Exercise sheet 9 for General Insurance: solutions

1. First we compute the development factors using the method from the lecture (notes). This yields:

for DY0 → DY1: \(DF_{0,1} = \frac{(210 + 450 + 110 + 316)}{(123 + 324 + 67 + 224)} = \frac{181}{123}\)

for DY1 → DY2: \(DF_{1,2} = \frac{(300 + 512 + 131)}{(210 + 450 + 110)} = \frac{943}{770}\)

for DY2 → DY3: \(DF_{2,3} = \frac{(350 + 551)}{(300 + 512)} = \frac{901}{812}\)

for DY3 → DY4: \(DF_{3,4} = \frac{365}{350} = \frac{73}{70}\).

Now we can fill the run-off triangle using these factors, this yields:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>2007</td>
<td>123</td>
<td>210</td>
<td>300</td>
<td>350</td>
<td>365</td>
</tr>
<tr>
<td>2008</td>
<td>324</td>
<td>450</td>
<td>512</td>
<td>551</td>
<td>(\frac{40223}{300})</td>
</tr>
<tr>
<td>AY 2009</td>
<td>67</td>
<td>110</td>
<td>131</td>
<td>(\frac{118031}{300})</td>
<td>(\frac{8616263}{56840})</td>
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<td></td>
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<tr>
<td>2010</td>
<td>224</td>
<td>316</td>
<td>(\frac{148994}{300})</td>
<td>(\frac{67121797}{136410})</td>
<td>(\frac{4899891181}{385})</td>
</tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>2011</td>
<td>406</td>
<td>(\frac{73486}{123})</td>
<td>120727</td>
<td>3750863</td>
<td>(\frac{273812999}{323400})</td>
</tr>
</tbody>
</table>

The required reserve is now computed by

\[
\frac{40223}{70} - 551 + \frac{8616263}{56840} - 131 + \frac{4899891181}{10941700} - 316 + \frac{273812999}{323400} - 406 \approx 617.
\]

2. We first compute the development factors:

\(DF_{0,1} = \frac{(200 + b)}{(100 + 200)} = \frac{(200 + b)}{300},\)

\(DF_{1,2} = a/200,\)

and we may fill the remaining entries of the run-off triangle:

<table>
<thead>
<tr>
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<th>0</th>
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<th>2</th>
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</thead>
<tbody>
<tr>
<td>2009</td>
<td>100</td>
<td>200</td>
<td>(a)</td>
</tr>
<tr>
<td>AY 2010</td>
<td>200</td>
<td>(b)</td>
<td>(ab/200)</td>
</tr>
<tr>
<td>2011</td>
<td>300</td>
<td>(200 + b)</td>
<td>(a(200 + b)/200)</td>
</tr>
</tbody>
</table>

With these entries the required reserve \(R\) can be expressed as

\[
R = \frac{ab}{200} - b + \frac{a(200 + b)}{200} - 300.
\]

Furthermore it is given that in 2011 a total of 700 was spent on claims. The amounts spent in 2011 appear in the triangle as well. As the entries in the triangle are cumulative (!), we need to take the cumulative amount in 2011 minus the cumulative amount in 2010
to get the amount spent in the year 2011. Furthermore we have to do this for all three product groups, namely with Accident Year 2009 (which yields $a - 200$), Accident Year 2010 (which yields $b - 200$) and Accident Year 2011 (which yields 300). This yields a total amount spent in 2011 of:

$$S = a - 200 + b - 200 + 300 = a + b - 100.$$ 

We are given that $R = 1300$ and $S = 700$. This gives us two equations to solve $a$ and $b$ from: $S = 700 \iff b = 800 - a$ and plugging this in $R = 1300$ yields the following equation for $a$: $a^2 - 1000a + 240000 = 0$. This gives $a = 400$ or $a = 600$. The first one gives $b = 800 - 400 = 400$ while the second gives $b = 800 - 600 = 200$. Since we are asked to find $a, b$ such that $a, b > 200$ the only solution is $a = 400, b = 400$.

3. a) Clearly $F(x) = 0$ for $x < 0$ since $X$ takes values in $\mathbb{R}_{\geq 0}$. Furthermore for $x \in [0, 100]$:

$$P(X \leq x) = P(\text{no fire}) + P(\text{fire occurs with damage} \leq x)$$

$$= P(\text{no fire}) + P(\text{fire occurs}) \cdot P(\text{damage} \leq x \mid \text{fire occurs})$$

$$= \frac{9}{10} + \frac{1}{10} \frac{x}{100} = \frac{9}{10} + \frac{x}{1000},$$

and finally for $x > 100$ we have $F(x) = 1$ (since $X$ takes only values of at most 100).

b) Note that $F$ has a jump of size $9/10$ at $x = 0$. To find $F^{-1}$ we thus have to be careful taking this into account (make a graph if you get confused). For $y \in [9/10, 1)$ we can find $F^{-1}(y)$ by solving the equation $y = F(x) = 9/10 + x/1000$ which yields $x = 1000(y - 9/10)$. Furthermore for $y \in (0, 9/10)$ we need to use

$$F^{-1}(y) = \inf \{x \mid F(x) \geq y\} = \inf \{x \mid x \geq 0\} = 0.$$ 

Thus

$$F^{-1}(y) = \begin{cases} 
1000 \left(y - \frac{9}{10}\right) & \text{if } y \in [9/10, 1) \\
0 & \text{if } y \in (0, 9/10).
\end{cases}$$

The inverse transform algorithm now looks as follows:
1. Generate a sample $u$ from a uniform distribution on $(0,1)$,
2. Return $F^{-1}(u)$ as sample from $X$, with $F^{-1}$ as above.

4. a) No, because we can’t find an explicit formula for $F^{-1}$.

b) We need to choose a rv $Y$ and a constant $c$. As $X$ takes values in $(0, 5)$ it is easiest (and best) to pick $Y$ taking the same values. You can pick any (suitable, i.e. one that we can easily produce samples from) such $Y$, for instance $Y \sim \text{Uniform}(0, 5)$ with pdf
\[
g(x) = \begin{cases} 
1/5 & \text{if } x \in (0, 5) 
0 & \text{if } x \not\in (0, 5). 
\end{cases}
\]

Next we need to find the smallest \( c \) such that \( cg(x) \geq f(x) \) for all \( x \in \mathbb{R} \). For \( x \not\in (0, 5) \) we have \( f(x) = g(x) = 0 \) and thus \( cg(x) \geq f(x) \) always holds. It remains to consider \( x \in (0, 5) \). For this we get

\[
cg(x) \geq f(x) \iff c \cdot \frac{1}{5} \geq \frac{6}{125} x^2(5 - x)^2 \iff c \geq \frac{6}{125} x^2(5 - x)^2
\]

and since we are looking for the \textit{smallest} \( c \) for which this holds, the best choice is

\[
c = \max_{x \in (0, 5)} \frac{6}{125} x^2(5 - x)^2.
\]

In order to compute this value we first find the derivative of \( h \) using the product rule:

\[
h'(x) = \frac{12}{125} x(5 - x)^2 - \frac{12}{125} x^2(5 - x) = \frac{12}{125} x(5 - x)(5 - 2x)
\]

and hence

\[
h'(x) \begin{cases} 
> 0 & \text{if } x \in (0, 5/2) 
= 0 & \text{if } x = 5/2 
< 0 & \text{if } x \in (5/2, 5)
\end{cases}
\]

i.e. \( h \) attains a maximum in \( x = 5/2 \). Going back to (1) we see that we should hence choose

\[
c = h(5/2) = 15/8.
\]

The algorithm thus looks as follows:

1. generate a sample \( y \) from \( Y \sim \text{Uniform}(0, 5) \) and \( u \) from (another) uniform distribution on \((0, 1)\).
2. if \( u \leq f(y)/(cg(y)) = 9y^2(5 - y)^2/100 \) then accept \( y \) as a sample from \( X \), otherwise reject it and go back to 1.

5. a) We can first compute the cdf \( F \) of the double exponential distribution. For \( x \leq 0 \):

\[
F(x) = \int_{-\infty}^{x} f(z) \, dz = \int_{-\infty}^{x} \frac{\lambda}{2} e^{\lambda z} \, dz = \frac{1}{2} e^{\lambda x}
\]

and for \( x > 0 \):

\[
F(x) = \int_{-\infty}^{x} f(z) \, dz = \int_{-\infty}^{0} \frac{\lambda}{2} e^{\lambda z} \, dz + \int_{0}^{x} \frac{\lambda}{2} e^{-\lambda z} \, dz = 1 - \frac{1}{2} e^{-\lambda x}.
\]
Hence we can explicitly find the inverse \( F^{-1} : (0,1) \to \mathbb{R} \) by standard means:

\[
F^{-1}(y) = \begin{cases} 
\frac{1}{\lambda} \log(2y) & \text{if } y \in (0, 1/2] \\
\frac{1}{\lambda} \log(2(1-y)) & \text{if } y \in (1/2, 1).
\end{cases}
\]

Since we have a formula for \( F^{-1} \) we can use the inverse transform method which looks as follows:

1. generate a sample \( u \) from a uniform distribution on \((0, 1)\)
2. return \( F^{-1}(u) \).

b) Let \( g \) denote the pdf of a standard normal distribution, i.e.

\[
g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

We need to find \( c > 0 \) such that \( cf(x) \geq g(x) \) for all \( x \in \mathbb{R} \). We know from the lecture (notes) that the most efficient choice, i.e. the choice for which the least amount of samples end up rejected, is the smallest such \( c \). Hence we need to find the smallest \( c > 0 \) such that

\[
\frac{c\lambda}{2} e^{-\lambda|x|} \geq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{for all } x \in \mathbb{R}
\]

or

\[
\frac{c\lambda\sqrt{2\pi}}{2} e^{x^2/2-\lambda|x|} \geq 1 \quad \text{for all } x \in \mathbb{R}.
\] (2)

Now the function \( h(x) = \exp(x^2/2-\lambda|x|) \) attains a global minimum in \( x = \pm \lambda \) with value \( h(\lambda) = \exp(-\lambda^2/2) \) as is easily checked by standard means. Hence (2) is equivalent to

\[
\frac{c\lambda\sqrt{2\pi}}{2} e^{-\lambda^2/2} \geq 1 \quad \text{or} \quad c \geq \frac{2}{\lambda\sqrt{2\pi}} e^{\lambda^2/2}.
\]

It is again easily checked that the right hand side attains its minimum in \( \lambda = 1 \), which results in \( c = 2e^{1/2}/\sqrt{2\pi} \) and these are hence the parameters we will be using.

The algorithm is now as follows:

1. generate a sample \( x \) from the double exponential distribution with \( \lambda = 1 \) and a sample \( u \) from the uniform distribution on \((0, 1)\)
2. if \( u < g(x)/(cf(x)) \) with \( c = 2e^{1/2}/\sqrt{2\pi} \) then return \( x \) else go back to step 1.

6. For any run of the algorithm, the probability that a sample from \( X \) is generated is the probability that in step 2. of the algorithm the sample of \( Y \) gets accepted, i.e.

\[
P \left( U \leq \frac{f(Y)}{cg(Y)} \right).
\]

In the lecture (notes) we have already computed this probability (cf. the proof of Algorithm 5.4.1) to be equal to \( 1/c \). Denote by \( N \) the number of runs the algorithm needs to produce a sample of \( X \). Then \( N \) takes values in \( \{1, 2, \ldots\} \) and for any \( k \in \{1, 2, \ldots\} \)
\[ \mathbb{P}(N = k) = \left(1 - \frac{1}{c}\right)^{k-1} \frac{1}{c} \]

(all the runs are independent, and we first need \(k - 1\) failures (i.e. rejected samples) and then a success (i.e. accepted sample) at the \(k\)-th run). You might want to note this is a geometric distribution. Hence the average number of runs of the algorithm to get a sample of \(X\) is indeed given by

\[
\mathbb{E}[N] = \sum_{k \geq 1} k \mathbb{P}(N = k) = \frac{1}{c} \sum_{k \geq 1} k \left(\frac{c - 1}{c}\right)^{k-1} \overset{(*)}{=} \frac{1}{c} \left(1 - \frac{c - 1}{c}\right)^{-2} = \frac{1}{c^2} = c,
\]

where we use the well known geometric sum \(f(x) = \sum_{k \geq 0} x^k = (1 - x)^{-1}\), which we may differentiate on both sides to find \(f'(x) = \sum_{k \geq 1} k x^{k-1} = (1 - x)^{-2}\) and plug this in (*) with \(x = (c - 1)/c\).