# VIEW FROM THE PENNINES: ALMOST AUTOMORPHISMS

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Dry stone walls are a striking man-made feature of the West Yorkshire countryside. Some of these walls date from the Elizabethan era, and may be evidence of land clearing rather than planned enclosure, but most were constructed around the beginning of the nineteenth century (1780-1820) [6]. Many of these walls have developed a lateral instability and are now decidedly wiggly, and it is natural to speculate on the period or wavelength of these wiggles. There are two obvious lengthscales involved: the length $L$ of the wall, and the average length $\ell$ of the stones themselves. Thinking of simple sinusoidal wiggles with fixed ends would suggests a period proportional to $L$, which seems far too large, whilst $\ell$ is much too small. Further reading [6] reveals another lengthscale. Stones which hold the two outside layers of stone together, called *throughs*, are placed regularly – twenty-one per rood of seven yards – along the wall. An enclosure act of 1788 specifies that twelve of these are to be low in the wall, and nine higher up. Since the lower part of the walls do not shift much, this suggests a period of twice $\frac{7}{9}$ yards, or approximately $1\frac{1}{2}$ metres. I had hoped for a number between two and three metres to match my observations, but at least this is the right order of magnitude.

Of course, the ratios of these three lengths give two independent non-dimensional quantities which further complicate the story, and the relative weight of the capstones on the top of the wall, and the gradient of the hill on which the wall is built may also be important. A next stage might be to think about superpositions of the natural wavelengths (quasi-periodic rather than periodic functions). Perhaps geographers have the answers already.

There are two motivations for this preamble. First, it is my standard device to indicate that I am writing about things that have interested me recently, rather than my own area of expertise. Second, it shows how the answers we obtain depend on where we look. I started off by thinking about periodic functions and looking for the period of the wiggles, i.e. a single lengthscale. I ended up wondering whether this had

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been the right set of functions to consider, and that I should, perhaps, broaden my search by thinking about a larger class of functions, an indication that my initial question (what is *the* period of the wiggles?) was too naive. This is where modelling can be useful: a well-posed problem has solutions in a well-defined class of functions and knowing this class can be helpful even if the exact solution is not known. Work by Yi (Georgia Tech) and Shen (Auburn, Alabama) [7, 8, 9] provides a fascinating example of this sort of result. Their analysis centres on a class of functions I had never heard of: *almost automorphic functions*. They show that this is the natural class for solutions to some partial differential equations which arise in biology and elsewhere [3]. Although their major review articles were published five years ago, the ideas are just starting to be exploited by others. Such time lags often occur when technically demanding new ideas are introduced into a subject.

Almost automorphic fuctions are part of a hierarchy of functions which starts with periodic functions. Periodic functions are defined by a notion of recurrence after a given time: a function $f(t)$ is periodic if there exists $p > 0$ called a period of $f$ such that

$$f(t+p) = f(t)$$

for all $t \in \mathbb{R}$. More general ideas of recurrence involve looking at sequences of 'almost periods' for which the function is approximately periodic. A continuous function is almost periodic if for all $\epsilon > 0$ there is an increasing sequence $(p_k)$ of positive integers depending on $\epsilon$ (the 'almost periods') and $N > 0$ such that

$$|f(t+p_k) - f(t)| < \epsilon \text{ for all } t \in \mathbb{R}$$

and $|p_{k+1} - p_k| < N$ [2]. Periodic functions are almost periodic since $f(t+kp) = f(t)$ for all integers $k$ and so we may take $p_k = kp$ and $|p_{k+1} - p_k| = p$. The next simplest examples are quasi-periodic functions, which involve only a finite number of independent frequencies. For example

$$q(t) = \cos t + \cos \sqrt{2}t$$

is almost periodic and quasi-periodic (see below for the distinction between these two).

Almost automorphic functions are a further generalization of this idea. They were introduced by Bochner [1] in the context of differential geometry. He began by deriving an equivalent definition of almost periodicity which can be generalized in an interestng way. His definition is that a continuous function $f$ is almost periodic if for any sequence $(t'_k)$ there is a subsequence $(t_k)$ and a function $g$ such that $\{f(t+t_k)\}$

converges uniformly to $g$. The generalization to almost automorphic funtions is now completely natural: a continuous function $f$ is almost automorphic if for any sequence $(t'_k)$ there is a subsequence $(t_k)$ and a function $g$ such that

$$f(t+t_k) \to g(t) \text{ and } g(t-t_k) \to f(t)$$

As ever there are all sorts of subtleties in the uniform versus non-uniform convergence of functions! The most commonly cited example [9] of a function that is almost automorphic but not almost periodic is the function (from the reals to the unit circle)

$$h_0(t) = \frac{2 + \mathrm{e}^{it} + \mathrm{e}^{i\sqrt{2}t}}{|2 + \mathrm{e}^{it} + \mathrm{e}^{i\sqrt{2}t}|}$$

which is easier to picture as a function of the reals to the circle by taking the argument of the right hand side:

$$h(t) = \tan^{-1}\left(\frac{\sin t + \sin\sqrt{2}t}{2 + \cos t + \cos\sqrt{2}t}\right)$$

where the branch of $\tan^{-1}$ is chosen according to the different signs of the numerator and denominator. Graphs of the functions $q$ and $h$ are shown in Figure 1: clearly the distinction between the two classes of function is not immediately obvious by inspection of their graphs!

The main results of Shen and Yi describe how solutions of a range of parabolic partial differential equations with almost periodic coefficients have almost automorphic solutions. Their examples include Fisher type equations that have biological significance as well as some quasi-periodically forced ordinary differential equations which have solutions that are almost automorphic but not almost periodic. Figure 1 demonstrates that the difference between almost periodic and almost automorphic solutions is less than obvious by inspection, so how might the differences be made clearer, and how might one recognise the need for almost automorphic functions in experiments or numerical simulations? This issue does not seem to have been addressed, possibly because the ideas have not filtered through to the more applied community, so I shall indulge in a little speculation, and to avoid becoming over-technical I shall not give the fully correct statements of the harmonic results below.

Almost automorphic functions can be represented as generalized Fourier series

$$f(t) \sim \sum_{\lambda\in\Lambda} a(\lambda)\mathrm{e}^{i\lambda t}$$

so it is possible that the power spectrum of the function might provide information about differences between almost automorphic and almost periodic functions. To some extent, the complexity of a function with a generalized Fourier representation can be gauged by the size of $\Lambda$, the spectrum of $f$, which is the countable set of $\lambda$ for which $a(\lambda) \neq 0$. For example the spectrum of a periodic function is contained in the set of all integer multiples of a single frequency $\omega$. The next level of complexity is the class of quasi-periodic functions: here the spectrum is contained in the set generated by a finite set of frequencies (two frequencies in the case of $q$ above). The spectrum of almost periodic functions can be generated by a countable set of independent frequencies, and the Fourier coefficients $a(\lambda)$ are uniquely defined and are equal to $b(\lambda)$, the standard Fourier coefficient

$$b(\lambda) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(t)\mathrm{e}^{-i\lambda t}\, dt$$

Nice almost automorphic functions also have a representation as a generalized Fourier series, but the coefficients are weighted multiples of the Fourier coefficients. The spectrum is a countable set $(\lambda_k)$ and $a(\lambda_k) = r_k b(\lambda_k)$ where $\sum_k r_k$ is absolutely convergent. Since the spectrum of an almost periodic and an almost automorphic function which is not almost periodic may coincide, the power spectrum is unlikely to be of much help. However, this weighted Fourier coefficient property might provide a method for distinguishing almost periodic from almost automorphic (but not almost periodic) functions.

A second approach is illustrated in Figure 2. This figure shows the graphs of $q$ and $h$ as in Figure 1, but with the time variable taken modulo $2\pi$, which is the period associated with one of the frequencies in the spectrum of both functions. The first picture, obtained from $q$, is clearly the projection of a torus (remember that the horizontal axis is circular). The second, obtained from the almost automorphic but not almost periodic function $h$, has many of the features of a torus. However, there are two striking features which differentiate it from the torus of $q$. First, the envelope is not smooth at $t = \pi$. Secondly, the range of $h$ is also a circle, and the values of $\{h((2n+1)\pi)\}$ are dense in this circle, so the object might not be the projection of a torus, but of a torus with two points pinched together. This, of course, is not a torus at all. It has two holes rather than one. It seems likely that at least one of these two features will turn out to be typical of almost automorphic functions that are not almost periodic, but I will leave the resolution of this to the experts.

The introduction of new pure mathematical concepts into applied mathematics can be a painful process. It takes time to come to grips with the technical machinery necessary to manipulate the ideas and to develop the intuition needed to apply them successfully. Almost automorphic functions are now being applied to other forced differential equations [4, 5]. I have not reached that level of expertise, but I will continue to follow future developments with interest.



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