VIEW FROM THE PENNINES: STICKY WAVES

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There is a lot of mud on the moors. Thankfully this does not mean that my boots need cleaning after every walk. If the weather has been dry or there is a light drizzle, then my boots are covered by a thin film of peaty dust which I can live with. If it is very wet then the water cascades down the paths and my boots return sodden but with only the odd stem of grass or heather stuck to them. All they need is time to dry. It is the intermediate case that is troublesome. Steady, undramatic rain penetrates the surface of the peat paths which rapidly revert to boggy quagmires. Mud adheres to mud, and with each step my boots get heavier until a critical mass is reached and gravitational effects remove a large chunk of mud. I stride on and the process begins anew. Mathematicians have recently been thinking about another sticky problem, but being mathematicians their description is cleaner and drier. Like the problem with my boots, though, the behaviour of their sticky liquid depends on the relative strengths of gravity, viscosity and surface tension.

Kelmanson and colleagues at Leeds and Cambridge have been working on a classic problem in fluid mechanics. They consider the motion of a sticky liquid coating the outside of a rotating cylinder when the axis of the cylinder is horizontal and gravity acts vertically downwards. This should be familiar to anyone who has used a honey spoon – to keep the honey on the spoon it is necessary to rotate the spoon, creating complicated waves on the surface of the honey as a drip begins to form and is then swept round the spoon by the rotation. The analysis of [2, 3] describes these waves, and in particular the different instabilities that form on different time scales.

The early advances in this problem are due to Moffatt [4] (also at Cambridge) and Pukhnachev [6]. Numerical simulations found surface waves, but it was hard to tell whether these were damped or undamped on the time scales that could reasonably be investigate [1, 5]. When computers cannot give clear answers a little more thought is called for, and this is what Hinch and Kelmanson have provided [2, 3].

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If (r, θ) are polar coordinates in the plane perpendicular to the axis of cylinder, then the goal is to describe the height, $H(\theta, \tau)$ of the liquid surface above the surface of the cylinder at time τ. This will depend upon a number of parameters: the radius a and constant angular velocity ω of the cylinder, the acceleration due to gravity g, and the surface tension σ, dynamic viscosity μ, density ρ and average thickness $\bar{h}$ of the fluid. The assumption at the heart of the analysis of Kelmanson *et al* is that the fluid layer is thin compared to the radius of the cylinder, so $\epsilon = \bar{h}/a$ is a small parameter. The equations are simplified by defining a rescaled height and time: $t = \frac{1}{\omega}\tau$ and $h(\theta, t) = \epsilon H(\theta, \tau)$. In these new variables the equation governing h, after adopting some standard approximations and assuming that the layer is initially uniform, is

$$h_t + h_\theta = \left(\gamma h^3 \cos\theta - \alpha h^3 (h_{\theta\theta} + h)_\theta\right)_\theta \quad h(\theta, 0) = 1 \tag{1}$$

where subscripts denote partial differentiation with respect to the relevant variable and

$$\gamma = \frac{\rho g \bar{h}^2}{3a\mu\omega} \qquad \alpha = \frac{\sigma \bar{h}^3}{3a^4\mu\omega}.$$

Note that γ is of order ϵ^2 and α is of order ϵ^3 and so, assuming that the other factors of γ and α are order one then

$$\gamma^2 \ll \alpha \ll \gamma \ll 1$$

which is the hierarchy adopted by [2]. Their approach to (1) in this case uses perturbation theory, and in particular, the method of multiple scales. This method separates out different time scales on which different mechanisms act. The natural scaling takes a bit of guessing, but it turns out that α is not very important (and could even be order one) and that the second natural timescale is $T = \gamma^2 t$. Substituting a solution of the form $h \sim 1 + \gamma\psi_1(\theta, t, T) + \gamma^2\psi_2(\theta, t, T) + \dots$ into (1) and identifying terms of different orders of γ gives a sequence of coupled equations. The first, order γ, equation for ψ_1 can be solved to give

$$\psi_1(\theta, t, T) = \cos\theta + A(T)\cos(\theta - t) + B(T)\sin(\theta - t) + O(\mathrm{e}^{-12\alpha t}) \tag{2}$$

where A and B are functions of the slow time T which must be determined at higher order. The behaviour of these functions determine the slow drift and decay of the waves. When the non-homogeneous terms are included the second order problem can be solved, and, rather unfortunately, there is nothing to determine the slow drift of the rotating solution, ψ_1. Most of us would give up at this stage – the third order problem has a right hand side containing 145 terms when calculated using a symbolic manipulator. This is where persistence and confidence come in to play: Hinch and Kelmanson are able to identify the terms (called the secular terms) which would give a particular integral which destroys the assumption that terms in the series solutions remain in order of importance. Setting these secular terms to zero yields a pair of coupled equations for the evolution of A and B that can be solved:

$$A(T) = -\mathrm{e}^{aT}\cos(bT) \qquad B(T) = \mathrm{e}^{aT}\sin(bT) \tag{3}$$

where

$$a = -\frac{81\alpha}{144\alpha^2+1}, \quad b = \frac{3(72\alpha^2+5)}{2(144\alpha^2+1)} \tag{4}$$

Recalling that $T = \gamma^2 t$ the lowest order solution can now be interpreted. On a timescale of order α^{-1} the last terms of (2) decay leaving an eccentric cylindrical shape which drifts in phase on the time scale γ^{-2} from the bT terms of (3) and decays exponentially on the time scale $\alpha^{-1}\gamma^{-2}$ (from the aT terms of (3)).

Hinch and Kelmanson do no stop there. Using a variety of perturbation methods they go on to consider other hierarchies of scaling, and in particular very small surface tension [3]. Again they produce a *tour de force* of perturbation theory which uses symbolic manipulators and mathematical insight to describe the folding of the liquid surface. The very slow decay to steady states means that this analysis is essential if we are to have confidence in numerical simulations.

Nothing in Hinch and Kelmanson's approach to this problem is conceptually new; multiple scales is a standard tool in perturbation theory. What makes this work particularly noteworthy is the ability of the authors to apply the technique in circumstances that would have discouraged most other researchers. The absence of a secularity condition at order γ^2 means that a morass of forcing terms needs to be waded through at third order. This would be a fruitless exercise without the use of symbolic manipulators, and even then it is a hard task. To have shaken off the mud to emerge with a clear, clean description of the mechanisms involved is a real achievement. I only wish I could do the same with my boots.

References

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