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Aspects of functional integration in Mathematical Physics

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In this lecture I will give a wholly *heuristic* calculation (i.e. one designed to produce the results obtained) of a large dimensional *quantum mechanical* partition function Z. Z is defined as a functional integral taken over all classical paths in a large dimensional symplectic manifold M^{2N} (N uncountable) the paths being weighted by $\exp S$, the exponential *classical* action S of a chosen dynamical system. The (large) class of dynamical systems I will (mostly) consider is the class of *integrable* field theories in $1+1$ (one time (t) and one space (x)) dimensions and the result obtained will be *exact*. The motions of these integrable systems in M^{2N} are not ergodic and are confined to large dimensional tori in M^{2N}. This latter fact will be exploited by canonical transformation to appropriate action-angle variables co-ordinatising M^{2N}. A (considerable) complication is that the real parameters t, x parametrising the paths in M^{2N} lie in a 2-torus, $0 \leq t < \beta$, $-\frac{1}{2}L \leq x < \frac{1}{2}L$ (so that the fields $\phi(x,t)$ considered satisfy periodic boundary conditions of period β in t, and of period L in x). Z defines a "free energy" $F = -\beta^{-1}\ell n Z$, and this must be "extensive", $\lim_{L\to\infty} FL^{-1} =$ finite and non-zero. Thus we need a "finite density" limit on Z as $L \to \infty$ and the paths on the large dimensional torus in M^{2N} must reflect this fact.

The integrable field theories in $1+1$ dimensions considered will mostly be the sine-Gordon and sinh-Gordon models (also briefly I will consider the nonlinear Schrödinger models): these are each nonlinear p.d.e.s in (x,t) of rather fundamental significance. Both the attractive NLS and the sine-Gordon equation have so-called *soliton* solutions. To this extent Z provides a quantum statistical mechanics for 'quantum solitons' [1].

All of the integrable field theories considered are 'completely integrable Hamiltonian systems' in the sense of Liouville-Arnold (hence the large torus and the action-angle variables): N independent constants of the motion commuting under a Poisson bracket on M^{2N} can be found but N *is* 'large' (uncountable). However the extension of the Liouville-Arnold theorem to this large dimensional case is not proved (I think). Nor is the validity of the methods for Z sketched in this lecture anywhere rigorously demonstrated[2].

It is therefore remarkable that such heuristic calculations[3], as this one is, can deliver results of physical interest. Very recently we have used not unrelated methods to calculate correlation functions for the quantum repulsive NLS system which seem to agree with those observed in experiments at 10^{-9} K (nano-Kelvin temperatures)!

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[1] Note that because the class of models we consider is a class of *integrable* models, the motions are not ergodic and the methods of statistical mechanics may seem in doubt. Indeed they might be in doubt *for any finite* N and any generic Hamiltonian system since in this case such generic Hamiltonian systems are neither integrable nor ergodic – because of the KAM theorem. For uncountable N's this aspect is (I think) wide open. However, it seems that our applications of statistical mechanics are valid because they rest ultimately on the quantum mechanical superposition principle. According to Dirac quantum mechanics is a theory of *linear* operators on a Hilbert space (of large dimension). This does *not* mean of course that the models we consider are linear! For the curious there is a a host of problems here requiring rigorous proof.

[2] Contact with algebraic geometry is made by the representation of the large dimensional torus as a punctured Riemann surface with $N\sqrt{\ }$ branch points. – Genus ∞ since $N = \infty$??

[3] Even rigorous proof would still present a problem – see e.g. E.P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", Comm. Pure and Appl. Math. **13**, 1 (1960)!