VIEW FROM THE PENNINES: ANALYSIS ON TIME SCALES

PAUL GLENDINNING

To be truly accepted in the village you need to be able to trace back your local roots for several generations. Everyone else is referred to as a 'comer-in'. The recent redevelopment of an abandoned mill has created an influx of comers-in, with some good and some bad effects. The moors and valley also have their share of botanical comers-in: rhodedendrons have occupied some moorland areas, and Himalayan balsam spreads speedily along the canal paths. Both these plants originate in the mountain ranges of China and the high moorland hereabout suits them. They are beautiful plants in their different ways but the fact that they are not local causes some concern.

For the past week I have been wrestling with what I am beginning to regard as a mathematical comer-in: analysis on time scales. Recently it has been the subject of a cover story in *New Scientist* [9], special issues of two international journals (*Journal of Computational and Applied Mathematics* [1] and *Dynamic Systems and Applications*, see [6]) and two books [3, 4]. Our library does not have the more recent book [4] and the introductory book [3] has been permanently on loan, but the other sources have in turn intrigued and infuriated me.

The basic idea seems reasonable and interesting. There are many similarities betwen results of differential calculus and discrete calculus. For example, the continuous time differential equation $\frac{dx}{dt} = ax$ has solution $e^{at}x_0$, whilst the analogous discrete time equation $x(t+1) - x(t) = ax(t)$ with $t \in \mathbb{Z}$ has solution $(1+a)^t x_0$. Provided $1+a > 0$ this can be written as $e^{bt}x_0$ where $b = \ln(1+a)$, so is there a formalism (or theory) in which the different cases can be treated uniformly? In the late 1980s, Stefan Hilger, then a graduate student at Augsburg in Germany, developed just such a formalism [7]. Using Hilger's ideas, applied mathematicians have started to develop and apply the theory. Many of the applications to date are from the biological sciences, since the techniques seem particularly well adapted to periods of latency or dormancy in systems, so that the dynamics is a mixture of continuous time development and discrete time development [9].

Date: September 9, 2003.

Before describing time scale analysis it is worth looking at the type of problem that it is designed to solve (the example is motivated by a diagram in [9]). Suppose that in the six months from April 1 of each year, the number N of insects in an isolated population grows exponentially with time:

$$\frac{dN}{dt} = aN, \qquad a > 0 \tag{1}$$

where the population is sufficiently large that N can be taken to be a continuous variable. In the following winter months the population is dormant, and the population at the beginning of the next April is some proportion, b say, of the population at the beginning of the winter. Here we are faced with a problem in two parts: a continuous time phase (summer) and a discrete time phase (winter). This problem is actually easy to solve using conventional techniques. Take the unit of time to be six months, with the origin on April 1 of year zero. If the population is initially $N(0)$, then integrating (1), the population at the end of the summer is $N(1) = \mathrm{e}^a N(0)$. At the beginning of the next April the population is therefore $N(2) = b\mathrm{e}^a N(0)$. More generally, if the population on April 1 of year k is $N(2k)$ then

$$N(2(k+1)) = b\mathrm{e}^a N(2k) \tag{2}$$

and since this linear difference equation is easy to solve, the population at any time in the summer months, $t \in [2k, 2k+1]$, can be calculated. Moreover, in this oversimplified model, if $b\mathrm{e}^a < 1$ the population will always die out, whilst if $b\mathrm{e}^a > 1$ it will increase from year to year.

The point about this example is that it is the type of example frequently appealed to by expositors of time scale analysis (e.g. [2]), but which is actually completely solvable using standard methods. My frustration is that this is not an isolated easy example – I have not come across any applications which could not be solved using standard methods.

So what is time scale analysis? A time scale is simply a closed subset $\mathbb{T}$ of the reals: the times at which the dependent variables will take values. Now for any $t \in \mathbb{T}$ define the forward jump function, σ, and the graininess, μ, by

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}, \qquad \mu(t) = \sigma(t) - t \tag{3}$$

with $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$. Roughly speaking, $\sigma(t)$ gives the 'next' point in $\mathbb{T}$ and μ describes how much further on it is. If $\sigma(t) = t$ then points in $\mathbb{T}$ accumulate on t from above and such a point is said to be *right dense*, otherwise it is *right scattered*. Left dense and left scattered are defined similarly. Differentiation is defined by the *delta derivative*, f^Δ,

$$f^\Delta(t) = \begin{cases} \lim_{s\to t,\ s\in\mathbb{T}} \frac{f(s)-f(t)}{s-t} & \text{if } \mu(t)=0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t)>0 \end{cases} \tag{4}$$

provided the limit exists. The delta derivative has many properties similar to standard differentiation. For example, if f and g are delta differentiable then the product rule is

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) \tag{5}$$

Antiderivatives (integrals) can also be defined: if $f^\Delta = g$ on $\mathbb{T}$ then define

$$\int_s^t g(u)\Delta u = f(t) - f(s) \tag{6}$$

A function is *rd-continuous* if it is continuous at every right dense point and if the left sided limit exists at every left dense point. One of Hilger's main results [7] is that every rd-continuous function has an antiderivative.

With this new calculus we can try to solve delta differential equations such as the linear initial value problem

$$y^\Delta = p(t)y, \qquad y(t_0) = 1 \tag{7}$$

with $t_0 \in \mathbb{T}$. If$\mathbb{T} = \mathbb{R}$ then (7) is the linear ordinary differential equation $\frac{dy}{dt} = p(t)y$, whilst if $\mathbb{T} = \mathbb{Z}$ then $y^\Delta = t(t+1) - y(t)$ so (7) is the difference equation $y(t+1) = (1+p(t))y(t)$. Hilger showed that (7) has a unique solution provided p is rd-continuous and $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$ (the latter condition is similar to the condition that $1 + a > 0$ at the beginning of this article) and hence that in this case a generalized exponential function, $\mathrm{e}_p(t, t_0)$ can be defined on $\mathbb{T}$. This in turn makes it possible to define generalized hyperbolic and trigonometric functions, and the same ideas can be applied to linear second order delta differential equations. In fact, there are neat equivalent results in time scale analysis for most of the classical theory of second order equations (Wronskians, integrating factors, self-adjoint equations,...). There are numerous other extensions and generalizations in the literature [2, 5, 8] – for example a natural alternative to (7) is obtained by replacing $y(t)$ on the right hand side of the equation by $y(\sigma(t))$, and solutions of this equation are related to the generalized exponential above [2].

The insect population model described above can be seen as a linear system of the form (7) on the time scale $\mathbb{T} = \cup_{k\in\mathbb{N}}[2k, 2k+1]$, with $p(2k+1) = b - 1$ and $p(t) = a$ otherwise, so p is rd-continuous. The theory of Hilger therefore allows us to state that there is a unique solution, but I cannot see how the generalized exponential form of the solution is any more useful than the standard approach above. All the papers on time scale analysis I have seen look at linear, or nearly linear, dynamics on time scales, and examples are based on relatively simple choices of $\mathbb{T}$ for which standard theory would work just as easily. To be useful the theory must be able to treat a case that is not obviously covered by classical theory, or do something much more simply than before – a unification of formalism isn't enough.

What might such a justification look like? One interesting possibility would be to do analysis on fractals. For example, what is the solution of the linear equation (7) with $p = 1$ if $\mathbb{T}$ is the middle third Cantor set, $\mathcal{C}$? This question is worth asking for two reasons: $\mathcal{C}$ is a non-trivial example of a time scale, and it also has potential applications – if the independent variable ('time') is thought of as a spatial variable then a Cantor set might model a medium with impurities or other fine structure. Although Bohner and Peterson [3] introduce the Cantor set as an example of a time scale, they do not solve dynamic equations on Cantor sets amongst their examples. Gard and Hoffacker [6] do consider Cantor sets, but they are interested in asymptotic behaviour as $t \to \infty$ in the case of $\mathbb{T} = \mathbb{Z} + \mathcal{C}$ (i.e. the union of $\mathcal{C}$ and its integer translates). However, assuming that solutions on approximations to $\mathcal{C}$ converge to the solution on $\mathcal{C}$ it is not too hard to solve the equation.

The middle thirds Cantor set is the closed subset of $[0, 1]$ obtained inductively by removing first the interval $(\frac{1}{3}, \frac{2}{3})$, then the middle thirds of the remaining two intervals, i.e. remove $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so on. So at the n^{th} stage 2^{n-1} intervals of length 3^{-n} are removed. The set that remains in the limit of this process is closed and uncountable but contains no intervals and no isolated points. It can be written explicitly using the representation of numbers in base three as those points with a ternary expansion that contains no ones, i.e.

$$\mathcal{C} = \{t \in [0, 1] \mid t = \sum_{1}^{\infty} a_n 3^{-n},\ a_k \in \{0, 2\}\} \tag{8}$$

So, what is the solution to

$$y^{\Delta} = y, \qquad y(0) = 1 \tag{9}$$

on the time scale $\mathcal{C}$? First, Hilger's theory shows that the solution *does* exist and is unique, which is by no means obvious. Let us assume that the solution is the limit of the solution of the same equation on the time scales $\mathcal{C}_m$ obtained in the limiting process of the construction of $\mathcal{C}$, so $\mathcal{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and so on, and for simplicity concentrate on the solution at $t = 1$. Start with $\mathbb{T} = \mathcal{C}_1$. Solving the differential equation

$\frac{dy}{dx} = y$ with $y(0) = 1$ on the interval $[0, \frac{1}{3}]$ gives $y(\frac{1}{3}) = \mathrm{e}^{\frac{1}{3}}$. Across the gap using (9) and the second of equations (4),

$$3\left(y(\tfrac{2}{3}) - y(\tfrac{1}{3})\right) = y(\tfrac{1}{3})$$

so $y(\frac{2}{3}) = (1 + \frac{1}{3})y(\frac{1}{3})$. Finally, solving the linear differential equation on $[\frac{2}{3}, 1]$ gives $y(1) = \mathrm{e}^{\frac{1}{3}}y(\frac{2}{3})$. Putting the pieces together we see that the solution at $t = 1$ on $\mathcal{C}_1$ is $Y_1 = (1 + \frac{1}{3})\mathrm{e}^{\frac{2}{3}}$. Repeating this argument on $\mathcal{C}_m$ shows that the solution at $t = 1$ on $\mathcal{C}_m$ is Y_m where

$$Y_m = \mathrm{e}^{2^m/3^m} \prod_{k=1}^{m} (1 + \tfrac{1}{3^k})^{2^{k-1}} \tag{10}$$

So assuming that the solution on $\mathcal{C}$ is the limit of the solutions on $\mathcal{C}_m$ we see that the value of the solution at $t = 1$ is given by the infinite product

$$\prod_{k=1}^{\infty} (1 + \tfrac{1}{3^k})^{2^{k-1}} < \mathrm{e} \tag{11}$$

where the inequality is obtained by taking logs and using $\ln(1+x) < x$ for $0 < x < 1$. So the solution to (7) on $\mathcal{C}$ at $t = 1$ is less then the corresponding solution to the standard differential equation with $\mathbb{T} = \mathbb{R}$.

More generally, a similar argument suggests that the general solution of (9) for $t \in \mathcal{C}$ is

$$y(t) = \prod_{k=1}^{\infty} \left(1 + \tfrac{1}{3^k}\right)^{\phi(t,k)} \tag{12}$$

where $\phi(t, k)$ is the number of middle third gaps of length 3^{-k} to the left of t. Note that $\phi(t, k)$ is less than or equal to 2^{k-1}, and it can be defined inductively. To do this first note that $\phi(0, k) = 0$ for all k and

$$\phi(\tfrac{1}{3^n}, k) = \begin{cases} 0 & \text{if } k \le n \\ 2^{k-n-1} & \text{if } k > n \end{cases} \tag{13}$$

since the subset of $\mathcal{C}$ in $[0, \frac{1}{3^n}]$ maps to the whole of $\mathcal{C}$ when multiplied by 3^n. If t is non-zero, $t \neq \frac{1}{3^n}$ for some n and $t \in \mathcal{C}$ then there is a smallest $r \ge 1$ such that $t \ge \frac{2}{3^r}$. In this case by considering the geometric construction of $\mathcal{C}$ we find that

$$\phi(t, k) = \begin{cases} 0 & \text{if } k < r \\ 1 + \phi(t - \frac{2}{3^r}, k) & \text{if } k = r \\ 2^{k-n-1} + \phi(t - \frac{2}{3^r}, k) & \text{if } k > r \end{cases} \tag{14}$$

In principle, these equations make it possible to evaluate the solution (12) for any $t \in \mathcal{C}$. Although this shows that dynamic equations on Cantor sets can be quite tractable, only conventional methods have been used to obtain the solution. It is possible that a more elegant method of solution exists using the formalism of [2], but I have not been able to find it.

Comers-in inevitably create tensions. The influx of families to our village has created congestion problems, but it has also brought economic regeneration, and the primary school no longer finds recruitment difficult. Time scale analysis evokes the same mixed feelings. I cannot decide whether the benefits (unified approach, theoretical framework) outweigh the disadvantages (formalism for its own sake, nothing that cannot be done on an *ad hoc* basis using standard methods). Perhaps it is just too early to decide – there certainly needs to be more work on nonlinearity and novel applications – but at the very least it will prevent researchers from duplicating each others efforts when considering mixed systems for which the theory is applicable.

Acknowledgements: I am grateful to Peter Kloeden (Frankfurt) for helping me with some references, and James Montaldi (UMIST) for discussions about (11).

References

[1] R.P Agarwal, M. Bohner & D. O'Regan (Eds.) (2002) Journal of Computational and Applied Mathematics **141**, Special Issue on 'Dynamic Equations on Time Scales'.

[2] R.P Agarwal, M. Bohner, D. O'Regan & A. Peterson (2002) Dynamic equations of time scales: a survey, in [1], 1-26.

[3] M. Bohner & A. Peterson (2001) *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston.

[4] M. Bohner & A. Peterson (Eds.) (2003) *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston.

[5] P.W. Eloe (2002) The method of quasilinearization and dynamic equations on compact measure chains, in [1], 159-167.

[6] T. Gard & J. Hoffacker (2003) Asymptotic behaviour of natural growth on time scales, *Dynamic Systems and Applications* **12** 131-148.

[7] S. Hilger (1990) Analysis on measure chains – a unified approach to continuous and discrete calculus, *Results Math.* **18** 18-56.

[8] V. Lakshmikantham & A.S. Vatsala (2002) Hybrid systems on time scales, in [1], 227-235.

[9] V. Spedding (2003) Taming nature's numbers, *New Scientist* **179** (2404) 28-31, 19 July 2003.

Department of Mathematics, UMIST, P.O. Box 88, Manchester M60 1QD, United Kingdom. E-mail: P.A.Glendinning@umist.ac.uk