On Poisson Geometry

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Poisson geometry has three principal sources:

1. It is a more convenient context in which to study many aspects of symplectic geometry.
2. It is the theory dual to Lie algebra theory and, more generally, Lie algebroid theory.
3. It is a (semi-)classical limit of quantum theory.

Formally, a Poisson structure on a manifold M is a Lie algebra structure on the real vector space of smooth functions $C(M)$ such that

$$\{f, gh\} = g\{f, h\} + h\{f, h\}$$

for all $f, g, h \in C^\infty(M)$.

If M is a symplectic manifold with symplectic form ω then each function $f \in C(M)$ defines the Hamiltonian vector field X_f by the condition that $\omega(X_f, Y) = Y(f)$ for all vector fields Y. In terms of these Hamiltonian vector fields, the classical Poisson bracket

$$\{f, g\} = -X_f(g)$$

is a Poisson structure on M. That $d\omega = 0$ is precisely what is needed for the Poisson bracket to satisfy Jacobi. Conversely any Poisson structure on a manifold M foliates it by symplectic leaves.

Many constructions in symplectic geometry destroy the symplectic structure but preserve the Poisson bracket. Put another way, the only morphisms in the symplectic category are the symplectomorphisms, which are diffeomorphisms, but the obvious concept of Poisson map allows a much greater flexibility and power.

If $\mathfrak{g}$ is a Lie algebra then its dual vector space $M = \mathfrak{g}^*$ has a natural Poisson structure. Namely take $f \in C(M)$ and any point $\phi \in M$. Then the (directional) derivative of f at ϕ is a linear map $M \to \mathbb{R}$; that is, it is an element of $\mathfrak{g}$. So we can define $\{f, g\} \in C(M)$ by

$$\{f, g\}(\phi) = \phi([D(f), D(g)])$$

where $D(f)$ is the directional derivative of f at ϕ. In this example the bracket of two linear functions is linear and this characterizes duals of Lie algebras amongst all Poisson structures. There is a lot of evidence that Lie actually thought more in terms of these duals than in terms of what we call a Lie algebra.

In this example the symplectic leaves are precisely the coadjoint orbits of any Lie group which integrates $\mathfrak{g}$.

If we replace $\mathfrak{g}$ by the tangent bundle to some manifold P and play the same game with the bracket of vector fields, we get the (Poisson structure corresponding to the) standard symplectic structure on T^*P.

I won't say much about how Poisson structures are a first stage in quantization, except that a Poisson bracket is the first term in the power series of a deformation quantization. Also, Poisson–Lie groups are very definitely a good approximation to quantum groups (Poisson groups also arise in complete integrability) and a good intro can be found in Chari and Pressley, for example.

I hope this is some help — and sounds sufficiently seductive. I will definitely recall the Marsden–Weinstein reduction at the start of my talk.