VIEW FROM THE PENNINES: NON-TRIVIAL PURSUITS

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Tennyson's 'Nature, red in tooth and claw' [10] paints a rather more vivid picture of the struggle for life than I encounter on the moors. Moorland deaths seem bloodless; the work of the weather, land and time rather than the result of battles between rival animals. The only regular wars I witness are the squabbles of finches and tits over the peanut dispenser, and the mobbing of crows and kestrels by smaller birds.

Perhaps I should pay more attention to insects. Tennyson (almost) wrote [11] of

> Dragon[flie]s of the prime,
> That tare each other in their slime,

and recent research reveals the dragonfly to be a rather more devious creature than I had imagined. Mizutani, Chahl and Srinivasen [7] have observed dragonflies moving during territorial disputes. Using reconstructions based on stereo camera images they have been able to show that these creatures operate a sophisticated form of stealth strategy that had been proposed by Srinivasen and Davey [9] in 1995. The aim of the stategy is for the aggressor to camouflage its approach by moving so that the aggressed dragonfly (the target) believes it to be stationary.

This is not as hard as it sounds. Even when moving, most animals have a good sense of the direction to a given fixed object at any time, and expect to see it on that line. If the aggressor moves so that at each moment it is on the line between the target and a given fixed point, which could be its initial position, then its relative motion in the eyes of the target is the same as that of the stationary reference point. The only way that the target can know that it is not stationary is to notice the change in size of the aggressor as it approaches. Mizutani, Chahl and Srinivasen [7] extrapolate the lines between the aggressor and the target at several different times and show that to a good approximation these all meet at a point, the fixed reference point or initial condition of the aggressor.

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Srinivasen and Davey [9] propose a number of different algorithms to model the way in which the aggressor can achieve this. Their favourite, 'algorithm 2', assumes that the aggressor moves so that it is '(i) always viewing the shadowee [target] frontally; and (ii) always pointing radially away from its starting point' [9]. This does not seem to have got us very far, since (i) and (ii) simply restate the assumption that the aggressor moves so as to remain on the line between the target and the starting point. Srinivasen and Davey turn this into a discrete time model as follows. Suppose that the aggressor is a distance ρ from the reference point, and that the angle between the lines from the target and the reference point at the start of the time step and at the end of the time step is $\delta\theta$. Then the aggressor should change the orientation of its body by $\delta\theta$ (a *yaw*) and move laterally through $\delta\lambda = \rho\delta\theta$ to place it on the new line from the reference point to the new position of the target. They then allow motion directly towards the victim before the next time step begins. This may sound a little clumsy, and is only correct to first order, but it gives perfectly reasonable results.

The algorithms of Srinivasen and Davey have been criticised by Anderson and McOwan [1] on the grounds that it is hard to imagine how the two pieces of data needed, ρ and $\delta\theta$, can be calculated by the dragonflies (Srinivasen and Davey have partial answers to this) and also that the errors over each time step would induce a lag which the victim would notice. They propose a biologically motivated control system and test it (successfully) on recorded hoverfly motion and numerically generated trajectories. A different tack would be to describe the ideal paths generated by this strategy. This is the direction taken below.

The active motion camouflage strategy described above can be seen as an extension of the classical theory of pursuit curves that has been studied for centuries. Davis [5] suggests that da Vinci knew of the problem, and the first serious mathematical formulation was in 1732. The standard pursuit strategy is simple: head directly towards the target as fast as you can. For the simplest case the aggressor (or predator) and the target (or prey) are assumed to have the same constant speed, which can be set to one by rescaling time. This means that if the target has position $\mathbf{z}(t)$ and the aggressor has position $\mathbf{r}(t)$ then the aggressor chooses to move on the solution to the differential equation

$$\dot{\mathbf{r}} = \frac{\mathbf{z}-\mathbf{r}}{|\mathbf{z}-\mathbf{r}|} \tag{1}$$

where the dot denotes differentiation with respect to time and $\mathbf{z}$ is given. The test case in which the target moves along a straight line in the plane with $\mathbf{z} = (0, t)$ can be solved explicitly in terms of special functions [12]. If $\mathbf{r} = (x, y)$ then the curve with initial condition (x_0, y_0) can also be expressed in the form $y = y(x)$ as

$$y = \frac{1}{4}\left\{(y_0 + R)\frac{x}{x_0} + (y_0 - R)\log\left(\frac{x}{x_0}\right) + 3y_0 - R\right\} \tag{2}$$

where $R = \sqrt{x_0^2 + y_0^2}$ [12]. A solution is shown in Figure 1. The equivalent formulation for active motion camouflage begins with the assumption that the position of the aggressor, $\mathbf{r}$, lies on the line between the target at $\mathbf{z}$ and some fixed reference point, $\mathbf{r}_0$, which we will take to be the initial position of the aggressor. In vector notation this means that

$$\mathbf{r} = \mathbf{r}_0 + u(\mathbf{z} - \mathbf{r}_0) \tag{3}$$

where $u(t)$ is a function of time which determines the position $\mathbf{r}$, and $u(0) = 0$ so that $\mathbf{r}(0) = \mathbf{r}_0$. The equation which determines $u(t)$ can be obtained by differentiating (3) to obtain $\dot{\mathbf{r}} = \dot{u}(\mathbf{z} - \mathbf{r}_0) + u\dot{\mathbf{z}}$ and then adding the constant speed constraint $|\dot{\mathbf{r}}|^2 = |\dot{\mathbf{z}}|^2 = 1$, which implies that

$$|\mathbf{z} - \mathbf{r}_0|^2\dot{u}^2 + 2[\dot{\mathbf{z}}.(\mathbf{z} - \mathbf{r}_0)]u\dot{u} + u^2 = 1 \tag{4}$$

This is a quadratic equation for $\dot{u}$ which, using the standard quadratic formula with the positive square root, gives a differential equation for u, solutions of which determine the active camouflage pursuit curve. The choice of the negative square root corresponds to a motion camouflage retreat curve. In this case $\mathbf{r}_0$ should be interpreted as the position of an obstacle which the prey, now playing the active role previously assigned to the aggressor, keeps between itself and a predator moving on a search path $\mathbf{z}$.

The equations obtained in this way are simple to integrate numerically, although they no longer attempt to explain how the creatures arrive at the ideal path. Capture corresponds to a solution with $u(\tau) = 1$ for some τ. In the test case with $\mathbf{z} = (0, t)$ the differential equation can be written in Cartesian coordinates with $\mathbf{r} = (x, y)$ and $\mathbf{r}_0 = (x_0, y_0)$ as

$$\dot{u} = \frac{-(t - y_0)u + \sqrt{x_0^2 + (t - y_0)^2 - x_0^2u^2}}{x_0^2 + (t - y_0)^2} \tag{5}$$

with $u(0) = 0$. Unfortunately this equation is not known to be integrable. After some changes of variable the solution can be related to solutions of an Abel equation of the first kind [8], $U' = (1+U^2)(A(s)+B(s)u)$, where s is a timelike variable, the prime denotes differentiation with respect to s, and A and B are rational functions of s. Equations of this sort have a long history – see [4] for a recent review – but in general solutions are not known in terms of standard special functions.

Of course, this does not make it any harder to compute solutions numerically, and an example is shown in Figure 1 alongside a solution of the standard pursuit problem (1) for comparison. Figure 2 shows a motion camouflage path and the corresponding pursuit path when the target moves on a circle.

The ideal active motion camouflage curves could be used to determine the efficiency of a creature's control system by making it possible to compare the ideal motion with the observed motion. There are also obvious questions about conditions to ensure capture and the effect of allowing the protagonists to move at different speeds. Analogies and comparisons with other strategies come to mind. For example, Körner [6] gives a wonderful description of Tizard's solution to interception paths for radar-guided fighters searching for bombers during the Second World War which has much in common with Srinivasen and Davey's geometric algorithm. This is a modification of the standard pursuit problem which uses long time steps but guarantees interception by a faster moving plane. A criterion used in sailing to detect boats on a collision course provides another comparison. If a boat appears to be stationary with respect to some distant reference point or has the same compass bearing from your boat over a period of time then it is on a collision course with you [3]. This is equivalent to active motion camouflage with the reference point at infinity, which requires a slightly modified formulation from the finite $\mathbf{r}_0$ case given above.

Whilst walking in a field last summer I watched a hoverfly out of the corner of my eye as it moved slowly along its mazy course. I can still remember the surprise I felt when I realized how close it had got to me. Had I gone to sleep for a second, lost concentration, or was there some other explanation? I think that it is quite possible that the hoverfly was employing active motion camouflage to investigate me; it is known that humans can be tricked in this way [2]. If that is true then it must have sufficient intelligence (or pre-programming) to realize that its path should be from a reference point to my eyes rather than some other part of me. I was not moving fast, and for a stationary target the active camouflage and classic pursuit strategies are the same, so perhaps I should not read too much into this event. On the other hand, the point about the eyes remains valid, although I won't become seriously worried until wasps start attacking out of the sun.

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