Actuarial Models: solutions example sheet 2

Answer to 1

(i) In this case $X_1, X_2, \ldots$ are independent and thus $X$ is trivially a Markov chain. (ii) In this case $X$ is a Markov chain as $X_n = X_{n-1} + Z_n$ and so given that we know the value of $X_{n-1}$, then $X_n$ depends only on $Z_n$ which does not depend on $X_1, X_2, \ldots, X_{n-1}$ as $Z_1, Z_2, \ldots, Z_n$ are independent (see also Example 11 in the notes).

(iii) In this case $X$ is not a Markov chain as now the history of the process does give us more information than just the present does about the future. In particular, if we know the values of $X_{n-1}, X_{n-2}, \ldots, X_2, X_1$, then we know the value of $Z_1, Z_2, \ldots, Z_{n-1}$ and since

$$X_n = S_n + S_{n-1} = S_{n-1} + S_{n-2} + Z_n + Z_{n-1} = X_{n-1} + Z_n + Z_{n-1}$$

this narrows the possible outcomes of $X_n$ to just 6. If we only know the value of $X_{n-1}$, then we cannot pinpoint the exact value of $Z_{n-1}$ and then the range of possible outcomes of $X_n$ becomes much wider.

Let us clarify this with an explicit example. Assume that we are told that $X_2 = 8$ and $X_1 = 2$. Then we know that $Z_1 = S_1 = X_1 = 2$ (note that $S_0 = 0$) and $Z_2 = X_2 - X_1 - Z_1 = 4$. Therefore, we know that $X_3 = X_2 + Z_3 + Z_2 = 12 + Z_3$ and so we know that $X_3 \in \{13, \ldots, 18\}$. But if we would only have been told that $X_2 = 8$, then we could not have ruled out the case that (a) $Z_1 = 3$ and $Z_2 = 2$ or (b) $Z_1 = 1$ and $Z_2 = 6$, which means that, for all we know, the future value $X_3$ could also be equal to e.g. 11, 12, 19 or 20. So the past and the present ($X_1 = 2$ and $X_2 = 8$) does give us more information than just the present ($X_2 = 8$) for finding out what the future ($X_3$) will bring. Hence the Markov property is not satisfied.

Answer to 2

(a) Transition matrices are stochastic and thus the elements on each row have to add up to one. From row 3, we deduce

$$\alpha - 7/20 + 2/5 + 1/5 = 1 \Rightarrow \alpha = 3/4$$

and from row 1 we have $\alpha + \beta = 1$, leading to $\beta = 1/4$.

(b) See Figure 1.

(c) The required probabilities are the elements of the 3-step transition matrix $P(3) = P^3$. As

$$P^3 = P^2 P = \begin{pmatrix} 12/16 & 3/16 & 1/16 \\ 53/80 & 23/80 & 4/80 \\ 17/25 & 9/50 & 7/50 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 3/4 & 0 & 1/4 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$$

$$= \begin{pmatrix} 466/640 & 136/640 & 38/640 \\ 1172/1600 & 297/1600 & 131/1600 \\ 701/1000 & 226/1000 & 73/1000 \end{pmatrix},$$

we get $p_{11}(3) = 466/640$, $p_{12}(3) = 136/640$, $p_{13}(3) = 38/640$, $p_{21}(3) = 1172/1600$, etc.
(d) From Theorem 2.2 in the notes,
\[ \text{Pr}(X_1 = 2, X_2 = 3, X_3 = 2|X_0 = 1) = p_{12}p_{23}p_{32} = 1/4 \times 1/4 \times 2/5 = 1/40. \]

**Answer to 3**

(a) We use a three-state (time homogeneous) MC whereby state 1, 2, 3 represent respectively the 0%, 25% and 40% discount rate. The transition matrix \( P \) is given by

\[
P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.2 & 0 & 0.8 \\ 0 & 0.2 & 0.8 \end{pmatrix}.
\]

The required probability is \( p_{22}(4) \), i.e. the (2, 2) element of the matrix \( P^4 \). Computing this matrix (note that by computing first \( P^2 \) and then \( P^2P^2 \) one has to compute just two matrix multiplications) and reading off this particular entry gives \( p_{22}(4) = 0.2112 \).

(b) Define the state space by \( S = \{1, 2, 3, 4, 5\} \), where state 1 represents the 0% discount level, state 2 the 25% level, state 3 the 40% level discount with no claims in the previous year, state 4 the 40% level with a claim in the previous year and state 5 represent the 60% discount level. With this state space this NCD model is a (time homogeneous) MC with transition matrix given by

\[
P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.8 \\ 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}.
\]
The required probability is \( p_{15}(5) \). Computing the matrix \( P^5 \) (this can be done with three matrix multiplications) and reading off the \((1, 5)\) entry gives \( p_{15}(5) = 0.67584 \).

**Answer to 4**

(a) Since, once a company is rated D, it is always rated as D thereafter (D is an absorbing state), the only way a company currently rated as A will never be rated B is by being rated as A for \( n \) consecutive years and then be rated as D the following year for \( n = 0 \) or \( n = 1 \) or \( n = 2 \), etc. Hence with \( E_n \) being the event that the company is rated A at the end of the years \( 1, 2, \ldots n \) and rated D at the end of year \( n + 1 \), we have

\[
\Pr(\text{the company will never be rated B}|\text{currently rated A}) = \Pr(\bigcup_{n=1}^\infty E_n|\text{currently rated A})
\]

\[
= \sum_{n=1}^\infty \Pr(E_n|\text{currently rated A})
\]

\[
= \sum_{n=1}^\infty (p_{AA})^{n-1}p_{AD}
\]

\[
= \sum_{n=0}^\infty (0.92)^n0.03 = \frac{0.03}{1-0.92} = 0.375,
\]

where in the second equality we used that the events \( E_n, n \geq 0 \) are disjoint and in the third we used Theorem 2.2 in the notes.

(b) (i) The two year transition probabilities are the elements of the matrix

\[
P(2) = P^2 = \begin{pmatrix}
0.8489 & 0.00885 & 0.0626 \\
0.0885 & 0.7250 & 0.1865 \\
0 & 0 & 1
\end{pmatrix}.
\]

(ii) Since \( p_{AD}(2) = 0.0626 \), the expected number of defaults by the end of year 2 amongst 100 companies all initially rated as A is \( 100 \times p_{AD}(2) = 6.26 \).

(c) Under the downgrade trigger policy any company that moves to the B category gets replaced by a company from the A category. Hence the ratings model becomes a two state MC (with states A and D) with transition matrix \( \hat{P} = \begin{pmatrix} 0.97 & 0.03 \\ 0 & 1 \end{pmatrix} \). Hence the expected number of defaults over the next two years given that the portfolio consists initially of 100 A-rated companies is \( 100 \times \hat{p}_{AD}(2) = 100 \times (\hat{p}_{AA}\hat{p}_{AD} + \hat{p}_{AD}\hat{p}_{DD}) = 5.91 \).