VIEW FROM THE PENNINES: NEWTONIAN CHOREOGRAPHIES

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Brass Band competitions are still held in our village. Local bands play marches at the Band Room and hymns outside one of the village pubs. Their performances are marked by a panel of judges. I never get to hear the marches, but this year the almost disorderly procession of bands between the two venues reminded me of some of the new periodic orbits which have been found for the $N-$body problem. This orbit is shown in Figure 1a (11 on a 10 chain, see [3]). Eleven particles move from left to right and back again, weaving up and down, miraculously avoiding collisions at the crossing points of the path. Although this analogy should not be taken too far, several other marches of Newtonian particles are shown in Figure 1.

The differential equations which determine the motion of N point masses subject to the forces of Newtonian gravity are easy to write down but hard to analyse. Most first year undergraduates learn that the two body problem is integrable and that the orbits can be described after noting some conservation laws and changing variables (does the transformation $u = \frac{1}{r}$ ring a bell?). However, if there are more than two bodies then the system can have chaotic orbits. Given this level of complexity it is natural to restrict attention either to simplifications of the general case (perturbation theory for example), or to special solutions of the full problem.

Periodic orbits are the simplest non-trivial special solutions which could be studied, and yet relatively few classes of periodic solutions are known. There are the Keplerian orbits (of which the two body problem gives the easiest examples), Euler orbits (three colinear bodies rotating about their centre of mass), Lagrangian orbits (the simple circular choreographies described below) and Hill's orbits (one body orbited by two tightly bound bodies), all of which were known by the end of the nineteenth century. In the last few years thousands of new and beautiful periodic solutions have been discovered by Carles Simó at the University of Barcelona, J. Gerver (Rutgers) and others. Examples of these orbits are shown in Figure 1: they all have the property that the periodic orbit involves N identical point masses which follow each other along a planar curve. Such solutions are examples of simple 'choreographies'.

The first new choreography – three bodies on a figure eight – was discovered numerically by Cris Moore (Santa Fé Institute) in 1993 [4]. However, the current explosion of interest is due to the rediscovery of this orbit, together with a proof of it's existence and stability, by Alain Chenciner (Paris VII) and Richard Montgomery (UCSC) [1]. Simó provided numerical verification of their theoretical results, and then went on to construct a host of choreographies using numerical shooting techniques [2, 5, 6]. Some of these orbits can be viewed on a Java applet written by Charlie McDowell [3] taken from an animation prepared by Simó [7]. The complicated procession of the particles along their paths is compulsive viewing.

It is not hard to describe choreographies on a circle of radius r, and this has some of the features of the proof by

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Chenciner and Montgomery. Place N identical particles with mass m at regular spacings around the circle and consider the gravitational forces acting on any one of the bodies. The forces perpendicular to the radial direction vanish pairwise by symmetry, so the net force on the body is radial. It is then just a short geometric exercise to calulate the total radial force F:

$$F = \frac{mk}{r^2} \sum_{n=1}^{N-1} \frac{1}{\sin \frac{\pi n}{N}}$$

where k is a constant. By choosing the angular velocity ω so that $mr\omega^2 = F$ for all the particles, the force F provides precisely the radial force required to maintain the particles in their symmetric positions, giving a simple choreography. Note that in this argument the only ingredients are symmetry and geometry. Chenciner and Montgomery need to work rather harder using variational techniques on the Newtonian action, but both symmetry and geometry are used to simplify their proof.

Although it is possible to imagine applications of these orbits – the storage of heavy units in space perhaps – it is unclear whether they will have immediate practical impact. The Moore-Chenciner-Montgomery orbit is stable, and researchers are currently examining the sensitivity of this orbits to perturbations which break the symmetries of the problem. Results from this work should help determine how likely it is that such orbits will be observed in nature. The other choreographies appear to be unstable and hence it is unlikely that they occur naturally, although this instability does not preclude them from being exploited by engineers in space missions (cf. slingshot orbits). This new class of periodic orbit should also stimulate research in other areas involving the interaction of large numbers of bodies.

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References

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