^2 dxdy$$
$$= 0.5m\int_0^{L_x}\int_0^{L_y}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\dot{A}_{ij}(t)\sin\frac{i\pi x}{L_x}\sin\frac{j\pi y}{L_y}\right)^2 dxdy \quad ....(10)$$
$$= \frac{1}{8}m L_x L_y \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\dot{A}_{ij}^2(t)$$

where m is the uniformly distributed mass of the floor.

The total strain energy is

$$U = 0.5D\int_0^{L_x}\int_0^{L_y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dxdy = \frac{1}{8}\pi^4 L_x L_y D\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}A_{ij}^2\left(\frac{i^2}{L_x^2} + \frac{j^2}{L_y^2}\right)^2 \quad ....(11)$$

where D is the flexural rigidity

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \quad ....(12)$$

The external work done by the load (eqn. (1)) is found to be

$$W = \int_0^{L_x}\int_0^{L_y}F(t)w(x,y,t)dxdy = \frac{4L_xL_yF(t)}{\pi^2}\sum_{i=1,3,...}^{\infty}\sum_{j=1,3,...}^{\infty}\frac{A_{ij}}{ij} \quad ....(13)$$

where A_{ij} exists only when subscripts i and j are odd numbers.

The energy dissipated by the system damping is

$$R = 0.5\int_0^{L_x}\int_0^{L_y}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}C_{ij}\left(\dot{A}_{ij}(t)\sin\frac{i\pi x}{L_x}\sin\frac{j\pi y}{L_y}\right)^2 dxdy$$

$$= \frac{1}{8}L_xL_y\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}C_{ij}\dot{A}_{ij}^2(t) \quad \text{....(14)}$$

where C_{ij} is the damping factor corresponding to the generalised co-ordinate A_{ij}.

Substituting eqns. (9-11, 13, 14) into eqn. (8) gives

$$\ddot{A}_{ij} + 2\zeta_{ij}\omega_{ij}\dot{A}_{ij} + \omega_{ij}^2 A_{ij} = \mu_{ij}F(t) \quad \text{....(15)}$$

$$\omega_{ij}^2 = \frac{\pi^4 D}{m}\left(\frac{i^2}{L_x^2}+\frac{j^2}{L_y^2}\right)^2 \quad \text{....(16)}$$

$$2\zeta_{ij}\omega_{ij} = C_{ij}/m \qquad \mu_{ij} = \frac{16}{ij\pi^2 m} \quad \text{....(17)}$$

Thus the continuous vibration model is transformed into a series of independent single degree-of-freedom equations. The solution of A_{ij} of eqn. (15) is null when either i or j is an even number. $F(t)$ is defined by eqn.s (6) and (7).

Solution of forced vibration

Eqn. (15) can be solved using the available theory for single degree-of-freedom systems[15], and the solution can be arranged into the following form:

$$A_{ij} = \frac{16G}{m\pi^2\omega_{11}^2}\,\frac{(1+\eta^2)^2}{ij(i^2+j^2\eta^2)^2}\left[1.0+\sum_{n=1}^{\infty}\frac{r_n\sin(n\omega_p t-\theta_{ijn}+\phi_n)}{\sqrt{(1-n^2\beta_{ij}^2)^2+(2n\zeta_{ij}\beta_{ij})^2}}\right]$$

$$= d_{11}S_{ij}\left[1.0+\sum_{n=1}^{\infty}D_{ijn}\right] = d_{11}S_{ij}(1.0+D_{ij}) \quad \text{....(18)}$$

similarly, the acceleration is

$$\ddot{A}_{ij} = -d_{11}S_{ij}\sum_{n=1}^{\infty}n^2f^2D_{ijn}$$

$$= -a_{11}S_{ij}\sum_{n=1}^{\infty}D_{ijn}^a = -a_{11}S_{ij}D_{ij}^a \quad \text{....(19)}$$

where

$$d_{11} = \frac{16}{\pi^2}\frac{G}{m\omega_{11}^2} \qquad a_{11} = \frac{16m_l}{\pi^2 m}g \qquad \theta_{ijn} = \tan^{-1}\left(\frac{2n\zeta_{ij}\beta_{ij}}{1-n^2\beta_{ij}^2}\right)$$

$$n = \frac{L_x}{L_y} \qquad \beta_{ij} = \frac{f_p}{f_{ij}} = \frac{\omega_p}{\omega_{ij}} \qquad \omega_p = 2\pi f_p = \frac{2\pi}{T_p} \qquad \omega_{ij} = 2\pi f_{ij}$$

$$S_{ij} = \frac{(1+\eta^2)^2}{ij(i^2+j^2\eta^2)^2} \qquad i,j = 1,3,5,\ldots \quad \text{....(20)}$$

$$D_{ij} = \sum_{n=1}^{\infty}D_{ijn} \qquad D_{ijn} = \frac{r_n\sin(n\omega_p t-\theta_{ijn}+\phi_n)}{\sqrt{\left(1-n^2\beta_{ij}^2\right)^2+\left(2n\zeta_{ij}\beta_{ij}\right)^2}} \quad \text{....(21)}$$

$$D_{ij}^a = \sum_{n=1}^{\infty}D_{ijn}^a \qquad D_{ijn}^a = \frac{r_n n^2\beta_{11}^2\sin(n\omega_p t-\theta_{ijn}+\phi_n)}{\sqrt{\left(1-n^2\beta_{ij}^2\right)^2+\left(2n\zeta_{ij}\beta_{ij}\right)^2}} \quad \text{....(22)}$$

Eqns. (18) and (19) give the steady state response which excludes transient terms, because the transient response decays quickly in a damped system and generally is of little interest[15]. In eqn. (18) d_{11} indicates the generalised displacement of the fundamental mode corresponding to the static load; the phase angle θ_{ijn} by which the response lags behind the applied load relates to the factors in the ij th mode and the order of the Fourier term n, while the phase angle θ_n only concerns the Fourier coefficients used to describe the load. a_{11} in eqn. (19) represents a scaled acceleration that relates to the ratio of the human mass m_l and the floor mass m. a_{11} is a relatively big acceleration – e.g. if the mass ratio is 1/4, a_{11}=0.4g. There are two parameters – the mode participating coefficient (S_{ij}) and the dynamic magnification factors (D_{ij} for displacement and D_{ij}^a for acceleration) – which differ from the conventional single degree-of-freedom vibration system under a single excitation frequency. These are discussed in the next two subsections.

TABLE 2 – Mode participating factors of rectangular plates

i	j	$S_{ij}(\eta=1.0)$	$S_{ij}(\eta=1/2)$	$S_{ij}(\eta=1/3)$	$S_{ij}(\eta=1/4)$
1	1	1.0000	1.0000	1.0000	1.0000
1	3	0.0133	0.0493	0.1029	0.1541
3	1	0.0133	0.0061	0.0050	0.0046
3	3	0.0014	0.0014	0.0014	0.0014

Mode participating factor. S_{ij} indicates the contribution of each mode of vibration to the overall response without considering the dynamic magnification factor, and relates to the order of modes (i,j) and the aspect ratio (η) of the floor. S_{11} always equals 1. For higher order modes S_{ij} can be evaluated for different aspect ratios; examples are presented in Table 2. It can be seen from Table 2 that

(1) S_{ij} reduces quickly as the subscript i or j increases;
(2) the bigger the aspect ratio ($0 < n \leq 1.0$), the quicker S_{ij} reduces as either subscript i or j increases;
(3) the first mode participating factor S_{11} is much bigger than others. (Considering the mode participating factor only, the contribution from the fundamental mode would represent over 90 % of the whole response when the aspect ratio $\eta > 1/3$, or 97% when $\eta = 1$).

Dynamic magnification factor. The dynamic magnification factors of the ij th mode for displacement and acceleration are defined by eqns. (21) and (22), respectively. They are functions of time t, the contact ratio α (through r_n), the frequency ratio β_{ij}, the damping ζ_{ij}, and the number of Fourier terms n.

The value of peak dynamic magnification factors of the n th Fourier component for displacement and acceleration for any mode can be defined as follows:

$$D_n = \frac{r_n}{\sqrt{\left(1-n^2\beta^2\right)^2+\left(2n\zeta\beta\right)^2}} \quad \text{....(23)}$$

$$D^a{}_n = \frac{r_n n^2\beta^2}{\sqrt{\left(1-n^2\beta^2\right)^2+\left(2n\zeta\beta\right)^2}} \quad \text{....(24)}$$

Figs 2 and 3 show a number of graphs relating D_n and $D^a{}_n$ (n =1,2,3,...., 10) to $1/\beta(=\omega/\omega_p)$ for various values of α (= 2/3, 1/2, 1/3, 1/4) with a selected damping factor ζ=0.02.

It can be concluded from these figures that

(1) resonance can occur when the natural frequency of the floor is equal to the dance frequency, or when it is equal to integer multiples of the dance frequency.
(2) for resonant excitation ($n\beta = 1$), $D_n = D^a{}_n$, i.e. D_n and $D^a{}_n$ have the same peak values.
(3) for displacement, the first Fourier term always provides a significant contribution to the magnification factor for every frequency ratio when $n > 3$ – e.g., when $\alpha = 2/3$, it provides a dominated contribution when $n > 3$.
(4) The factor for the acceleration for each component is dominated by resonant response (i.e. when $n\beta = 1$). Unlike the factor for the displacement, the factor of each component has little effect on its neighbouring components – e.g. when $\alpha = 2/3$, this effect is negligible for $n > 3$.

For dance floors, the fundamental frequency f_{11} is usually higher than the excitation frequency f_p, but situations where f_{11} equals f_p, $2f_p$ or $3f_p$ should be avoided. This situation has been observed in practice[16,17] and coincides with the first conclusion. However, α_{11} is a relatively big-scaled acceleration, thus even a relatively small dynamic magnification factor $D^a{}_n$, when $nf_p = f_{11}$ (n = 4,5,6,...), can produce significant accelerations. Consequently, more Fourier terms may need to be taken into account in order to calculate acceleration.

Determination of the number of the Fourier terms required. As it is not desirable to use all the terms in the Fourier series to describe the load, it is necessary to select how many terms of the series should be used in an analysis. The number should be selected according to the significance of the resulting response rather than the accuracy of the load description.

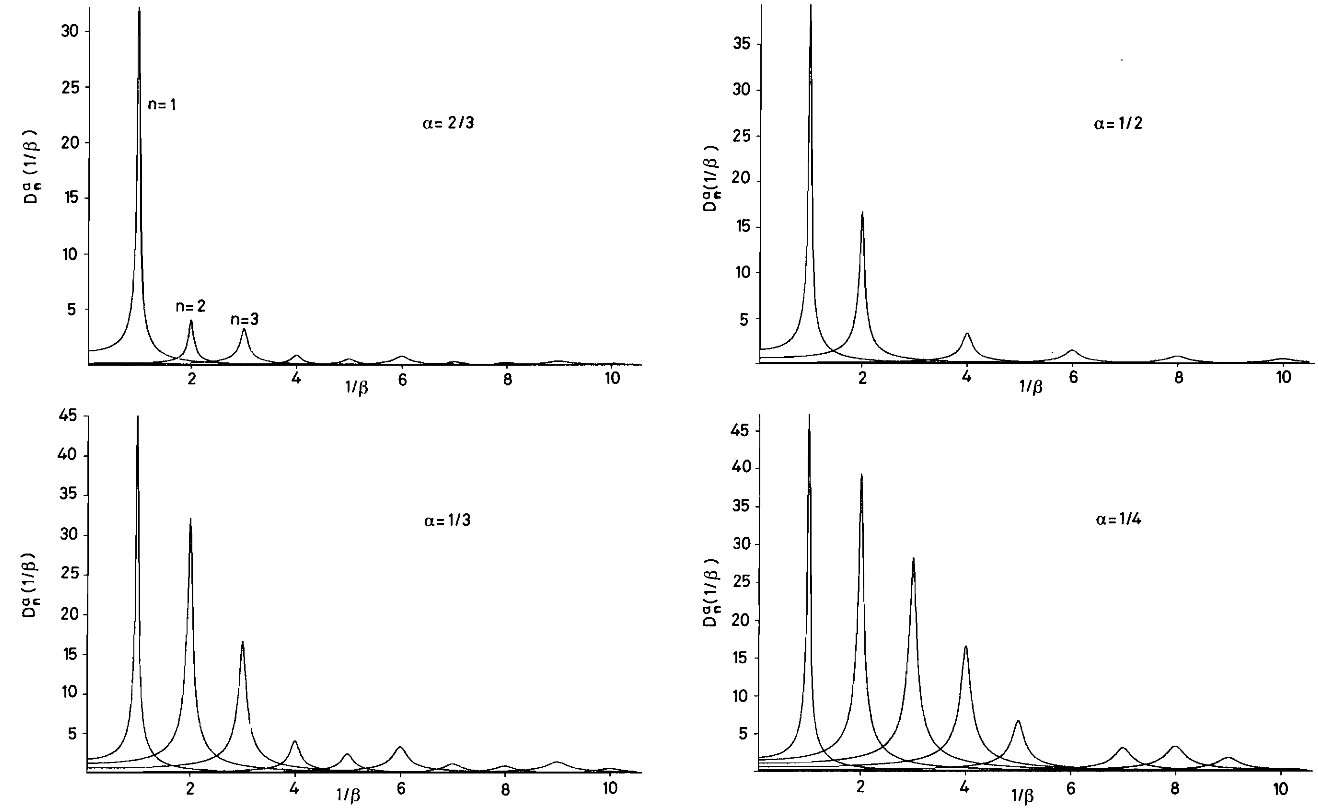


Fig 2. Dynamic magnification factors for displacements

Examining the characteristics of the dynamic magnification factors, the number of Fourier terms to be taken into account depends on their contribution to the response, including any possible resonance. It is suggested that the first I Fourier terms should be included in the analysis, where I is defined as the first integer bigger than ω_{11}/ω_p, i.e.

$$I = Int(\omega_{11} / \omega_p) = Int(1/\beta) \quad ....(25)$$

This ensures that any resonant excitation of the fundamental mode will be considered. For instance, when $1/\beta = 2.5$, $I = 3$, and the first three Fourier terms should be used in the analysis. Eqn. (25) implies that, the stiffer the floor, the more Fourier terms need to be considered.

Simplification of the solution

For the simply supported floor under a symmetric load, no antisymmetric mode is involved in the floor vibration. Therefore, the first higher mode

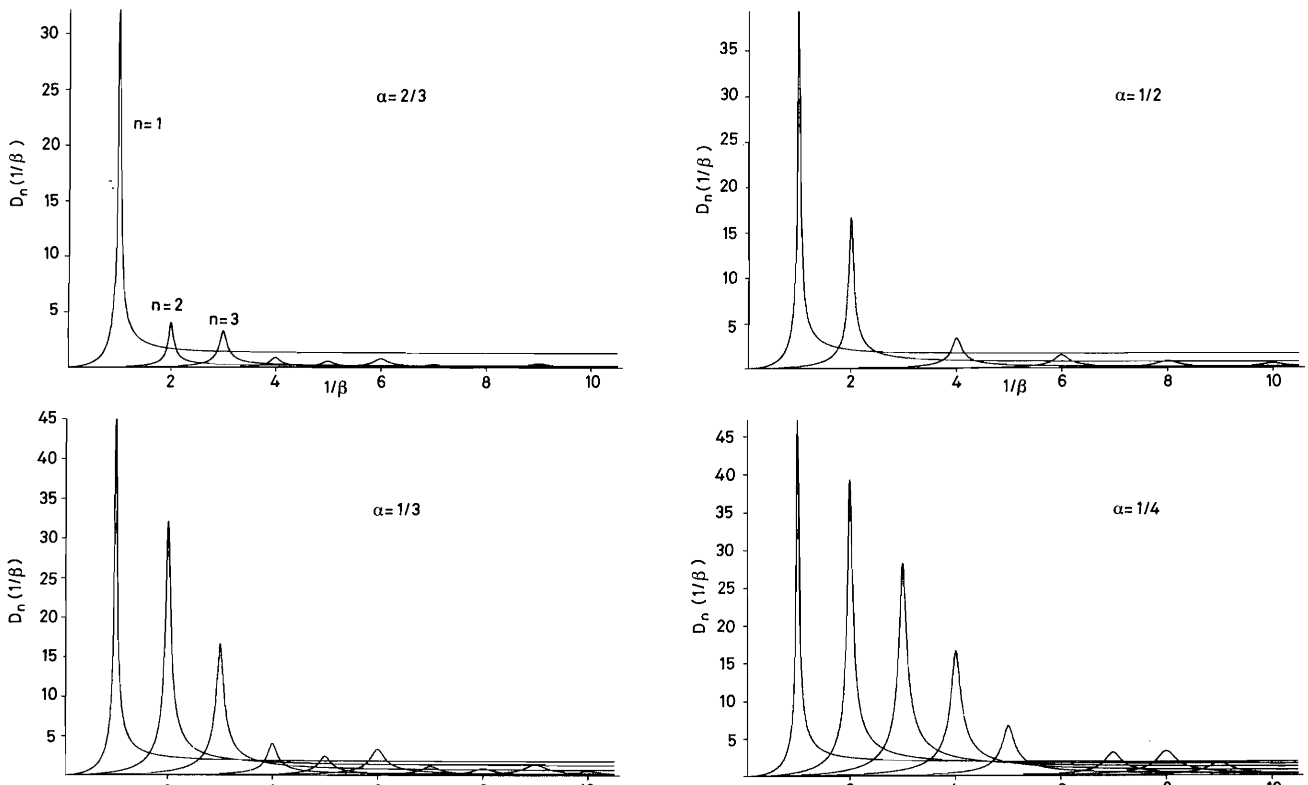


Fig 3. Dynamic magnification factor for accelerations

involved in the floor vibration is the second symmetric mode with the frequency ω_{13}. The frequency ratio of the first mode and the second symmetric mode can be expressed using eqn. (16) as follows:

$$\frac{\omega_{13}}{\omega_{11}} = \frac{1+9\eta^2}{1+\eta^2} = \begin{cases} 5 & \text{when} \quad \eta = 1 \\ 3.46 & \text{when} \quad \eta = 2/3 \\ 1.8 & \text{when} \quad \eta = 1/3 \end{cases} \qquad \text{....(26)}$$

Therefore, for floors where $\omega_{11} > f_p$, which is the usual case,

$$|D_{11}|_{\max} > |D_{13}|_{\max} \qquad |D^a_{11}|_{\max} > |D^a_{13}|_{\max} \qquad \text{....(27)}$$

where $|D_{ij}|_{\max}$ and $|D^a_{ij}|_{\max}$ mean the maximum absolute magnification factors of the ij th mode for displacement and acceleration, respectively. The values for other higher modes are even smaller than $|D_{13}|_{\max}$ and $|D^a_{13}|_{\max}$. It has been shown earlier that

$$S_{11} >> S_{13} \qquad \text{....(28)}$$

Substituting eqns. (27) and (28) into eqns. (21) and (22) gives

$$|A_{11}|_{\max} >> |A_{13}|_{\max} \qquad |\ddot{A}^a_{11}|_{\max} >> |\ddot{A}^a_{13}|_{\max} \qquad \text{....(29)}$$

which indicates that the floor response is dominated by the contribution of the fundamental mode and the response from the higher modes is negligible for a symmetric dance floor with simply supported boundary conditions under a symmetric dynamic load. Therefore, only the response from the fundamental mode needs to be considered.

Extension of the solution

The simplification discussed previously can be applied to other structures with different boundary conditions. The difficulty encountered is in choosing suitable displacement functions. However, the simplified solution for the simply supported floor is applicable for a floor under other symmetric boundary conditions since, under a uniformly distributed load, no antisymmetric modes are involved in the floor vibration. Therefore, only the fundamental mode needs to be considered, and the shape of this mode is relatively easy to choose with sufficient accuracy for many common cases. The response of a floor or other structure can be approximated by the contribution of the fundamental mode

$$w(x,y,t) = A_{11}(t)\, W_{11}(x,y) \qquad \text{....(30)}$$

where $W_{11}(x,y)$ is the dimensionless fundamental mode with unit peak value and A_{11} is the generalised co-ordinate corresponding to the first mode. Using the procedure shown in the last subsection, the solution has the following form

$$A_{11} = d_{11}(1.0 + D_{11}) = B_{11}\frac{G}{m\omega_{11}^2}(1.0 + D_{11}) \qquad \text{....(31)}$$

$$\ddot{A} = a_{11}D^a_{11} = B_{11}\frac{m_l}{m}gD^a_{11} \qquad \text{....(32)}$$

where D_{11} and D^a_{11} are the dynamic magnification factors for the displacement and acceleration defined in eqns. (21) and (22); B_{11} is defined as the *structural factor* which relates to the fundamental mode and depends only on the type of structure (beams, plates or shells) and boundary conditions (free, simple and clamped supports). Comparing eqns. (31) and (18) it can be seen that $B_{11}=16/\pi^2$ for simply supported plates. If the dynamic load is only a function of time, and the floor mass is uniformly distributed in space, the structural factor can, according to the solution procedure, be defined as follows

$$B_{11} = \frac{\iint_s W_{11}(x,y)dxdy}{\iint_s W^2_{11}(x,y)dxdy} \qquad \text{....(33)}$$

Sometimes the mode shape $W_{11}(x,y)$ is difficult to find, but an assumed one that satisfies the boundary conditions can be used as an approximation, such as a normalised deflection under uniform static load. Table 3 provides approximate structural coefficients for several symmetric structures.

Using dynamic measurements in the analysis

For checking the safety and serviceability of existing dance floors, it is desirable to perform dynamic tests on the floors. This will provide feedback from the actual structure, including accurate values of the fundamental frequency, damping, modal stiffness, and mode shape. The fundamental frequency of a bare floor is relatively easy and inexpensive to measure[18]. On the other hand, it may prove difficult or inconvenient to measure the actual response when a crowd of people is involved.

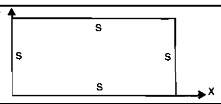
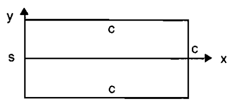
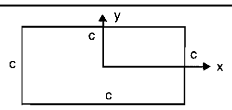
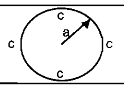
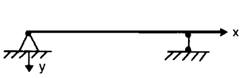
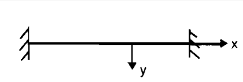
Eqns (31) and (32) can be expressed in the following form to accommodate the measurements

$$A_{11} = \frac{G\iint_s W_{11}(x,y)dxdy}{k*}(1.0 + D_{11}) \qquad \text{....(34)}$$

$$\ddot{A}_{11} = \frac{G\omega^2_{11}\iint_s W_{11}(x,y)dxdy}{k*}D^a_{11} \qquad \text{....(35)}$$

To use the above equations the measured modal stiffness $k*$, frequency

TABLE 3 – Approximate structural factors for several common cases

Structures and boundaries	Assumed fundamental mode	Approximate fundamental frequency (ω)	Structural factors
Plate 	$\sin(\pi x/L_x)\sin(\pi y/L_y)$	$\pi^2\lambda(1+\eta^2)/L_x^2$	$(4/\pi)^2 \doteq 1.62$
	$\sin(\pi x/L_x)(1-4y^2/L_y^2)^2$	$22.45\lambda\left(1+\frac{\pi^2}{21}\eta^2+\frac{\pi^4}{504}\eta^4\right)^{1/2}/L_x^2$	$\frac{4}{\pi}\frac{21}{16} \doteq 1.67$
	$(1-4x^2/L_x^2)^2\ (1-4y^2/L_y^2)^2$	$22.45\lambda\left(1+\frac{4}{7}\eta^2+\eta^4\right)^{1/2}/L_x^2$	$\left(\frac{21}{16}\right)^2 \doteq 1.72$
	$(1-r^2/a^2)^2$	$10.22\lambda/a^2$	$5/3 \doteq 1.67$
Beams 	$\sin(\pi x/L_x)$	$\pi^2\lambda/L_x^2$	$4/\pi \doteq 1.27$
	$(1-4x^2/L_x^2)^2$	$22.37\lambda/L_x^2$	$21/16 \doteq 1.31$

S – simply supported boundary C – clamped boundary $\lambda = (D/m)^{1/2}$ D – flexural rigidity m – floor mass $\eta = L_x/L_y$

w_{11} and mode shape W_{11} (x,y) are required. Alternatively, eqns. (31) and (32) can be used where the floor mass density needs to be estimated, but the mode shape W_{11} (x,y) is not required. It is also preferable to use the measured damping value.

If one person jumps at the centre of a floor or a beam, the response can be obtained from eqns. (34) and (35)

$$A_{11} = \frac{G_s}{k*}(1.0 + D_{11}) \qquad ....(36)$$

$$\ddot{A} = \frac{G_s \omega_{11}^2}{k*} D_{11}^a \qquad ....(37)$$

where G_s is the weight of the person. These two formulae are particularly useful for comparing calculation with measurements, and are used in ref. 6.

Resonance caused by higher Fourier components

An example is provided here to demonstrate a possible resonance induced by higher Fourier components of the load, and the importance of selecting the appropriate number of Fourier terms to describe the load when calculating acceleration. The basic data for a square floor clamped at its four edges are summarised:

dimension of the floor:	$L_x = L_y = 8.0$m
damping ratio:	$\zeta_{11} = 0.02$
floor material:	reinforced concrete (2400 kg/m³)
floor thickness:	$h = 0.15$ m
frequency of excitation load:	$f_p = 2.33$ Hz
frequency of the bare floor:	$f_{11} = 13.99$ Hz
dancing load:	$G = 1177.2$ N/m²
contact ratio:	$\alpha = 0.5$

The load frequency is chosen to be one-sixth of the floor frequency. In the following calculation using eqns. (31) and (32), three Fourier terms, $N = 3$ (according to refs. 16, 19 and 10) and six Fourier terms, $N = 6$ (according to eqn. (25)) are adopted, respectively, for comparison. The corresponding loads are described in Fig 1. Fig 4 shows the displacements and accelerations at the centre of the floor. The upper figure shows the response calculated when the load is described using the first three Fourier terms ($N = 3$). The lower figure shows the response to the load described by the first six Fourier terms ($N = 6$). The following observations can be made:

(1) *Displacement*
The maximum displacement calculated using $N = 6$ is approximately 30 % bigger than that using $N = 3$. However, the values are so small that the difference would not affect the safety criterion, since it is a relatively stiff floor.
(2) *Acceleration*
The acceleration using $N = 6$ is over 10 times that using $N = 3$. The significant difference is due to the resonance caused by the sixth Fourier component of the load (i.e. $\omega/f = 6$). This resonance produces accelerations as high as 0.88 g.
(3) *Load and response*
There is little difference between the loads described using the first three or six Fourier terms (Fig 1). However, significant differences can result in the calculated accelerations.

This calculation suggests that it is possible for resonance to occur on a relatively stiff dance floor ($f_{11} > 10$ Hz).

Conclusions

This paper presents a method for calculating the response of floors to loads from dancing involving jumping or aerobics-type exercises. The conclusions to be drawn from this study are:

(1) Dance-type loads where jumping is involved can be expressed analytically using Fourier series in which the Fourier coefficients, or dynamic load factors, are a function of the contact ratio α which relates to the dance activities and dance frequency.
(2) For the vibration of a simply supported rectangular floor under symmetric dynamic loads, the response of the fundamental mode can represent the whole response reasonably accurately. Furthermore, by defining the structural factor (eqn. (33)), the solution can be applied to other structures with different boundary conditions.
(3) A possible resonance, due to higher Fourier components ($n > 3$) of the load, is predicted which is particularly important if calculations of floor accelerations are required. This resonance can occur on a relatively stiff floor ($f_{11} > 10$ Hz).
(4) The number of Fourier terms required in an analysis can be determined using eqn. (25) and is dependent on the load frequency and the fundamental frequency of the floor.

The results from the proposed method, including the prediction of a

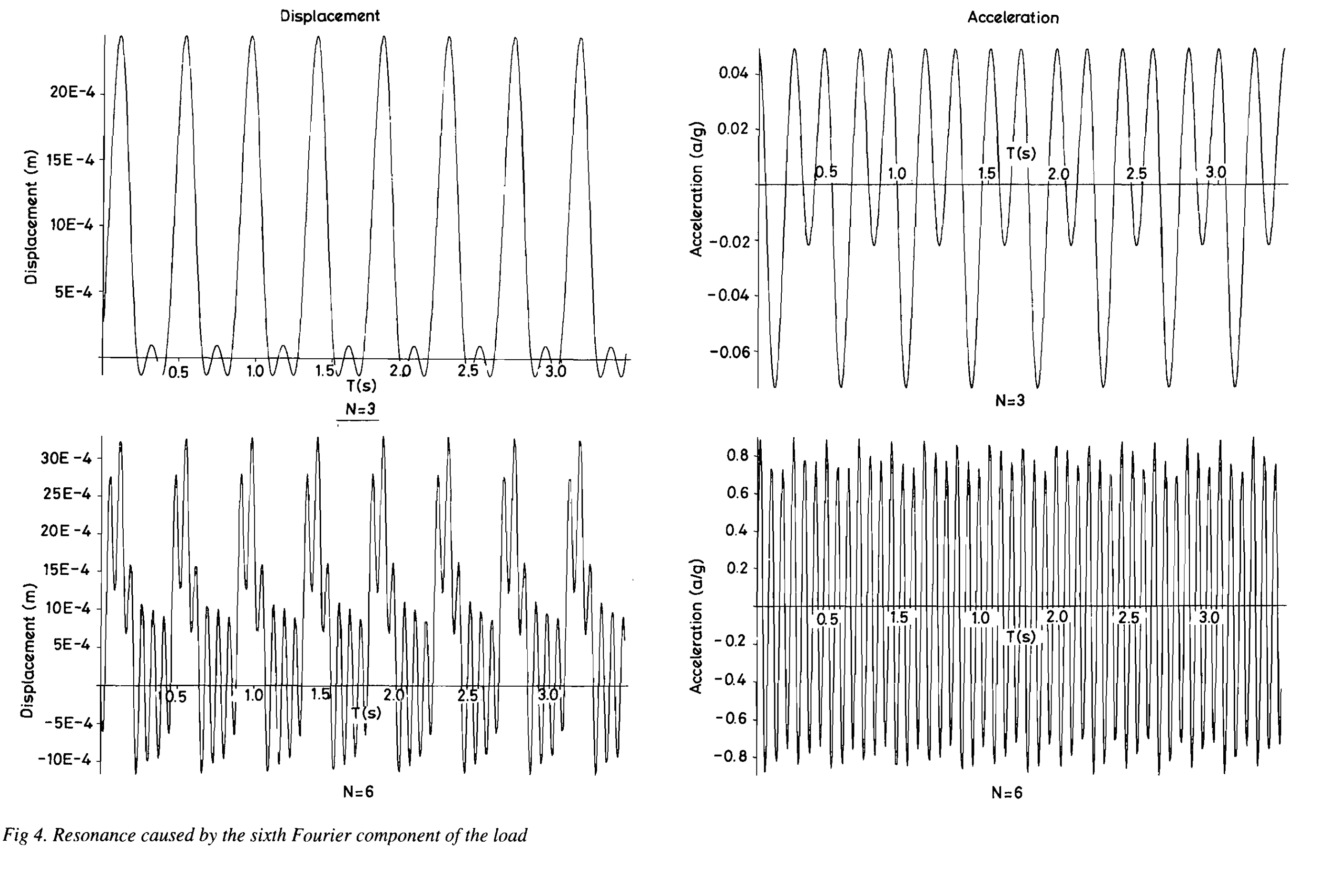


Fig 4. Resonance caused by the sixth Fourier component of the load

possible resonance, are verified by experimental measurements, numerical analysis and published data in a separate paper[6]. Further theoretical work is required to develop the proposed method to deal with the case where the dance-type loads are non-uniformly distributed and other structures, such as grandstands, where dance-type loads are encountered.

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Appendix A. The procedure for calculating the response of a floor to dance-type loads

The procedure required for calculating the response at the centre of a floor is summarised here. The calculated displacements are likely to provide information for safety assessments, whereas the accelerations provide information for serviceability.

(1) *Evaluation of the characteristics of the bare floor*
This includes

— the fundamental frequency ω_{11} (or $2\pi f_{11}$)
— the structural coefficient B_{11},
— the mass m (excluding the mass of the dancers)
— the damping value ζ

Table 3 gives typical values of the first two items for floors with symmetric boundary conditions. Alternatively, for an existing floor, measurements could be used.

(2) *Evaluation of the loads*
This requires the following:

(a) The load density G, or the number and weight of dancers in a given area.
(b) The dance frequency f_p, which is likely to be in the range 1.5–3.5 Hz. Several dance frequencies may need to be evaluated, with the dance period ($1/f_p$) which is integer number multiple of the floor period likely to produce the largest accelerations.
(c) The contact ratio α, for each dance activity. This may be chosen from Table 1 in ref. 6 for different activities, and several values may need to be evaluated.
(d) The dynamic crowd effect. The likely attenuation of loading due to the crowd effect should be considered[13]

(3) *Calculation of the dynamic response*
Having determined the structural characteristics, the response can be calculated for each load condition in turn. The following items enable the response time history to be calculated over a period t, and considers only the fundamental mode.

(a) Determine the number of Fourier terms to use in the calculation from eqn. (25).
(b) Determine the dynamic magnification factors for displacement D_{11} and acceleration D^a_{11} using eqns. (21) and (22).
(c) Calculate the displacement and acceleration time histories of the centre of the floor using eqns. (31) and (32). If test results are available, either eqns. (31) and (32) or eqns. (34) and (35) can be used.