Exercise sheet 8 for General Insurance: solutions

1. a) The aggregate claim amount for the insurer, $S_I$, is given by $S_I = \sum_{i=1}^{N} Y_i$, where $Y_i = \min\{X_i, K\}$ and the aggregate claim amount for the reinsurer, $S_R$, is given by $S_R = \sum_{i=1}^{N} Z_i$, where $Z_i = \max\{X_i - K, 0\}$. Both the $Y_i$'s and the $Z_i$'s are iid since the $X_i$'s are, and hence we can apply Lemma 3.6.9 from the notes:

$$E[S_I] = E[N]E[Y_1] \quad \text{and} \quad \text{Var}(S_I) = \text{Var}(N)(E[Y_1])^2 + E[N]\text{Var}(Y_1),$$

$$E[S_R] = E[N]E[Z_1] \quad \text{and} \quad \text{Var}(S_R) = \text{Var}(N)(E[Z_1])^2 + E[N]\text{Var}(Z_1).$$

Now it is a matter of working out all the terms involved. From q1 we know that $E[N] = \text{Var}(N) = \lambda$. Furthermore, using that $Y_1 = \min\{X_1, K\}$ we can compute

$$E[Y_1] = \int_{R} \min\{x, K\} f(x) \, dx = \int_{0}^{100} \min\{x, K\} \frac{1}{100} \, dx$$

$$= \frac{1}{100} \int_{0}^{100} x \, dx + \frac{1}{100} \int_{K}^{100} K \, dx = \frac{1}{100} K^2 + \frac{1}{100} K(100 - K) = K - K^2/200$$

and

$$E[Y_1^2] = \int_{R} \min\{x, K\}^2 f(x) \, dx = \int_{0}^{100} \min\{x, K\}^2 \frac{1}{100} \, dx$$

$$= \frac{1}{100} \int_{0}^{K} x^2 \, dx + \frac{1}{100} \int_{K}^{100} K^2 \, dx = \frac{1}{100} \frac{K^3}{3} + \frac{1}{100} K^2(100 - K) = K^2(1 - K/150),$$

hence, plugging this in the above equations:

$$E[S_I] = \lambda K \left( 1 - \frac{K}{200} \right)$$

and

$$\text{Var}(S_I) = \lambda \left( K - K^2/200 \right)^2 + \lambda K^2(1 - K/150) - \lambda \left( K - K^2/200 \right)^2 = \lambda K^2(1 - K/150).$$

By similar means, using that $Z_1 = \max\{X_1 - K, 0\}$:

$$E[Z_1] = \int_{R} \max\{x - K, 0\} f(x) \, dx = \frac{1}{100} \int_{K}^{100} (x - K) \, dx = \frac{(100 - K)^2}{200}$$

and

$$E[Z_1^2] = \int_{R} \max\{x - K, 0\}^2 f(x) \, dx = \frac{1}{100} \int_{K}^{100} (x - K)^2 \, dx = \frac{(100 - K)^3}{300},$$
so that
\[ \mathbb{E}[S_R] = \lambda \left( \frac{(100 - K)^2}{200} \right) \]
and
\[ \text{Var}(S_R) = \lambda \left( \frac{(100 - K)^4}{200^2} + \frac{(100 - K)^3}{300} \right) - \lambda \left( \frac{(100 - K)^4}{200^2} \right) = \lambda \left( \frac{(100 - K)^3}{300} \right). \]

b) Since \( S_N = S_I + S_R \) we have
\[ \mathbb{E}[S_N] = \mathbb{E}[S_I] + \mathbb{E}[S_R] = \lambda K \left( 1 - \frac{K}{200} \right) + \lambda \left( \frac{(100 - K)^2}{200} \right) = 50\lambda. \]

c) The amount the reinsurer pays is \( S_R \) and is given by \( S_R = \sum_{i=1}^{N} Z_i \), where \( Z_i = \max \{ X_i - K, 0 \} \). We are asked to compute \( \mathbb{P}(S_R = 0) \). As usual we use the law of total probability, which yields
\[ \mathbb{P}(S_R = 0) = \sum_{n \geq 0} \mathbb{P}(S_R = 0 \mid N = n) \mathbb{P}(N = n). \tag{1} \]

Now the rv \( S_R \mid N = n \) equals 0 when \( n = 0 \), while for \( n > 0 \) it equals \( \sum_{i=1}^{n} Z_i \). For \( n > 0 \), from \( Z_i = \max \{ X_i - K, 0 \} \) we see that \( Z_i \) takes the value 0 whenever \( X_i \) takes a value less than or equal to \( K \) while \( Z_i > 0 \) whenever \( X_i \) takes a value larger than \( K \). This means that the rv \( S_R \mid N = n \) takes the value 0 iff \( \sum_{i=1}^{n} Z_i = 0 \) iff \( Z_1 = 0, Z_2 = 0, \ldots, Z_n = 0 \) iff \( X_1 \leq K, X_2 \leq K, \leq X_n \leq K \). Hence, we have \( \mathbb{P}(S_R = 0 \mid N = 0) = 1 \) while for \( n \geq 1 \) we have
\[ \mathbb{P}(S_R = 0 \mid N = n) = \mathbb{P} \left( \sum_{i=1}^{n} Z_i = 0 \right) = \mathbb{P}(Z_1 = 0, Z_2 = 0, \ldots, Z_n = 0) \]
\[ = \mathbb{P}(X_1 \leq K, X_2 \leq K, \leq X_n \leq K) \overset{\text{indep.}}{=} \mathbb{P}(X_1 \leq K) \cdot \mathbb{P}(X_2 \leq K) \cdot \ldots \cdot \mathbb{P}(X_n \leq K) \tag{2} \]
and since the \( X_i \)'s have a common uniform distribution on the interval \([0, 100]\) we can compute
\[ \mathbb{P}(X_i \leq K) = \int_0^K \frac{1}{100} \, dx = \frac{K}{100}. \]

Plugging this back in (2) we find
\[ \mathbb{P}(S_R = 0 \mid N = n) = \left( \frac{K}{100} \right)^n. \]

(Note that this formula is also valid for \( n = 0 \), since in that case the rhs equals 1 as needed). We may now use this together with \( \mathbb{P}(N = n) = \lambda^n e^{-\lambda} / n! \) into (1) to get indeed
\[ P(S_R = 0) = \sum_{n \geq 0} \left( \frac{K}{100} \right)^n \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda \sum_{n \geq 0} \frac{(K\lambda/100)^n}{n!}} e^{-\lambda} e^{K\lambda/100} = e^{\lambda(K/100-1)}, \]

where (*) uses the power series of the exponential: \( e^x = \sum_{k \geq 0} \frac{x^k}{k!} \) with \( x = K\lambda/100 \).

2. a) For excess of loss we have that \( Y_i = \min\{X_i, K\} = \min\{X_i, 4\} \) for \( i = 1, 2 \). If \( X_i = 1 \), then \( Y_i = \min\{1, 4\} = 1 \); if \( X_i = 3 \) then \( Y_i = \min\{3, 4\} = 3 \); if \( X_i = 5 \) then \( Y_i = \min\{5, 4\} = 4 \) and if \( X_i = 7 \) then \( Y_i = \min\{7, 4\} = 4 \). Hence \( Y_i \) takes values in \( \{1, 3, 4\} \) with common pmf \( p_Y \) given by \( p_Y(1) = p_X(1) = 1/4, p_Y(3) = p_X(3) = 1/4, p_Y(4) = p_X(5) + p_X(7) = 1/2 \).

For the \( Z_i \)'s, since we already determined the distribution of the \( Y_i \)'s the most convenient expression to use is now (cf. lecture (notes)) \( Z_i = X_i - Y_i \), which yields the following. If \( X_i = 1 \) then \( Z_i = 1 - 1 = 0 \); if \( X_i = 3 \) then \( Z_i = 3 - 3 = 0 \); if \( X_i = 5 \) then \( Z_i = 5 - 4 = 1 \) and if \( X_i = 7 \) then \( Z_i = 7 - 4 = 3 \). Hence \( Z_i \) takes values in \( \{0, 1, 3\} \) with common pmf \( p_Z(0) = p_X(1) + p_X(3) = 1/2, p_Z(1) = p_X(5) = 1/4 \) and \( p_Z(3) = p_X(7) = 1/4 \).

b) Recall that we have \( S_I = \sum_{i=1}^N Y_i \) and \( S_R = \sum_{i=1}^N Z_i \). Since both the \( Y_i \)'s and the \( Z_i \)'s are idd, we are allowed to use our standard results for compound distributions (Theorem 3.6.8 and Lemma 3.6.9). This yields for \( S_I \) that its mgf \( m_{S_I} \) is given by

\[ m_{S_I}(t) = m_N(\log(m_Y(t))) \quad \text{for all } t \geq 0, \quad (3) \]

where \( m_Y \) is the common mgf of the \( Y_i \)'s. We know from q1 on sheet 7 that

\[ m_N(t) = \left( 1 + \frac{1}{4} (e^t - 1) \right)^2 = \left( \frac{3}{4} + \frac{1}{4} e^t \right)^2 \]

and we can compute

\[ m_Y(t) = \mathbb{E}[e^{tY}] = \sum_{y \in \{1,3,4\}} e^{ty} p_Y(y) = (1/4) e^t + (1/4)e^{3t} + (1/2)e^{4t}, \]

which yields with (3)

\[ m_{S_I}(t) = \left( \frac{3}{4} + \frac{1}{4} e^{\log(m_Y(t))} \right)^2 = \left( \frac{3}{4} + \frac{1}{4} m_Y(t) \right)^2 = \left( \frac{3}{4} + \frac{1}{16} e^t + \frac{1}{16} e^{3t} + \frac{1}{8} e^{4t} \right)^2. \]

Furthermore, we can use Lemma 3.6.9 to compute the mean and variance of \( S_I \) (alternatively, of course you could differentiate the above mgf) using the formulae

\[ \mathbb{E}[S_I] = \mathbb{E}[N] \mathbb{E}[Y_i] \]

and

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Var(\(S_I\)) = Var(N)(E[Y_1])^2 + E[N]Var(Y_1).

Using the results from q1 on sheet 7 again we know that
\[ E[N] = 2 \cdot \frac{1}{4} = \frac{1}{2}, \quad \text{Var}(N) = 2 \cdot \frac{1}{4} \cdot \left( 1 - \frac{1}{4} \right) = \frac{3}{8}. \]

Using the results from part a) we get:
\[ \mathbb{E}[Y_1] = 1 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} = 3, \quad \text{Var}(Y_1) = (1-3)^2 \cdot (1/4) + (3-3)^2 \cdot (1/4) + (4-3)^2 \cdot (1/2) = 3/2. \]

Hence
\[ \mathbb{E}[S_I] = \frac{1}{2} \cdot 3 = \frac{3}{2}, \quad \text{Var}(S_I) = \frac{3}{8} \cdot 3^2 + \frac{1}{2} \cdot \frac{3}{2} = \frac{33}{8}. \]

In exactly the same way we can find the required quantities for \(S_R\), using
\[ S_R = \sum_{i=1}^{N} Z_i. \]

You should find
\[ m_{S_R}(t) = \left( \frac{7}{8} + \frac{1}{16} e^t + \frac{1}{16} e^{3t} \right)^2, \quad \mathbb{E}[S_R] = \frac{1}{2}, \quad \text{Var}(S_R) = \frac{9}{8}. \]

\(c)\) First note that \(\hat{N}\) takes the same values as \(N\), i.e. in \(\{0,1,2\}\). Furthermore, as we saw in the lecture (notes) we may express \(\hat{N}\) as
\[ \hat{N} = \sum_{i=1}^{N} I_i, \]
where the indicator variables \(I_i\)’s take values in \(\{0,1\}\). In particular, \(I_i\) takes the value 0 if \(X_i \leq K\) while it takes the value 1 if \(X_i > K\). Hence the \(I_i\)’s are iid with common pmf \(p_I\) given by
\[ p_I(0) = \mathbb{P}(X_i \leq K) = \mathbb{P}(X_i \leq 4) = p_X(1) + p_X(3) = 1/2 \] and \(p_I(1) = 1 - p_I(0) = 1/2\).

In order to find the distribution of \(\hat{N}\) we go down the by now very usual road of using the law of total probability. For all \(k \in \{0,1,2\}\) we have
\[ \mathbb{P} (\hat{N} = k) = \sum_{n=0}^{2} \mathbb{P} (\hat{N} = k \mid N = n) \mathbb{P} (N = n) \]
and we note that we have:

the rv \(\hat{N} \mid N = 0\) is 0, possible values of this rv: 0
the rv \(\hat{N} \mid N = 1\) is \(I_1\), possible values of this rv: 0, 1
the rv \(\hat{N} \mid N = 2\) is \(I_1 + I_2\), possible values of this rv: 0, 1, 2.

This yields:
\[
P(\hat{N} = 0) = P(\hat{N} = 0 \mid N = 0)P(N = 0) + P(\hat{N} = 0 \mid N = 1)P(N = 1) + P(\hat{N} = 0 \mid N = 2)P(N = 2) \\
= 1 \cdot p_N(0) + P(I_1 = 0)p_N(1) + P(I_1 + I_2 = 0)p_N(2),
\]
where using the pmf of \( N \) (cf. q1 on sheet 7) we can compute \( p_N(0) = \frac{9}{16}, p_N(1) = \frac{3}{8} \) and \( p_N(2) = \frac{1}{16} \).

Similarly we get
\[
P(\hat{N} = 2) = P(\hat{N} = 2 \mid N = 0)P(N = 0) + P(\hat{N} = 2 \mid N = 1)P(N = 1) + P(\hat{N} = 2 \mid N = 2)P(N = 2) \\
= 0 \cdot p_N(0) + P(I_1 = 2)p_N(1) + P(I_1 + I_2 = 2)p_N(2),
\]
which using \( P(I_1 = 2) = 0 \) (since \( I_1 \) can not take the value 2) and \( P(I_1 + I_2 = 2) = P(I_1 = 1, I_2 = 1) \overset{\text{iid}}{=} p_I(1)^2 = \frac{1}{4} \) boils down to
\[
P(\hat{N} = 2) = 0 \cdot \frac{9}{16} + 0 \cdot \frac{3}{8} + 0 \cdot \frac{1}{16} = \frac{1}{64}.
\]
Finally, \( P(\hat{N} = 1) = 1 - P(\hat{N} = 0) - P(\hat{N} = 2) = 1 - \frac{49}{64} - \frac{1}{64} = \frac{7}{32} \) and we have now fully specified the distribution of \( \hat{N} \).

3. a) As done in the lecture (notes), we can write \( \hat{N} = \sum_{i=1}^{N} I_i \) where
\[
I_i = \begin{cases} 
1 & \text{if } X_i > K \\
0 & \text{if } X_i \leq K.
\end{cases}
\]
In particular the \( I_i \)'s are iid with common mgf given by
\[
m_I(t) = E[e^{tI_i}] = e^{t0}P(I_i = 0) + e^{t1}P(I_i = 1) = P(X_i \leq K) + e^tP(X_i > K) \\
= 1 - q + e^tq = 1 + q(e^t - 1)
\]
where \( q = P(X_i > K) \). Thus we may apply Theorem 3.6.8 to compute the mgf of \( \hat{N} \) as follows, also using the formula for \( m_I \) above:
\[
m_{\hat{N}}(t) = m_N(\log(m_I(t))) = \left(1 + p(\log(m_I(t)) - 1)\right)^n \\
= (1 + p(m_I(t) - 1))^n = (1 + pq(e^t - 1))^n.
\]
Now this is the mgf of a binomial distribution with parameters $n$ and $pq$. However, using the fact we are allowed to use, this must mean that $\hat{N}$ indeed has a binomial distribution with parameters $n$ and $pq$.

Intuitively this may be explained as follows: $N$ following a binomial distribution means that each product in the portfolio independently yields a claim with common probability $p$. For each of those claims there is the same probability that the resulting claim is larger than $K$, namely $q$. Hence the reinsurer is facing a group of $n$ products, each of which generating a claim larger than $K$ (i.e. generating a claim where he has to contribute to the claim payment) with probability $pq$ (prob. that a claim at all occurs times the probability this claim is larger than $K$). Hence the number of non-zero payments he has to do follows a binomial distribution with parameters $n$ and $pq$.

b) This works in the same way as part a). Using the same notation & Theorem as in part a) we now get

$$m_N(t) = m_N(\log(m_I(t))) = \exp(\lambda (e^{\log(m_I(t))} - 1)) = \exp(\lambda (m_I(t) - 1)) = \exp(q\lambda(e^t - 1))$$

and again using the given fact it follows that $\hat{N} \sim \text{Poisson}(q\lambda)$.

4. a) The aggregate claim amount for the insurer, $S_I$, is given by $S_I = \sum_{i=1}^{N_i} Y_i$, where as usual $Y_i$ denotes the part of the $i$-th incoming claim to be paid by the insurer. From the description of the contract we deduce that we have

$$Y_i = \begin{cases} X_i & \text{if } X_i \leq K \\ X_i - a(X_i - K) = (1 - a)X_i + aK & \text{if } X_i > K \end{cases}$$

that is to say, $Y_i = h(X_i)$ for the function $h$ given by

$$h(x) = \begin{cases} x & \text{if } x \leq K \\ (1 - a)x + aK & \text{if } x > K. \end{cases}$$

This means that the $Y_i$'s are iid since the $X_i$'s are, and thus we may use Lemma 3.6.9 from the notes to deduce that

$$\mathbb{E}[S_I] = \mathbb{E}[N]\mathbb{E}[Y_1].$$

So let us compute both $\mathbb{E}[N]$ and $\mathbb{E}[Y_1]$. For the former we have

$$\mathbb{E}[N] = \sum_{k \geq 0} k \mathbb{P}(N = k) = \sum_{k \geq 1} k \cdot \frac{2}{5} \left(\frac{3}{5}\right)^k = \frac{2}{5} \cdot \frac{3}{5} \cdot \sum_{k \geq 1} k \left(\frac{3}{5}\right)^{k-1} \leq \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{1}{(1 - 3/5)^2} = \frac{3}{2}.$$
where (*) uses the identity \( \sum_{k \geq 1} k x^{k-1} = 1/(1 - x)^2 \) for all \( x \) with \( |x| < 1 \), which follows from the geometric series \( \sum_{k \geq 0} x^k = 1/(1 - x) \) by differentiating both sides wrt \( x \) where on the lhs we differentiate inside the sum (non-examinable note: there is a slight technical issue here wrt differentiating infinite sums. To do it completely correct it is nicer to first look at the identity \( \sum_{n=1}^{\infty} k x^{k-1} = (1 - x^n)/(1 - x) \), differentiate both sides — which is now fine since the lhs is a finite sum — and then consider the limit for \( n \to \infty \)). For the latter we have

\[
\mathbb{E}[Y_i] = \mathbb{E}[h(X_i)] = \int_{\mathbb{R}} h(x) f(x) \, dx = 2 \int_{1}^{\infty} h(x)x^{-3} \, dx
\]

\[
= 2 \int_{1}^{K} x \cdot x^{-3} \, dx + 2 \int_{K}^{\infty} ((1 - a)x + aK)x^{-3} \, dx
\]

\[
= 2 \int_{1}^{K} x^{-2} \, dx + 2(1 - a) \int_{K}^{\infty} x^{-2} \, dx + 2aK \int_{K}^{\infty} x^{-3} \, dx
\]

\[
= 2(-K^{-1} + 1) + 2(1 - a)K^{-1} + 2aK \cdot \frac{1}{2}K^{-2} = 2 - \frac{a}{K}.
\]

Plugging this all back in (4) yields indeed

\[
\mathbb{E}[S_I] = \frac{3}{2} \cdot \left( 2 - \frac{a}{K} \right) = 3 - \frac{3a}{2K}.
\]

b) First, let us denote the part paid by the reinsurer for the \( i \)-th claim by \( Z_i \), which is according to the description of the reinsurance contract given by

\[
Z_i = \begin{cases} 
  a(X_i - K) & \text{if } X_i > K \\
  0 & \text{if } X_i \leq K.
\end{cases}
\]

We see that the reinsurer has to pay a non-zero amount only if it happens that \( Z_i > 0 \), or \( X_i > K \), in which the case the payment is \( a(X_i - K) \). So, if we denote by \( \hat{Z}_j \)'s the non-zero payments done by the reinsurer, then we can find the common cdf \( F \) of the \( \hat{Z}_j \)'s as follows (imitating the proof of Lemma 3.3.22 from the notes). For \( z < 0 \) we have \( F(z) = 0 \), since the \( \hat{Z}_j \)'s take only positive values. For any \( z \geq 0 \) we have

\[
F(z) = \mathbb{P}(\hat{Z} \leq z) = \mathbb{P}(Z \leq z \mid Z > 0) = \frac{\mathbb{P}(Z \leq z \& Z > 0)}{\mathbb{P}(Z > 0)} = \frac{\mathbb{P}(0 < Z < z)}{\mathbb{P}(Z > 0)}
\]

\[
= \frac{\mathbb{P}(K < X < z/a + K)}{\mathbb{P}(X > K)} = \frac{\int_{K}^{z/a+K} x^{-3} \, dx}{\int_{K}^{\infty} x^{-3} \, dx} = \frac{-(z/a + K)^{-2} + K^{-2}}{K^{-2}} = 1 - K^2 \left( \frac{z}{a} + K \right)^{-2}.
\]

We can find the pdf by differentiating the cdf. Hence, \( f_{\hat{Z}}(z) = 0 \) for \( z \leq 0 \) while for \( z > 0 \):

\[
f_{\hat{Z}}(z) = F'(z) = \frac{2K^2}{a(z/a + K)^3}.
\]
as required.

c) Following the same idea as in the notes, the number of non-zero claim payments done by the reinsurer, \( \hat{N} \), can be expressed as

\[
\hat{N} = \sum_{i=1}^{N} I_i,
\]

where

\[
I_i = \begin{cases} 
1 & \text{if } X_i > K \\
0 & \text{if } X_i \leq K.
\end{cases}
\]

Note that the \( I_i \)'s are iid since the \( X_i \)'s are, and hence we may apply Theorem 3.6.8 from the notes to derive that the mgf \( m \) of \( \hat{N} \) is given by

\[
m(t) = m_N \left( \log(m_I(t)) \right),
\]

where \( m_N \) is the mgf of \( N \) and \( m_I \) the common mgf of the \( I_i \)'s. Computing the latter is quite straightforward since (6) shows that the \( I_i \)'s take only the values 0 and 1, with probabilities:

\[
P(I_1 = 0) = P(X_1 \leq K) = \int_{1}^{K} 2x^{-3} \, dx = 1 - K^{-2}, \quad P(I_1 = 1) = 1 - P(I_1 = 0) = K^{-2}.
\]

Hence

\[
m_I(t) = E[e^{tI_1}] = e^{t0}P(I_1 = 0) + e^{t1}P(I_1 = 1) = 1 - K^{-2} + K^{-2}e^t.
\]

We can compute the mgf of \( N \) making use of the geometric series:

\[
m_N(t) = E[e^{tN}] = \sum_{k \geq 0} e^{tk}P(N = k) = \sum_{k \geq 0} e^{tk} \frac{2}{5} \left( \frac{3}{5} \right)^k = \frac{2}{5} \sum_{k \geq 0} \left( \frac{3}{5}e^t \right)^k = \frac{2}{5} \frac{1}{1 - \frac{3}{5}e^t} = \frac{2}{5 - 3e^t}.
\]

Plugging these two back in (7) gives

\[
m(t) = \frac{2}{5 - 3e^{\log(m_I(t))}} = \frac{2}{5 - 3m_I(t)} = \frac{2}{2 - 3K^{-2}(e^t - 1)} \quad \text{for all } t \geq 0.
\]

To guess the mgf of \( N \) from this one, note that if we set \( K = 1 \) then the reinsurer pays a positive amount for every incoming claim, i.e. then we have that \( \hat{N} = N \). Thus the mgf \( m_N \) of \( N \) should be \( m \) with \( K = 1 \) substituted:

\[
m_N(t) = \frac{2}{2 - 3(e^t - 1)} \quad \text{for all } t \geq 0.
\]

d) As noted in part c) above we may write
\[ \hat{N} = \sum_{i=1}^{N} I_i \tag{8} \]

where the \( I_i \)'s are iid, each taking the value 0 with prob. \( 1 - K^{-2} \) and the value 1 with prob. \( K^{-2} \). Now, as done several times before, when we need to compute a probability for a rv like this we use the law of total probability. (Recall that before we used this same method to compute probabilities for \( S_N \)). The law of total probability yields (where we use \( \{N = 0\}, \{N = 1\}, \ldots \) as partition):

\[ P(\hat{N} = n) = \sum_{k \geq 0} P(\hat{N} = n \mid N = k)P(N = k) = \sum_{k \geq 0} P(Y_k = n)P(N = k), \tag{9} \]

where, analogue to before, we denote by \( Y_k \) the rv \( \hat{N} \mid N = k \) for all \( k = 0, 1, \ldots \). Staring at (8) we see that \( Y_k = \sum_{i=1}^{k} I_i \). The neat thing is now that \( Y_k \) is a sum of \( k \) iid rv’s, each taking the value 0 or 1. This means two things. Firstly, the largest value \( Y_k \) can take is \( k \) and hence if \( k < n \) then \( P(Y_k = n) = 0 \). Secondly this means that \( Y_k \sim \text{Binomial}(k, p) \), where \( p \) is the probability that the \( I_i \)'s take the value 1, which we computed in part c) to be equal to \( K^{-2} \). As a consequence we get for all \( k \geq n \), using the given pmf of a binomial distribution:

\[ P(Y_k = n) = \binom{k}{n} K^{-2n}(1 - K^{-2})^{k-n}. \]

Plugging this back in (9) yields

\[ P(\hat{N} = n) = \sum_{k \geq n} \binom{k}{n} K^{-2n}(1 - K^{-2})^{k-n} \cdot 2 \cdot \left( \frac{3}{5} \right)^k \]

\[ = \frac{2}{5} K^{-2n}(1 - K^{-2})^{-n} \sum_{k \geq n} \binom{k}{n} \left( \frac{3}{5} (1 - K^{-2}) \right)^k \]

\[ = \frac{2}{5} (K^2 - 1)^{-n} \sum_{k \geq n} \binom{k}{n} \left( \frac{3}{5} (1 - K^{-2}) \right)^k. \]

5. a) We have

\[ G_S(r) \overset{def.}{=} \mathbb{E}[r^{S_N}] = \mathbb{E}[r^{\sum_{i=1}^{N} X_i}] = \mathbb{E} \left[ \prod_{i=1}^{n} r^{X_i} \right] \overset{\text{indep.}}{=} \prod_{i=1}^{n} \mathbb{E}[r^{X_i}] = \prod_{i=1}^{n} G_X(r) = G_X(r)^n. \]

b) Taking the logarithm of both sides of the result in part a) yields

\[ \log (G_S(r)) = n \log (G_X(r)). \]
Differentiating both sides yields
\[
\frac{G'_S(r)}{G_S(r)} = n \frac{G'_X(r)}{G_X(r)}
\]
or indeed
\[
G'_S(r) = nG_S(r) \frac{G'_X(r)}{G_X(r)}.
\]

c) **Note:** I’m sorry there’s a mistake in the equation (2) on the exercise sheet, it should be \(G_X(r)F(r) = G_S(r)G'_X(r)\) and not \(G_X(r)F(r) = G_S(r)G'_S(r)\)!

First we write \(G_S\) and \(G_X\) in terms of power series, just by using their definition:
\[
G_S(r) = \mathbb{E}[r^{S_n}] \overset{\text{def.}}{=} \sum_{k=0}^{nN} p_S(k)r^k
\]
and
\[
G_X(r) = \mathbb{E}[r^{X_1}] \overset{\text{def.}}{=} \sum_{k=0}^{N} p_X(k)r^k.
\]

By differentiating the latter we get
\[
G'_X(r) = \sum_{k=1}^{N} p_X(k)kr^{k-1} = \sum_{k=0}^{N-1} (k+1)p_X(k+1)r^k.
\]

With \(F(r) = \sum_{k=0}^{nN-1} a_kr^k\) we hence have that
\[
G_X(r)F(r) = G_S(r)G'_X(r) \iff \left( \sum_{k=0}^{N} p_X(k)r^k \right) \cdot \left( \sum_{k=0}^{nN-1} a_kr^k \right) = \left( \sum_{k=0}^{nN} p_S(k)r^k \right) \cdot \left( \sum_{k=0}^{N-1} (k+1)p_X(k+1)r^k \right). \tag{10}
\]

Now we want to apply Lemma 3.6.19. You may complain that that Lemma is about power series of the form \(\sum_{m \geq 0} b_mr^m\), i.e. \(m\) runs to \(\infty\) while our expressions above all only have a finite number of terms. However this is easily resolved by simply setting all the remaining coefficients equal to 0. For instance, if we define \(p_S(k) = 0\) for all \(k = nN + 1, nN + 2, \ldots\) then we have
\[
G_S(r) = \sum_{k=0}^{nN} p_S(k)r^k = \sum_{k=0}^{\infty} p_S(k)r^k.
\]

Proceeding like like this, Lemma 3.6.19 (ii) tells us that
\[
\left( \sum_{k=0}^{N} p_X(k)r^k \right) \cdot \left( \sum_{k=0}^{nN-1} a_k r^k \right) = \sum_{m \geq 0} \gamma_m^{(1)} r^m
\]

where
\[
\gamma_m^{(1)} = \sum_{i=0}^{m} a_i p_X(m - i) \quad \text{for all } m \geq 0,
\]

and similarly it tells us that
\[
\left( \sum_{k=0}^{nN} p_S(k)r^k \right) \cdot \left( \sum_{k=0}^{N-1} (k+1)p_X(k+1)r^k \right) = \sum_{m \geq 0} \gamma_m^{(2)} r^m
\]

where
\[
\gamma_m^{(2)} = \sum_{i=0}^{m} p_S(i)(m - i + 1)p_X(m - i + 1) \quad \text{for all } m \geq 0.
\]

Hence (10) reads as \( \sum_{m \geq 0} \gamma_m^{(1)} r^m = \sum_{m \geq 0} \gamma_m^{(2)} r^m \), which according to Lemma 3.6.19 (ii) implies that \( \gamma_m^{(1)} = \gamma_m^{(2)} \) for all \( m \geq 0 \), i.e.
\[
\sum_{i=0}^{m} a_i p_X(m - i) = \sum_{i=0}^{m} p_S(i)(m - i + 1)p_X(m - i + 1).
\] (11)

Using that
\[
\sum_{i=0}^{m} a_i p_X(m - i) = a_m p_X(0) + \sum_{i=0}^{m-1} a_i p_X(m - i)
\]

we get in (11)
\[
a_m p_X(0) = \sum_{i=0}^{m} p_S(i)(m - i + 1)p_X(m - i + 1) - \sum_{i=0}^{m-1} a_i p_X(m - i)
\]
\[
= \sum_{i=-1}^{m-1} p_S(i+1)(m - i)p_X(m - i) - \sum_{i=-1}^{m-1} a_i p_X(m - i)
\]
\[
= \sum_{i=-1}^{m-1} (p_S(i+1)(m - i)p_X(m - i) - a_i p_X(m - i))
\]
\[
= \sum_{i=-1}^{m-1} p_X(m - i) (p_S(i+1)(m - i) - a_i)
\]

which yields the result of we divide both sides by \( p_X(0) \). Note that for (*) we replace \( i \) by \( i - 1 \) in the first sum, while in the second sum we can add the term without any problems since it is 0 by virtue of \( a_{-1} = 0 \).
d) From part b) we know
\[ G'_S(r) = nG_S(r) \frac{G'_X(r)}{G_X(r)} \]
and we have defined \( F \) is such a way in part c) that
\[ G'_S(r) = nF(r). \]  \hspace{1cm} (12)

Now plugging in the power series:
\[ G_S(r) = \sum_{k=0}^{nN} p_S(k)r^k \quad \Rightarrow \quad G'_S(r) = \sum_{k=0}^{nN-1} (k + 1)p_S(k + 1)r^k \]
and \( F(r) = \sum_{k=0}^{nN-1} a_k r^k \) we see that (12) reads
\[ \sum_{k=0}^{nN-1} (k + 1)p_S(k + 1)r^k = \sum_{k=0}^{nN-1} a_k r^k. \]

According to Lemma 3.6.19 this yields
\[ (k + 1)p_S(k + 1) = a_k r^k \quad \text{for} \quad k = 0, \ldots, nN - 1 \]
or indeed
\[ p_S(k) = \frac{na_k}{k} \quad \text{for} \quad k = 1, \ldots, nN. \]

The formula for \( p_S(0) \) is clear by itself, since \( S_n = \sum_{i=1}^{n} X_i \) where the \( X_i \)'s are iid and take values in \( \{0, 1, \ldots, N\} \). So \( S_n \) takes the value 0 exactly if all the \( X_i \)'s take the value 0. Since the \( X_i \)'s are independent this yields
\[ p_S(0) = \mathbb{P}(S_n = 0) = \mathbb{P}(X_1 = 0, X_2 = 0, \ldots, X_n = 0) = \prod_{i=1}^{n} \mathbb{P}(X_i = 0) = \prod_{i=1}^{n} p_X(0) = p_X(0)^n. \]