VIEW FROM THE PENNINES: PATTERN COMPLEXITY

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Textbook examples of pre-Enclosure Act strip holdings with their gentle reversed S shape characteristic of medieval plough rigs [7] can be seen in the neighbouring valley. These are complemented by more recent (eighteenth and nineteenth century) rectangular enclosures. The situation in our valley is a little more complicated. The fields are less regular both in outline and in size, and the deep cloughs that rake the valley sides appear to distort the line of the walls. I wonder whether there is a general principle: the size of each field may be determined as that which can be made using a constant amount of time and effort, for example? So the fields on gentle slopes are large and uniform, whilst those on the edge of the high moorlands are small and ragged. It is as though a hyperbolic metric were operating on these regions, creating smaller and more complicated fields close to the moorland edge like one of Escher's tilings of the Poincaré disc.

An obvious first step towards understanding this in more detail would be to look at distributions of the areas of fields and lengths of straight walling in the different valleys. The problem would be doubly difficult if the walls were themselves shifting in time, creating and destroying new fields. With what mathematics could this process be described, and how would one determine which movements were more complicated?

This is almost precisely the sort of question posed by Konstantin Mischaikow of Georgia Tech and co-workers in the context of patterns generated by (for example) mixing fluids or the concentrations of reacting chemicals. Patterns generated by a model of excitable media in two spatial dimensions are shown in Figure 1. The first (upper left) is a stable spiral of the sort found in many experiments in biology, chemistry and physics [8]. The other three patterns are more complicated; they are snapshots at different times of the same solution as it evolves in a simulation of the same model as the stable spiral pattern, but at a different parameter value. Mischaikow's group consider the time evolution of these patterns as creating a three dimensional geometric object (the

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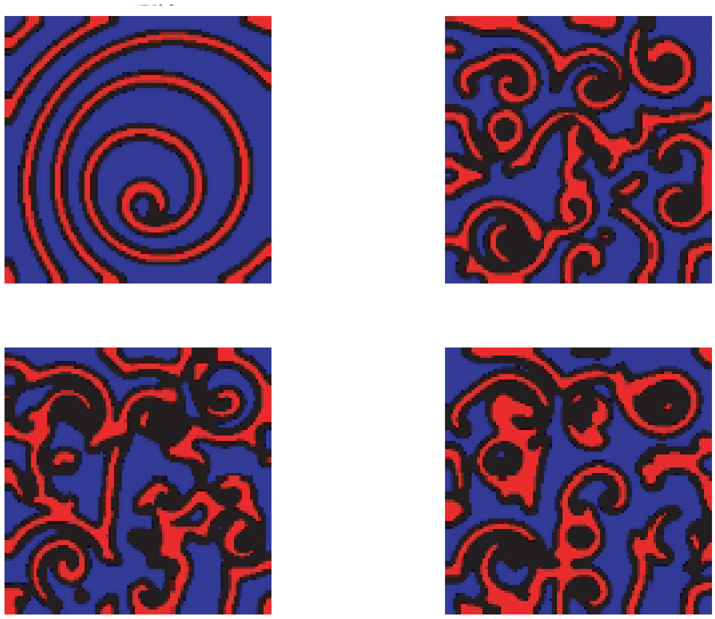

FIGURE 1. Evolution of patterns in a model of excitable media. The figures are obtained from simulations of (1) with $\epsilon = 14$ in the top left hand figure, and $\epsilon = 12$ in the remaining cases. See the text for further details (adapted from [3] with permission).

pattern is in two dimensions and time is the third dimension; three dimensional flows would generate four dimensional geometry) which can then be treated using the language of algebraic topology. Part of their achievement is to make this language easily accessible via computer programmes, and this is described in their forthcoming book [5].

Before describing their approach in more detail the idea of what we mean by *pattern* needs to be defined more precisely. In standard English usage a pattern is 'a decorative design: a particular disposition of forms and colours: a design or figure repeated indefinitely' [2]. The first of these is not helpful to a computer, the third is too prescriptive and the second, which is closest to the sense in which the word is used by physicists, is too vague. Mischaikow *et al* standardize what they mean by pattern using thresholds. The model used in [3] is a modified

FitzHugh-Nagumo equation

$$\begin{array}{ll} u_t & = \Delta u + \epsilon u(1-u)\left(u - \frac{v+\gamma}{\alpha}\right) \\ v_t & = u^3 - v \end{array} \qquad (1)$$

where the subscript t denotes partial derivatives with respect to time and Δ is the Laplacian operator. This is solved on the square $\Omega = [0,80] \times [0,80]$ with Neumann boundary conditions, $\alpha = 0.75$ and $\gamma = 0.06$. The parameter ϵ can take any specified value, although the interesting threshold behaviour considered below has $11 \leq \epsilon \leq 13$ (note that ϵ here corresponds to ϵ^{-1} in [3]). The greyscales of Figure 1 represent different levels of activity as expressed by the magnitude of the variable u. Mischaikow *et al* take $u \geq 0.9$ as an indication of high activity, and colour every point with high activity black, and every other point white. In this way they obtain a clear contrast and create an object (the black locus of high activity) which expresses what is meant by a pattern in stark terms. If time is considered as a third dimension as described above then this produces a three dimensional locus of high activity in space and time, and it is this object that they focus upon. Choosing different thresholds would of course create different geometric objects – the assumption is that any sensible choice of threshold will give roughly similar results.

The pattern is now the region of high activity, stored in the computer as a set of voxels (the three dimensional analogue of pixels: three dimensional cubes reflecting the spatial and temporal discretization used to simulate the model) coloured black. As such it is static and although this might have interesting properties in itself, the aim is to characterize dynamical properties of the patterns. To create dynamically interesting objects Mischaikow *et al* use a trick familiar to anyone who has seen the times series analysis of dynamical systems developed in the 1980s: time delay [1]. After rescaling the time step and spatial discretization each voxel is represented by three integers (the first two are spatial, and the third is in time) so voxel V_{ijk} denotes the voxel at discrete position (i,j) at discrete time k and the pattern is the black object created by taking the union of those voxels which are coloured black in the time for which data is available. Let these black voxels be the set V_{ijk} with $(i,j,k) \in B$ (so B is a subset of the integer lattice). We now take time slices: let

$$T_{n,b} = \{V_{ijk} \mid (i,j,k) \in B,\ n \leq k \leq n+b\} \qquad (2)$$

where b is chosen small compared to the total time of the simulation but sufficiently large so that $T_{n,b}$ can have interesting topology (again, it is assumed that this can be done in a sensible way). For fixed b the time evolution of the pattern is given by the map from $T_{n,b}$ to $T_{n+1,b}$. The question that remains is to determine which features of $T_{n,b}$ can be used to characterize the dynamics. This is where algebraic topology comes in.

For the example given here the only bits of information we need are the Betti numbers of the sets $T_{n,b}$, although in more general situations the full homology of the pattern might be important. The Betti numbers of a set S, $\beta_i(S)$, $i = 0, 1, 2, \ldots$ describe different topological features of the set: $\beta_0(S)$ is the number of connected components, $\beta_2(S)$ the number of enclosed cavities and $\beta_1(S)$ is (essentially) the number of tunnels. In this three dimensional example all the other Betti numbers are zero. The Betti numbers sound intuitively appealing, but computing them for complicated examples is not easy. The standard definitions of homology generally start from ideas of triangulation and simplices, which are the bodies that can be built from line segments, triangles, tetrahedra, ..., and the formal definition of the Betti numbers involves the number of distinct non-boundary cycles of different dimensions. These methods do not seem to translate easily to computer manipulations, and so Mischaikow *et al* have developed computer programmes using lines, squares (pixels), cubes (voxels), ... instead; the so-called *cubical homology* [4, 5]. This has made it possible for them to compute homological quantities such as the Betti numbers for many complicated patterns. For the pattern evolution of Figure 1 they find that β_2 is zero and β_0 small and piecewise constant for the $T_{n,b}$ they have considered, but that the behaviour of β_1 is much more interesting.

Fixing $b = 1000$ they computed β_1 for the first 10000 of the sets $T_{10m,b}$ – advancing time by ten units each step so as to see a reasonably fast evolution [3]. The results for $\epsilon = 11.5$ and $\epsilon = 12$ look like a standard chaotic time series with β_1 in the range 150 to 400 and they are even able to measure quantities like Liapounov exponents which describe how chaotic the system is. Moreover, as the parameter ϵ of the system (1) is decreased through a critical value, the mean value of β_1, $\bar{\beta}_1$ increases from zero (or at least, from a value that appears to be zero on the scale of the diagram in [3]) at a rate that is certainly faster than linear, and looks like a phase transition (i.e. the increase appears to be a power law). This is not hard to check. A very crude set of data points can be obtained from [3] by simply measuring the height of points in their diagram with a ruler – this gives thirteen data points with positive mean values $\bar{\beta}_1(\epsilon)$ at different values of ϵ. Two of these data points are a little out of line with the others and were discarded (perhaps the asymptotic mean had not been reached with the available data) leaving eleven points $(\epsilon_k, \bar{\beta}_1(\epsilon_k))$, $k = 1, \ldots, 11$. The threshold at which the mean starts to increase from zero, ϵ_c, lies between 12.5 and 12.625 and a hypothesis for growth of the form

$$\bar{\beta}_1(\epsilon) = \kappa(\epsilon_c - \epsilon)^\delta \tag{3}$$

leads to a linear log-log plot:

$$\log \bar{\beta}_1(\epsilon) = \delta \log(\epsilon_c - \epsilon) + \log \kappa \tag{4}$$

Linear regression (using the `reglin` command in Scilab) on the mean square differences from a straight line with different values of ϵ_c shows that the best fit is for $\epsilon_c \approx 12.509$ and at this value $\delta \approx 0.166$, $\log \kappa \approx 5.67$ and the standard deviation from a straight line is 0.0058 (compared to a standard deviation of 0.249 for the values of $\log \bar{\beta}_1$ at the data points). This is such a good fit that it is not really worth showing the comparison of the straight line with the data points. It suggests that in this model the mean value $\bar{\beta}_1$ of the first Betti number scales roughly as $(\epsilon_c - \epsilon)^{\frac{1}{6}}$ close to the onset of complexity as measured by positive mean β_1. It would be interesting to know whether this is born out by more careful calculations, and if so, whether this is generally the case.

Mischaikow *et al* are not the first to use homology in the analysis of dynamics. Muldoon *et al* [6] (for example) use homology to describe attractors from experimental data, although their attractors are static. What Mischaikow *et al* have produced is a wonderfully efficient computer programme using a sophisticated analysis of cubical homology, and this is making it possible to describe the dynamics and complexity of patterns via Betti numbers. The scaling law for the mean first Betti number that I derived above suggests a strong connection between homology and dynamics (although this scaling is highly conjectural; I relied on a ruler to obtain the data points from [3] and assumed that $\bar{\beta}_1 = 0$ for $\epsilon > \epsilon_c$). Whether the scaling law is a fluke or a symptom of something deeper remains to be seen. What is clear is that Mischaikow *et al* are in the process of creating a powerful new tool for the analysis of patterns in dynamics.

Fields are not the only feature to pattern the valley, there are many reservoirs (from service reservoirs to small mill pools). It would be nice to feel that the pattern techniques could take these into account as being different, but still part of the same process. Do multi-phase or multi-fluid patterns need to be seen as multi-dimensional patterns (taking the cross products of the different threshold models designed to capture the relevant phase), should each be treated separately, or do they all contain essentially the same dynamical information because of their interactions? There are still several aspects to this new approach that are not understood (how to choose b systematically, how to choose the thresholds, ...) but the real question remains: does homology describe patterns sufficiently to be a really good characterization when considering dynamics?

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