Exercise sheet 2 for General Insurance: solutions

1. a) Player A aims for the payoff

\[
\max_{i \in \{1,2,3,4\}} \min_{j \in \{1,2,3\}} P_{ij} = \max\{\min\{P_{11}, P_{12}, P_{13}\}, \min\{P_{21}, P_{22}, P_{23}\}, \min\{P_{31}, P_{32}, P_{33}\}, \min\{P_{41}, P_{42}, P_{43}\}\}
\]

\[
= \max\{\min\{1, -1, -5\}, \min\{4, -4, 2\}, \min\{3, -3, -10\}, \min\{5, -5, -4\}\}
\]

\[
= \max\{-5, -4, -10, -5\} = -4,
\]

hence player A aims for the payoff -4 by playing strategy 2. Similarly for player B:

\[
\min_{j \in \{1,2,3\}} \max_{i \in \{1,2,3,4\}} P_{ij} = \min\{\max\{1, 4, 3, 5\}, \max\{-1, -4, -4, -5\}, \max\{-5, 2, -10, -4\}\} = \min\{5, -1, 2\} = -1,
\]

hence player B aims for the payoff -1 by playing strategy 2. Since both players aim for different payoffs there is no saddle point.

b) Player A aims for the payoff

\[
\max_{i \in \{1,2\}} \min_{j \in \{1,2\}} P_{ij} = \max\{\min\{P_{11}, P_{12}\}, \min\{P_{21}, P_{22}\}\}
\]

\[
= \max\{\min\{1, 1\}, \min\{2, -4\}\} = \max\{1, -4\} = 1,
\]

hence player A aims for the payoff 1 by playing strategy 1. For player B we get

\[
\min_{j \in \{1,2\}} \max_{i \in \{1,2\}} P_{ij} = \min\{\max\{1, 2\}, \max\{1, -4\}\} = \min\{2, 1\} = 1,
\]

hence player B also aims for the payoff 1 by playing strategy 2. Since both players aim for the same payoff there is saddle point, using their optimal strategies we know that \(P_{12}\) is the saddle point.

c) Player A aims for the payoff

\[
\max_{i \in \{1,2\}} \min_{j \in \{1,2,3\}} P_{ij} = \max\{\min\{P_{11}, P_{12}, P_{13}\}, \min\{P_{21}, P_{22}, P_{23}\}\}
\]

\[
= \max\{\min\{-2, 1, -3\}, \min\{-2, 3, -2\}\} = \max\{-3, -2\} = -2,
\]

hence player A aims for the payoff -2 by playing strategy 2. For player B we get

\[
\min_{j \in \{1,2,3\}} \max_{i \in \{1,2\}} P_{ij} = \min\{\max\{-2, 2\}, \max\{1, 3\}, \max\{-3, -2\}\} = \min\{-2, 3, -2\} = -2,
\]
hence player B also aims for the payoff $-2$ by playing either strategy 1 or 3. Since both players aim for the same payoff there is a saddle point, using their optimal strategies we know there are in fact two saddle points, namely $P_{21}$ and $P_{23}$.

2. a) Under the minimax criterion player A aims for the payoff

$$\max\{\min\{a, b\}, \min\{c, d\}\} \quad (*)$$

while player B aims for the payoff

$$\min\{\max\{a, c\}, \max\{b, d\}\} \quad (**)$$

First suppose that both $a \leq b$ and $a \geq c$ hold. Then:

$$(* \quad a \leq b \quad \Rightarrow \quad \max\{a, \min\{c, d\}\} \quad a \geq c \quad \Rightarrow \quad \min\{a, b\} \quad \Leftrightarrow \quad a$$

hence player A aims for payoff $a$ by playing strategy 1 (plus possibly strategy 2). In a similar way it follows that player B aims for payoff $a$ by playing strategy 1 (plus possibly strategy 2), hence $a$ is indeed a saddle point.

For the other direction, suppose that $a$ is a saddle point. Then $(*)$ has to equal $a$ and player A has to pick strategy 1 or strategy 1 and 2, i.e. the maximum in $(*)$ has to be attained in the first term. Hence it follows that $\min\{a, b\} = a$ and $\min\{c, d\} \leq a$. The former yields $a \leq b$. Furthermore $(**) \quad a$ has to equal $a$ and player B has to pick strategy 1 (or 1 and 2). Using $(**)$ this implies $\max\{a, c\} = a$ and $\max\{b, d\} \geq a$, the former yields $c \leq a$. Hence the result follows.

b) Note that there are two scenarios possible to reduce the game to the $1 \times 1$-game given. The first one is that first row 2 can be left out and then column 2 can be left out, the second one is that first column 2 can be left out and then row 2 can be left out.

For the first scenario, we can only leave out row 2 if player A never plays row 2, i.e. if $a \geq c$ and $b \geq d$. After row 2 is left out, we arrive at the following matrix:

\[
\begin{array}{c|c}
B: & 1 & 2 \\
A: & 1 & (a \ b) \\
\end{array}
\]

Now column 2 has to go, i.e. player B never plays column 2 which implies $b \geq a$. So, this scenario can only happen if in particular $a \geq c$ and $b \geq a$. Using a) this indeed implies that $a$ is a saddle point in the original game.

For the second scenario a similar argument applies.

c) Any choice were $a = b = c = d$ works.

3. a) The optimal pure action under the minimax criterion is found by computing

$$\min_{j \in \{1, 2\}} \max_{i \in \{1, 2, 3\}} L(\theta_i, a_j),$$
for which we find \( \min\{\max\{1, 3, 0\}, \max\{2, 1, 2\}\} = \min\{3, 2\} = 2 \), i.e. the optimal pure action is \( a_2 \) (because the minimum is attained for \( j = 2 \)).

b) Since there are two possible actions, \( a_1 \) and \( a_2 \), we can describe all possible randomised actions by the random variable \( X_{(x,1-x)} \) for \( x \in [0,1] \) with distribution:

\[
X_{(x,1-x)} = \begin{cases} 
    a_1 & \text{with prob. } x \\
    a_2 & \text{with prob. } 1-x
\end{cases}
\]

Under the minimax criterion we now have to compute

\[
\min_{x \in [0,1]} \max_{i \in \{1,2,3\}} \mathbb{E}[L(\theta_i, X_{(x,1-x)})].
\]

First we compute \( \mathbb{E}[L(\theta_i, X_{(x,1-x)})] \) for \( i = 1, 2, 3 \):

\[
\mathbb{E}[L(\theta_1, X_{(x,1-x)})] = L(\theta_1, a_1)\mathbb{P}(X_{(x,1-x)} = a_1) + L(\theta_1, a_2)\mathbb{P}(X_{(x,1-x)} = a_2)
= 1 \cdot x + 2 \cdot (1-x)
= 2 - x
\]

and similarly we find \( \mathbb{E}[L(\theta_2, X_{(x,1-x)})] = 3 \cdot x + 1 \cdot (1-x) = 1 + 2x \) and \( \mathbb{E}[L(\theta_3, X_{(x,1-x)})] = 0 \cdot x + 2 \cdot (1-x) = 2 - 2x \). Hence (1) boils down to

\[
\min_{x \in [0,1]} \max\{2 - x, 1 + 2x, 2 - 2x\}.
\]

You can e.g. make a graph (see Figure 1) of the functions \( x \mapsto 2 - x, \ x \mapsto 1 + 2x \) and \( x \mapsto 2 - 2x \) on \([0,1]\) to find that \( f(x) := \max\{2 - x, 1 + 2x, 2 - 2x\} \) attains its minimum in \( x = 1/3 \), with \( f(1/3) = 5/3 \). Hence we conclude that the optimal randomised action under the minimax criterion is to choose \( a_1 \) with prob. \( 1/3 \) and to choose \( a_2 \) with prob. \( 2/3 \), the expected value (or expected loss) is \( 5/3 \).

c) Under Bayes criterion we have to compute

\[
\min\limits_{j \in \{1,2\}} \mathbb{E}[L(\Theta, a_j)].
\]

For \( j = 1 \) we have \( \mathbb{E}[L(\Theta, a_1)] = L(\theta_1, a_1)\mathbb{P}(\Theta = \theta_1) + L(\theta_2, a_1)\mathbb{P}(\Theta = \theta_2) + L(\theta_3, a_1)\mathbb{P}(\Theta = \theta_3) = 1 - 1/4 + 3 \cdot 1/2 + 0 - 1/4 = 7/4 \), similarly for \( j = 2 \): \( \mathbb{E}[L(\Theta, a_2)] = 2 \cdot 1/4 + 1 \cdot 1/2 + 2 - 1/4 = 3/2. \) Hence the minimum in (2) is attained for \( j = 2 \), so \( a_2 \) is optimal and \( 3/2 \) is the expected value (or expected loss).

d) We can use reduction by dominance to get rid of action \( a_3 \) (action \( a_2 \) is always better) and of action \( a_4 \) (action \( a_4 \) is always better). The simplified game has only the actions \( a_2 \) and \( a_4 \) left, with the same loss matrix as we started this exercise with. Hence we can just copy the answers from a)-c) above: the optimal pure action under minimax criterion is \( a_4 \); the optimal randomised action under minimax criterion is to choose action \( a_1, a_2, a_3 \) and \( a_4 \) with prob. \( 0, 1/3, 0 \) and \( 2/3 \) resp. (don’t forget to include here the actions you eliminated using reduction by dominance!); under the Bayes criterion with prior distribution given by \( \Theta \) action \( a_2 \) is optimal.
4. a) First we have to decide what the states of nature $\theta_i$ and what our available actions $a_j$ are. You might e.g. take

$\theta_1$: it will rain in 15 mins
$\theta_2$: it won’t rain in 15 mins
$a_1$: go borrow umbrella
$a_2$: don’t borrow an umbrella

From the story it follows that the losses are as follows: $L(\theta_1, a_1) = 0$, $L(\theta_1, a_2) = 5$, $L(\theta_2, a_1) = 1$ and $L(\theta_2, a_2) = 0$. Hence we get the following loss matrix:

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

As usual by now, we can represent any randomised action by a random variable $X_{(x,1-x)}$ taking the value $a_1$ with prob. $x$ and value $a_2$ with prob. $1 - x$, for $x \in [0, 1]$. For the minimax criterion we have to compute

$$\min_{x \in [0,1]} \max_{i \in \{1, 2\}} \mathbb{E}[L(\theta_i, X_{(x,1-x)})].$$

For $i = 1$ we get $\mathbb{E}[L(\theta_1, X_{(x,1-x)})] = x \cdot 0 + 5 \cdot (1 - x) = 5 - 5x$ and for $i = 2$ we get $\mathbb{E}[L(\theta_2, X_{(x,1-x)})] = 1 \cdot x + 0 \cdot (1 - x) = x$, hence (3) boils down to

$$\min_{x \in [0,1]} \max\{5 - 5x, x\}.$$
and you can use your favorite method (e.g. a graph of \( x \mapsto \max\{5 - 5x, x\} \)) to deduce that the minimum is attained for \( x = 5/6 \) with value \( 5/6 \). Hence the requested expected value/(loss) is \( 5/6 \) and the optimal randomised action is to choose \( a_1 \) with prob \( 5/6 \) and to choose \( a_2 \) with prob \( 1/6 \).

b) We are given that \( \Theta \) takes values \( \theta_1 \) and \( \theta_2 \) with probs \( p \) and \( 1 - p \) resp. For the Bayes criterion we have to compute

\[
\min_{j \in \{1, 2\}} \mathbb{E}[L(\Theta, a_j)].
\]

For \( j = 1 \) we get

\[
\mathbb{E}[L(\Theta, a_1)] = L(\theta_1, a_1)\mathbb{P}(\Theta = \theta_1) + L(\theta_2, a_1)\mathbb{P}(\Theta = \theta_2) = 0 \cdot p + 1 \cdot (1 - p) = 1 - p
\]

and similarly \( \mathbb{E}[L(\Theta, a_2)] = 5p \). It is optimal to choose \( a_1 \) if and only if

\[
\mathbb{E}[L(\Theta, a_1)] \leq \mathbb{E}[L(\Theta, a_2)],
\]

i.e. if and only if \( 1 - p \leq 5p \). So action \( a_1 \) is optimal if and only if \( p \in \left[ \frac{1}{6}, 1 \right] \).

c) Denote the outcome of the experiment (namely, looking outside) by the random variable \( Z \) taking values in \( \{z_1, z_2\} \), where \( z_1 \) : the sky is bright and \( z_2 \) : the sky is cloudy. Furthermore we are given the following distributional properties. If \( \theta_1 \) is the true state of nature, then \( \mathbb{P}(Z = z_2) = 1 \) and hence \( \mathbb{P}(Z = z_1) = 0 \), while if \( \theta_2 \) is the true state of nature then \( \mathbb{P}(Z = z_1) = 9/10 \) and hence \( \mathbb{P}(Z = z_2) = 1/10 \).

Now let us first list all possible decision functions, i.e. all mappings from \( \{z_1, z_2\} \) to \( \{a_1, a_2\} \):

\[
d_1(z) = \begin{cases} a_1 & \text{when } z = z_1, \\ a_1 & \text{when } z = z_2, \end{cases} \quad d_2(z) = \begin{cases} a_1 & \text{when } z = z_1, \\ a_2 & \text{when } z = z_2, \end{cases} \quad d_3(z) = \begin{cases} a_2 & \text{when } z = z_1, \\ a_1 & \text{when } z = z_2, \end{cases} \quad d_4(z) = \begin{cases} a_2 & \text{when } z = z_1, \\ a_2 & \text{when } z = z_2. \end{cases}
\]

and finally

\[
d_4(z) = \begin{cases} a_2 & \text{when } z = z_1, \\ a_2 & \text{when } z = z_2. \end{cases}
\]

Next let us for every combination of a state \( \theta_i \) and a decision function \( d_j \) compute the risk \( R(\theta_i, d_j) \):

\[
R(\theta_i, d_j) = \mathbb{E}[L(\theta_i, d_j(Z))]
\]

(\text{be aware that } Z \text{ follows different distributions dependent on whether } \theta_1 \text{ or } \theta_2 \text{ is the true state of nature}). We find

\[
R(\theta_1, d_1) = \mathbb{E}[L(\theta_1, d_1(Z))] = L(\theta_1, d_1(z_1))\mathbb{P}(Z = z_1) + L(\theta_1, d_1(z_2))\mathbb{P}(Z = z_2)
\]

\[
= L(\theta_1, a_1)\mathbb{P}(Z = z_1) + L(\theta_1, a_1)\mathbb{P}(Z = z_2) = 0 \cdot 0 + 0 \cdot 1 = 0,
\]
\[ R(\theta_1, d_2) = \mathbb{E}[L(\theta_1, d_2(Z))] = L(\theta_1, d_2(z_1))\mathbb{P}(Z = z_1) + L(\theta_1, d_2(z_2))\mathbb{P}(Z = z_2) \\
= L(\theta_1, a_1)\mathbb{P}(Z = z_1) + L(\theta_1, a_2)\mathbb{P}(Z = z_2) = 0 \cdot 0 + 5 \cdot 1 = 5 \]

and similarly \( R(\theta_1, d_3) = 0, R(\theta_1, d_4) = 5 \). Now we turn to computing \( R(\theta_2, .) \) and hence use the other distribution of \( Z \):

\[ R(\theta_2, d_1) = \mathbb{E}[L(\theta_2, d_1(Z))] = L(\theta_2, d_1(z_1))\mathbb{P}(Z = z_1) + L(\theta_2, d_1(z_2))\mathbb{P}(Z = z_2) \\
= L(\theta_2, a_1)\mathbb{P}(Z = z_1) + L(\theta_2, a_2)\mathbb{P}(Z = z_2) = 1 \cdot 9/10 + 1 \cdot 1/10 = 1, \]

\[ R(\theta_2, d_2) = \mathbb{E}[L(\theta_2, d_2(Z))] = L(\theta_2, d_2(z_1))\mathbb{P}(Z = z_1) + L(\theta_2, d_2(z_2))\mathbb{P}(Z = z_2) \\
= L(\theta_2, a_1)\mathbb{P}(Z = z_1) + L(\theta_2, a_2)\mathbb{P}(Z = z_2) = 1 \cdot 9/10 + 0 \cdot 1/10 = 9/10 \]

and similarly \( R(\theta_2, d_3) = 1/10, R(\theta_2, d_4) = 0 \).

We thus end up with the following risk matrix:

\[
\begin{array}{cccc}
\text{DM:} & d_1 & d_2 & d_3 & d_4 \\
\text{Nature:} & \theta_1 & (0 & 5 & 0 & 5) \\
\theta_2 & (1 & 9/10 & 1/10 & 0)
\end{array}
\]

Finally we need to compute the optimal randomised decision function under the minimax criterion for the above risk matrix. First we see if we can apply reduction by dominance. We see that \( d_3 \) dominates \( d_1 \) and that \( d_4 \) dominates \( d_2 \), hence we can reduce to

\[
\begin{array}{cccc}
\text{DM:} & d_3 & d_4 \\
\text{Nature:} & \theta_1 & (0 & 5) \\
\theta_2 & (1/10 & 0)
\end{array}
\]

and we compute the optimal randomised decision function in the now usual way: suppose \( D_{(x,1-x)} \) is a random variable taking value \( d_3 \) with prob. \( x \) and value \( d_4 \) with prob. \( 1-x \).

We need to find

\[
\min_{x \in [0,1]} \max_{i \in \{1,2\}} \mathbb{E}[R(\theta_i, D_{(x,1-x)})]. \tag{4}
\]

We may compute

\[
\mathbb{E}[R(\theta_1, D_{(x,1-x)})] = R(\theta_1, d_3)\mathbb{P}(D_{(x,1-x)} = d_3) + R(\theta_1, d_4)\mathbb{P}(D_{(x,1-x)} = d_4) \\
= 0 \cdot x + 5 \cdot (1-x) \\
= 5 - 5x
\]
and
\[ E[R(\theta_2, D_{(x,1-x)})] = R(\theta_2, d_3)P(D_{(x,1-x)} = d_3) + R(\theta_2, d_4)P(D_{(x,1-x)} = d_4) \\
= 1/10 \cdot x + 0 \cdot (1 - x) \\
= x/10, \]
which yields that (4) equals
\[ \min_{x \in [0,1]} \max \{5 - 5x, x/10\}. \]
Again by a graph e.g. we deduce that this minimum is attained for \( x = 50/51 \), with expected value 5/51.

d) Indeed the expected value/loss under c) is 5/51 which is less than the 5/6 we found under a). The explanation is that including this experiment is giving us some extra information about the state of nature and apparently we can indeed use this to our advantage.

5. Denote as usual by \( X_{(x,1-x)} \) the rv that takes value \( a_1 \) with prob. \( x \) and value \( a_2 \) with prob. \( 1 - x \), for \( x \in [0,1] \). Under the minimax criterion we look for the value of \( x \) for which the minimum in
\[ \min_{x \in [0,1]} \max_{i \in \{1,2\}} E[L(\theta_i, X_{(x,1-x)})] \]  
(5)
is attained. We may compute
\[ E[L(\theta_1, X_{(x,1-x)})] = L(\theta_1, a_1)P(X_{(x,1-x)} = a_1) + L(\theta_1, a_2)P(X_{(x,1-x)} = a_2) \\
= 1 \cdot x + b \cdot (1 - x) = (1 - b)x + b =: f_b(x) \]
and
\[ E[L(\theta_2, X_{(x,1-x)})] = L(\theta_2, a_1)P(X_{(x,1-x)} = a_1) + L(\theta_2, a_2)P(X_{(x,1-x)} = a_2) \\
= b \cdot x + 2 \cdot (1 - x) = 2 + (b - 2)x =: g_b(x), \]
hence (5) boils down to
\[ \min_{x \in [0,1]} \max \{f_b(x), g_b(x)\}. \]
Now \( f_b \) is the straight line between the points \((0,b)\) and \((1,1)\), while \( g_b \) is the straight line between the points \((0,2)\) and \((1,b)\). Using the assumption that \( b < 1 \) and making a graph of \( f_b \) and \( g_b \) e.g., we see that \( \max \{f_b(x), g_b(x)\} \) attains its minimum on the interval \( [0,1] \) for \( x = 3/4 \) if and only if \( f_b(3/4) = g_b(3/4) \). This yields the equation
\[(1 - b)3/4 + b = 2 + (b - 2)3/4 \]
and this is uniquely solved by \( b = 1/2 \). Hence this is the
value of $b$ we were looking for.

6. a) (Eq. numbers here refer to equations on the exercise sheet) For (2) you minimising
the expected loss over all randomised actions, whereas for (1) you minimising the expected
loss over all pure actions, which is a smaller set of actions since any pure action belongs
to the set of randomised actions. So (2) must be less than or equal to (1).

b) Clearly you cannot base your randomised action $X^A_{\vec{p}}$ on $\Theta$ since you do not observe the
latter random variable. Hence the random variables $X^A_{\vec{p}}$ and $\Theta$ are independent. Therefore,

$$
E \left[ L \left( \Theta, X^A_{\vec{p}} \right) \right] = \sum_{j=1}^{m} \sum_{i=1}^{n} L(\theta_i, a_j) \mathbb{P}(\Theta = \theta_i, X^A_{\vec{p}} = a_j)
$$

$$
= \sum_{j=1}^{m} \sum_{i=1}^{n} L(\theta_i, a_j) \mathbb{P}(\Theta = \theta_i) \mathbb{P}(X^A_{\vec{p}} = a_j)
$$

$$
= \sum_{j=1}^{m} E \left[ L \left( \Theta, a_j \right) \right] p_j
$$

Now clearly we have for all $j = 1, \ldots, m$,

$$
E \left[ L \left( \Theta, a_j \right) \right] \geq \min_{k \in \{1, \ldots, m\}} E \left[ L \left( \Theta, a_k \right) \right]
$$

and combining the above equality and inequality we get

$$
E \left[ L \left( \Theta, X^A_{\vec{p}} \right) \right] \geq \sum_{j=1}^{m} \min_{k \in \{1, \ldots, m\}} E \left[ L \left( \Theta, a_k \right) \right] p_j
$$

$$
= \min_{k \in \{1, \ldots, m\}} E \left[ L \left( \Theta, a_k \right) \right] \sum_{j=1}^{m} p_j
$$

$$
= \min_{k \in \{1, \ldots, m\}} E \left[ L \left( \Theta, a_k \right) \right].
$$

Hence (1) is less than or equal to (2).

c) It follows from a) and b) that in fact (1) equals (2). Hence the decision maker can not
improve his expected loss by using randomised actions, so it is not useful to introduce
them.