Actuarial Models: example sheet 7

*=easy, **=intermediate, ***=difficult

** Exercise 1

Consider the employment-unemployment MJP model with two states $e$: employed and $u$: unemployed and constant transition rates $\sigma, \eta$ where $\sigma$ is the transition rate from employment to unemployment and $\eta$ is the transition rate from unemployment to employment. For $j = e, u$ let $w_j$ to be the total waiting time spent in state $j$ and $\delta_j$ to be the total number of transitions out of state $j$ during the observation interval $[0, t]$.

(a) Suppose you observe a person’s history of employment throughout $[0, t]$ and you record the corresponding sample path. Write down the corresponding likelihood function of the pair of parameters $(\sigma, \eta)$ and find the mles of $\sigma$ and $\eta$.

(b) Given the following data from a study:

- Waiting times: $w_e = 9600, \quad w_u = 1200$
- Transition numbers: $\delta_e = 100, \quad \delta_u = 20$,

estimate

(i) the probability that an individual, who is initially employed, remains employed for a full year;
(ii) the probability that an individual, who is employed at the beginning of a year, is in employment at the end of the year.

** Exercise 2

Consider the following four state model for the mortality of a married pair:

\[
\begin{array}{cccc}
\text{State } a & & \text{State } b \\
\mu_1 & \nu_1 & \mu_2 & \\
\text{State } c & & \text{State } d \\
& \nu_2 & \\
\end{array}
\]

State $a$ means both husband and wife are alive, state $b$ stands for husband alive, wife dead, state $c$ stands for husband dead, wife alive and state $d$ represents both are dead.
(a) Suppose you observe the mortality of a married pair throughout some time interval, hereby recording the corresponding sample path. Write down the likelihood function of $\nu_1, \nu_2, \mu_1$ and $\mu_2$ in terms of the observed number of transitions and the waiting times in each state (during this time interval) and find the mles of the parameters in the model.

(b) Suppose you are told that $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. What are now the mles of the parameters?

** Exercise 3

Consider the four state model with states $a, b, c,$ and $d$, in which the only possible transitions are from $a$ to either $b$ or $c$ or $d$ with respective (constant) rates $\alpha, \beta$ and $\mu$. The following data are available on 1000 individuals, all of the same age, observed for exactly one calendar year.

- Total waiting time in state $a =$ 750 years
- Total number of transitions from $a$ to $b =$ 180
- Total number of transitions from $a$ to $c =$ 175
- Total number of transitions from $a$ to $d =$ 18

(a) Compute the mles of $\alpha, \beta$ and $\mu$.

(b) Derive 95% asymptotic confidence intervals of $\alpha, \beta$ and $\mu$.

(c) Estimate the transition probabilities $p_{aa}(1)$ and $p_{ab}(0.5)$.

*** Exercise 4 (not so important)

(The formulas in this exercise appear in the contingencies course units.) Consider a time homogeneous MJP $X = \{X_t : t \geq 0\}$ with transition rates $\mu_{jk}$ and transition probabilities $p_{ij}(t)$. Let $W_j$ be the total length of time the MJP visits the state $j$ in the interval $[0, 1]$ and let $\Delta_{jk}$ be the total number of transitions from state $j$ to state $k$ (with $k \neq j$) in the interval $[0, 1]$. Assume the MJP starts in state $i$.

(a) Prove that $\mathbb{E}[W_j] = \int_0^1 p_{ij}(u)du$.

(b) Define for $m \geq 1$ the discrete time stochastic process $Y^{(m)} = \{Y^{(m)}_n : 0 \leq n \leq m\}$ by $Y^{(m)}_n = X_{n/m}$ and let $\Delta_{jk}^{(m)}$ be the total number of times the process $Y$ jumps from state $j$ to state $k$. Prove that $\mathbb{E}\left[\Delta_{jk}^{(m)}\right] = \sum_{n=1}^{m} p_{ij}\left(\frac{n-1}{m}\right) p_{jk}(1/m)$.

(c) Explain why $\mathbb{E}[\Delta_{jk}] = \mu_{jk} \int_0^1 p_{ij}(u)du$. 

2