(a) Because of the higher discount level, the policyholders are more inclined to want this discount and therefore will less likely report a claim to the insurance company in order to stay or get to the highest discount level. Of course policyholders will only not report a claim if the value of the claim is small; it makes no sense to not let the insurance company pay a large a claim just to have a relative small discount on the premium in the next years. So since only small claims will not be reported, the average claim size will go up.

(b) The stationary distribution is \( \vec{\pi} = (0.0323, 0.0312, 0.0260, 0.1301, 0.7804) \). So with \( \beta = (100, 80, 70, 70, 50) \) the vector of premiums per year corresponding to each discount state, the expected profit per policyholder per year in the long run equals

\[
\sum_{i=1}^{d} \pi_i \beta_i - 320 \times \frac{1}{6} = 2.338.
\]

(c) The higher discount level is more attractive to policyholders and so the number of policyholders is likely to increase. This means that the total profit per year in the long run might well be higher under the new regime.

Answer of 2

(a) The transition graph is given in Figure 1. Further all the states are communicating with each other (by following the arrows in the diagram you see that from any arbitrary state you can go to any other state in some finite number (note that this number does not necessarily have to be equal to one) of steps). So the MC is irreducible. Further, all states are aperiodic, since clearly state 1 is aperiodic as \( p_{11} > 0 \) and so by Proposition 3.5 all states are aperiodic and we say that the MC is aperiodic.

![Figure 1: Transition graph corresponding to Exercise 2(a).](image-url)
Figure 2: Transition graph corresponding to Exercise 2(b).

(b) The transition graph is given in Figure 2. We easily see that all states are communicating since there are arrows from 1 to 2, from 2 to 4, from 4 to 3 and from 3 to 1. Hence the entire state space is a communicating class and thus the MC is irreducible. Since state 4 is not aperiodic (once in state 4, the MC can only return to state 4 in an even number of steps), it follows by Proposition 3.5 that all states are not aperiodic.

Answer of 3

(a) The matrix of two step transition probabilities is given by \( P(2) = P^2 \) since the MC is time homogeneous. We have

\[
P^2 = \begin{pmatrix}
3/8 & 3/8 & 2/8 \\
5/16 & 10/16 & 1/16 \\
5/16 & 6/16 & 5/16
\end{pmatrix}.
\]

(b) There is a unique stationary distribution given by \( \vec{\pi} = (\pi_1, \pi_2, \pi_3) = (2/6, 3/6, 1/6) \).

(c) Let \( X_n \) be the site at which the car is at time \( n \). Then by an application of Theorem 2.2 in the notes,

\[
\Pr(X_3 = 3, X_2 = 2, X_1 = 1|X_0 = 1) = p_{11}p_{12}p_{23} = 0.50 \times 0.25 \times 0 = 0.
\]

Answer of 4

Assume \( i \leftrightarrow j \) and \( j \leftrightarrow k \). We need to prove \( i \leftrightarrow k \). We first prove \( i \rightarrow k \). Since \( i \rightarrow j \) and \( j \rightarrow k \), we know that there exists \( n, m \) such that one can get in \( n \) steps from \( i \) to \( j \) and in \( m \) steps from \( j \) to \( k \), i.e. \( p_{ij}(n) > 0 \) and \( p_{jk}(m) > 0 \). Then from the Markov property, one can guess that it should be possible to get in \( n + m \) steps from \( i \) to \( k \). In order to prove this rigorously, note that by Theorem 2.2 in the notes

\[
p_{ik}(n + m) = \Pr(X_{n+m} = k|X_0 = i) \geq \Pr(X_{n+m} = k, X_n = j|X_0 = i)
\]

\[= \Pr(X_{n+m} = k|X_n = j)\Pr(X_n = j|X_0 = i)\]

\[= p_{jk}(m)p_{ij}(n) > 0.
\]

Hence \( p_{ik}(n + m) > 0 \) and so \( i \rightarrow k \). (Note that one can also use the Chapman-Kolmogorov equations as in the proof of Proposition 3.5 in order to prove \( i \rightarrow k \).) Similarly, we can prove \( k \rightarrow i \) by reversing the roles of \( i \) and \( k \).
The transition diagrams are given in Figures 3 and 4. In case (a) \(\{1, 2\}\) and \(\{3, 4\}\) are the communicating classes and so the MC is not irreducible. For case (b), \(\{1, 2\}, \{3\}\) and \(\{4, 5\}\) are the communicating classes. Hence the MC is not irreducible in this case. Now we turn to finding the stationary distributions.

(a) Let’s write the system of 5 linear equations that need to be satisfied in the following form: \(Ax = b\) with \(A\) a 5 by 4 matrix, \(x\) a 4-dimensional column vector and \(b\) a 5-dimensional column vector. Then

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-1/2 & 2/3 & 0 & 0 & 0 \\
1/4 & -1 & 0 & 0 & 0 \\
1/4 & 1/3 & -1/3 & 1/2 & 0 \\
0 & 0 & 1/3 & -1/2 & 0
\end{pmatrix}
\begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

One way of solving this system (note that there are several other and perhaps quicker ways) is by writing down the augmented matrix \(A|b\) and then perform elementary row operations to transform the augmented matrix into row echelon form or reduced row echelon form. Note that, as mentioned in the notes, we can remove one of the last four equations; let’s remove the last but one. Then using elementary row operations

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-1/2 & 2/3 & 0 & 0 & 0 \\
1/4 & -1 & 0 & 0 & 0 \\
0 & 0 & 1/3 & -1/2 & 0
\end{pmatrix}
\sim 
\begin{pmatrix}
1 & 0 & 4/7 & 4/7 & 4/7 \\
0 & 1 & 3/7 & 3/7 & 3/7 \\
0 & 0 & 2/7 & 2/7 & 2/7 \\
0 & 0 & 1/3 & -1/2 & 0
\end{pmatrix}
\sim 
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & -5/6 & -1/3
\end{pmatrix}
\sim 
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 2/5
\end{pmatrix}
\]

We have brought the augmented matrix into reduced row echelon form and we can read of the unique solution given in the most right column. Since this vector contains only positive elements, it follows that there exist exactly one stationary distribution which is given by \((\pi_1, \pi_2, \pi_3, \pi_4) = (0, 0, 3/5, 2/5)\).

(b) Similarly, writing down the augmented matrix representing the system of linear equations (hereby removing one equation) that needs to be solved and performing elemen-
tary row operations, we get
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-1/2 & 1/3 & 0 & 0 & 0 \\
1/2 & -1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/3 & 1/2 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 5/6 & 1/2 & 1/2 & 1/2 \\
0 & -5/6 & -1/2 & -1/2 & -1/2 \\
0 & 0 & 0 & -1/3 & 1/2 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 0 & 2/5 & 2/5 & 2/5 \\
0 & 1 & 3/5 & 3/5 & 3/5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/3 & 1/2 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 2/5 & 0 & 1 & 2/5 \\
0 & 1 & 3/5 & 0 & 3/2 & 3/5 \\
0 & 0 & 0 & 1 & -3/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

We see that the system of five linear equations consists of three independent linear equations. Further, we have that \( \vec{\pi} \) is a stationary distribution if all its elements are positive and
\[
\begin{align*}
\pi_1 + \frac{2}{5}\pi_3 + \frac{2}{5}\pi_5 &= \frac{2}{5}, \\
\pi_2 + \frac{3}{5}\pi_3 + \frac{3}{5}\pi_5 &= \frac{3}{5}, \\
\pi_4 - \frac{3}{5}\pi_5 &= 0.
\end{align*}
\]
Setting \( \pi_5 = \frac{2}{5}\gamma \), leads to \( \pi_4 = \frac{3}{5}\gamma \) and setting \( \pi_3 = \beta \) leads to \( \pi_2 = \frac{3}{5}(1 - \gamma - \beta) \) and \( \pi_1 = \frac{2}{5}(1 - \gamma - \beta) \). Hence there exist infinitely many stationary distributions and all of them are of the form
\[
(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = \left( \frac{2}{5}(1 - \gamma - \beta), \frac{3}{5}(1 - \gamma - \beta), \beta, \frac{3}{5}\gamma, \frac{2}{5}\gamma \right),
\]
whereby \( \gamma, \beta \geq 0 \) such that \( \beta + \gamma \leq 1 \).

**Answer of 6**

As the MC is irreducible and not aperiodic, we must have that all states are not aperiodic. This implies that \( p_{AA} = p_{BB} = p_{CC} = 0 \), since if e.g. \( p_{AA} > 0 \), then state A would be aperiodic. Further \( p_{AA} + p_{AB} + p_{AC} = 1 \) and \( p_{AB} = p_{AC} \) (together with \( p_{AA} = 0 \)) imply that \( p_{AB} = p_{AC} = 1/2 \).

Now if both \( p_{BA} = 0 \) and \( p_{CA} = 0 \), then the MC would not be irreducible since in that case state A would not communicate with state B and/or C. Hence (i) \( p_{BA} > 0 \) or (ii) \( p_{CA} > 0 \).

Now suppose that (i) \( p_{BA} > 0 \). Suppose further that \( p_{CB} > 0 \). Then \( p_{AA}(2) > 0 \) and \( p_{AA}(3) > 0 \), i.e. with strictly positive probability (wspp) one can return in state A in two and in three steps. Consequently, one can return wspp in A in \( 2n + 3m \) number of steps, where \( n, m \) are strictly positive integers. This means that one can return wspp in A in \( k \) number of
Figure 3: Transition graph corresponding to Exercise 5(a).

Figure 4: Transition graph corresponding to Exercise 5(b).
steps, where $k$ is any integer bigger or equal to 2. Hence state A would be aperiodic which forms a contradiction. It follows that $p_{CB} = 0$, which implies further that $p_{CA} = 1$ (recalling that $p_{CC} = 0$).

Now suppose that (ii) $p_{CA} > 0$. Then by the same arguments as for case (i), we must have $p_{BC} = 0$ and $p_{BA} = 1$.

It follows that $p_{BA} > 0$ or $p_{CA} > 0$ implies that $p_{BA} > 0$ and $p_{CA} > 0$. Hence $p_{CB} = p_{BC} = 0$ and $p_{CA} = p_{BA} = 1$.

The transition graph is given in Figure 5.