** Exercise 1

Consider the following three state time homogeneous model for a terminal illness:

\[ \begin{array}{ccc}
  & \lambda & \\
\text{h: healthy} & \mu & \text{i: ill} \\
\text{d: dead} & \nu & \\
\end{array} \]

Let \( P(t) = \begin{pmatrix} p_{hh}(t) & p_{hi}(t) & p_{hd}(t) \\
p_{dh}(t) & p_{hi}(t) & p_{hd}(t) \\
p_{dh}(t) & p_{di}(t) & p_{dd}(t) \end{pmatrix} \), \( t \geq 0 \), be the associated homogeneous transition matrix function.

(a) By using Kolmogorov’s forward equations show that \( p_{id}(t) = 1 - e^{-\nu t}, \ t \geq 0 \).

(b) Write down the forward equations for \( p_{hh}(t), p_{hi}(t), p_{hd}(t) \) and solve them.

** Exercise 2

(This exercise more or less provides the proof of Theorem 2.2 in the notes.) Let \( X = \{X_t : t \geq 0\} \) be a Markov jump process with state space \( S \) and denote \( p_{ik}(s, t) = \Pr(X_t = k | X_s = i) \).

Show that for any \( n \geq 1 \)

\[ \Pr(X_{t_n} = k_n, X_{t_{n-1}} = k_{n-1}, \ldots, X_{t_1} = k_1 | X_{t_0} = k_0) = \prod_{i=1}^{n} p_{k_{i-1}k_i}(t_{i-1}, t_i), \]

where \( 0 \leq t_0 < t_1 < \ldots < t_{n-1} < t_n \) and \( k_0, k_1, \ldots, k_{n-1}, k_n \in S \). (Hint: first consider the case \( n = 2 \) and then with the gained insight prove the general case either directly or by induction.)

** Exercise 3

Prove the Chapman-Kolmogorov equations for an MJP.

** Exercise 4

(This exercise justifies that in order to solve the Kolmogorov forward equations with a fixed backward state \( i \), one can always replace one of the differential equations by the equation...
which says that the $i$th rowsum of the matrix solution equals one.) Let for each $t \geq 0$, $Q(t) = (\mu_{ij}(t))_{i,j=1}^d$ be a Q-matrix. Fix $i$ and $l \in \{1, \ldots, d\}$ and suppose that $p_{ik}(s,t)$ satisfies for $0 \leq s < t$,

$$\frac{\partial}{\partial t} p_{ik}(s,t) = \sum_{j=1}^d p_{ij}(s,t) \mu_{jk}(t), \quad (*)$$

for all $k \in \{1, \ldots, d\}\setminus\{l\}$. Suppose in addition that $\sum_{j=1}^d p_{ij}(s,t) = 1$. Show that $(*)$ is satisfied for $k = l$ as well.