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Transitivity and blowout bifurcations

in a class of globally coupled maps

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Abstract

A class of globally coupled one dimensional maps is studied. For the uncoupled one dimensional map it is possible to compute the spectrum of Liapunov exponents exactly, and there is a natural equilibrium measure (Sinai-Ruelle-Bowen measure), so the corresponding `typical' Liapunov exponent may also be computed. The globally coupled systems thus provide examples of blowout bifurcations in arbitrary dimension. In the two dimensional case these maps have parameter values at which there is a transitive (topological) attractor which is a filled-in quadrilateral and, simultaneously, the synchronized state is a Milnor attractor.

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Blowout bifurcations and their associated riddled basins of attraction have been studied for some time [1-17], and it has been recognised that coupled identical systems provide a good supply of examples. Most of the examples considered are two dimensional, which makes it possible to use the theory of critical curves [18-21], but there are relatively few examples for which bifurcation values may be calculated explicitly, since to do this the natural equilibrium measure of the uncoupled system must be known. The coupled systems considered here have synchronized states in which all variables behave identically, and the blowout bifurcations are associated with the loss of transverse stability of these states. Roughly speaking, as a parameter is varied an asymptotically stable chaotic synchronized state loses asymptotic stability as some orbits in the attractor become transversely unstable. There is then an interval of parameter values for which typical synchronized orbits are transversely stable, but some are not, leading to riddled basins of attraction (either locally or globally) and the synchronized state is a Milnor attractor, e.g. [4,8,12,13]. At a critical parameter value typical synchronized orbits lose transverse stability (the blowout bifurcation) although some synchronized orbits may remain transversely stable. Finally, the synchronized state becomes completely unstable.

In this note we consider a class of coupled skew tent maps introduced by Pikovsky and Grassberger [3], see also [12,22]. For these examples it is possible to find all the bifurcation values explicitly and determine which orbits lose transverse stability and when. We then consider (in the case of two coupled maps and a restricted class of skew tent maps) the dynamics of the map at parameters between the loss of asymptotic stability of the synchronized state (the diagonal in the $(x,y)-$plane) and the blowout bifurcation. Hasler and Maistrenko [12] have already shown that the synchronized state is a Milnor attractor (and so can be thought of as the measure-theoretic attractor, attracting almost all nearby points). We sketch a proof that at the same time, the topological attractor (which attracts open sets) is completely different: it is a filled-in quadrilateral containing the Milnor attractor, and the dynamics on this region of the plane, $\mathcal{D}$, is transitive (i.e. there is a dense orbit) and periodic points are dense in $\mathcal{D}$. After the blowout bifurcation, $\mathcal{D}$ is both the measure-theoretic and the topological attractor, and so this result shows that there is some continuity of the topological attractor as parameters pass through the blowout bifurcation value; it is only the measure-theoretic attractor which changes drastically. This confirms a conjecture of Pikovsky and Grassberger [3].

The class of map we consider determines the evolution of M real variables, $x_1,\dots,x_M$, as a function of the discrete time variable, $n \in \mathbf{N}$. Later in this note we shall restrict to the simplest case: $M = 2$, which is the case discussed in [3], but until this becomes necessary we shall treat the general case. Let $x_i(n)$ represent the value of the i^{th} variable at time n, then the evolution of each variable is determined by the difference equation

$$x_i(n+1) = (1-\varepsilon)\, f_a(x_i(n)) + \frac{\varepsilon}{M}\sum_{k=1}^{M} f_a(x_k(n)), \quad i = 1,\dots M, \tag{1}$$

where $\varepsilon \in (0,1)$ and f_a is a continuous map of the interval, parametrized by the real parameter a. These globally coupled maps have been studied by many authors and have many interesting features [2,10,23]. Such systems have *synchronized* states: states in which $x_i(n) = X(n)$, $i = 1, \ldots, M$, i.e. each variables behaves in the same way over time, and (substituting in to (1)) the variable $X(n)$ evolves as a standard one dimensional map, usually referred to as the base map:

$$X(n+1) = f_a(X(n)), \quad n \in \mathbf{N}. \tag{2}$$

There are two non-degenerate ways in which a stable synchronized state can lose stability as the parameters a and ε are varied. Either it may lose stability to another synchronized state, or it can lose stability in a transverse direction. It is the relationship between these two possibilities which causes blowout bifurcations. Whilst the attractor may be stable in the transverse directions, some orbits in the attractor may be transversely unstable, leading to local riddling (the `holes' in the basin of attraction contain the transverse unstable manifolds of such orbits together with their preimages). As a parameter is varied the attractor itself may become unstable in the transverse direction: a blowout bifurcation.

Let $\mathcal{A}$ be an invariant set of the base map f_a containing a dense orbit ($\mathcal{A}$ might be a periodic orbit or a chaotic set). Assuming that $\mathcal{A}$ is uniquely ergodic then the Liapunov exponent of $\mathcal{A}$ is $\Lambda(\mathcal{A})$ where

$$\begin{aligned} \Lambda(\mathcal{A}) &= \lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} \log|f_a'(X(k))| \\ &= \int_{\mathcal{A}} \log|f_a'(x)|dm \end{aligned} \tag{3}$$

where m is the invariant measure associated with $\mathcal{A}$ and the first sum converges for almost every $x \in \mathcal{A}$ (with respect to the invariant measure m). A standard result, going back to [24,25], is that the synchronized state corresponding to $\mathcal{A}$ for the globally coupled maps (1) has a Liapunov exponent equal to $\Lambda(\mathcal{A})$ in the synchronized direction, and $N-1$ equal transverse Liapunov exponents, $\Lambda_T(\mathcal{A})$ where

$$\Lambda_T(\mathcal{A}) = \log|1-\varepsilon| + \Lambda(\mathcal{A}). \tag{4}$$

If $\lambda(a)$ is the Liapunov exponent of the attractor of f_a then (4) implies that the synchronized state with this attractor is stable provided $\varepsilon > \varepsilon_a$ where

$$\varepsilon_a = \max\{0, 1 - \mathrm{e}^{-\lambda(a)}\}. \tag{5}$$

(Recall that ε is restricted to the interval $(0,1)$ here, although extensions to $\varepsilon < 0$ are straightforward.) If $\lambda(a) < 0$ then (5) shows that the attractor of the synchronized state is transversely stable for all $\varepsilon \in (0,1)$.

For the remainder of this note we shall work with the family of base maps parametrized by $a > 1$ and defined by

$$f_a(x) = \begin{cases} ax & \text{if } 0 \le x \le a^{-1} \\ \frac{a(1-x)}{a-1} & \text{if } a^{-1} < x \le 1 \end{cases} \tag{6}$$

This is a generalization of the usual tent map which corresponds to the case $a = 2$. Similar maps and variants of the coupling given in (1) have been considered in [3,12,22] for the case $M = 2$, and the stability results presented below are essentially a restatement or simple elaboration of results in [3,12]. Note that the analysis below does suggest a marked difference between the case $a < 2$ and the case $a > 2$ (cf. [12]). For each $a > 1$, the dynamics of f_a restricted to the unit interval is semi-conjugate to the one-sided shift on two symbols as indicated below. If required we can extend the map continuously outside the unit interval so that the attractor is the unit interval. Moreover, if we assign a coding of $[0,1]\backslash\{a^{-1}\}$ with $c(x) = 0$ if $x \in [0,a^{-1})$ and $c(x) = 1$ if $x \in (a^{-1},1]$ then kneading sequences may be defined for any $x \in [0,1]$ which is not a preimage of a^{-1} by

$$k(x) = c(x)c(f_a(x))c(f_a^2(x))\dots \tag{7}$$

and this may be extended to primages of a^{-1} by taking limits from above and from below respectively, giving (distinct) upper and lower kneading sequences. It is then straightforward to show that for any infinite sequence of 0s and 1s there is a unique $y \in [0,1]$ such that this sequence is the kneading sequence (possibly the upper or lower kneading sequence) of y (see [26] for example). Of course, any preimage of a^{-1} will have upper and lower kneading sequences which end 0^∞ (an infinite sequence of 0s). These will be ignored as a minor technical complication henceforth.

Now let K_ρ, $\rho \in [0,1]$, be the set of infinite sequence $c_1c_2c_3\dots$ with $c_k \in \{0,1\}$, $k \in \mathbf{N}$, and such that

$$\rho = \lim_{n\to\infty} \frac{1}{n}\sum_1^n c_k. \tag{8}$$

Clearly K_ρ is non-empty for all $\rho \in [0,1]$ and if $s \in K_\rho$ then there is $x \in [0,1]$ such that $k(x) = s$. For any such point, the sum on the right hand side of the first equation (3) with $X(0) = x$ clearly converges to

$$(1-\rho)\log a + \rho \log\left(\frac{a}{a-1}\right).$$

If ρ is rational, then by choosing any periodic orbit $\mathcal{P}$ with the given proportion of 1s per period in its symbolic code we obtain a uniquely ergodic invariant set with

$$\Lambda(\mathcal{P}) = \log a - \rho\log(a-1). \tag{9}$$

With a little more work (taking limits of appropriate periodic orbits for example) it is possible to show that uniquely ergodic invariant sets of f_a exist for all irrational $\rho \in (0,1)$. Thus the set of all possible Liapunov exponents of the map is a closed interval $[\alpha,\beta]$ where

$$\alpha = \begin{cases} \log a & \text{if } 1 < a \le 2 \\ \log a - \log(a-1) & \text{if } a \ge 2 \end{cases} \quad \text{and} \quad \beta = \begin{cases} \log a - \log(a-1) & \text{if } 1 < a \le 2 \\ \log a & \text{if } a \ge 2 \end{cases}$$

or in other words

$$\alpha = \min\{\log a, \log a - \log(a-1)\}, \quad \beta = \max\{\log a, \log a - \log(a-1)\}. \tag{10}$$

Note that if $a = 2$ this interval collapses to a single point (this is the reason why the usual tent map has no riddled basins in this model).

Beck and Schlögel [27] show that the natural (SBR) invariant measure for this map is just Lebesgue measure, independent of the value of a and hence

$$\begin{aligned}\lambda(a) &= \int_0^{a-1} \log a \, dx + \int_{a^{-1}}^{1} [\log a - \log(a-1)] \, dx \\ &= \log a - (1-a^{-1}) \log(a-1).\end{aligned}$$

In particular, if $a \neq 2$ then $\lambda(a) \in (\alpha, \beta)$. This is the crucial statement which makes it possible to follow the arguments of [8] explicitly for this example. Note that (11) corresponds to a value of ρ equal to $1-a^{-1}$ in (9).

Any invariant set $\mathcal{A}$ of f_a with well-defined Liapunov exponent $\Lambda(\mathcal{A})$ (and in particular, any of the invariant sets which satisfy (9)) also exists as a synchronized state for the full multi-dimensional system (1), and this synchronized state will have a Liapunov exponent $\Lambda(\mathcal{A})$ in the synchronized direction and $M-1$ transverse Liapunov exponents given by (4). This invariant set is therefore transversely stable if $\varepsilon > \varepsilon_{\mathcal{A}}$ where

$$\varepsilon_{\mathcal{A}} = 1 - e^{-\Lambda(\mathcal{A})}. \tag{12}$$

Since $\lambda(a) > 0$ from (11), (5) implies that

$$\varepsilon_a = 1 - e^{-\lambda(a)} = \frac{a - (a-1)^{\frac{1}{a}(a-1)}}{a}.$$

This stability boundary is indicated by the curve labelled S in Figure 1. Let α and β be as defined in (10). Then the corresponding transverse stability boundaries for sets having α (resp. β) as Liapunov exponent is ε_α (resp. ε_β) which may be obtained frpom (4) and (10):

$$\varepsilon_\alpha = \min\left(\frac{1}{a}, \frac{a-1}{a}\right), \quad \varepsilon_\beta = \max\left(\frac{1}{a}, \frac{a-1}{a}\right). \tag{13}$$

From these considerations it is easy to obtain the following results.

- If $\varepsilon > \varepsilon_\beta$ then every synchronized state is transversely stable. In the language of [8], ε_β is a point of loss of asymptotic stability.
- If $\varepsilon_\beta > \varepsilon > \varepsilon_a$ then every uniquely ergodic invariant set with code in K_ρ and

$$\rho < \frac{\log(1-\varepsilon) + \log a}{\log(a-1)} \quad \text{if } a > 2 \tag{14}$$

or

$$\rho > \frac{\log(1-\varepsilon) + \log a}{\log(a-1)} \quad \text{if } a < 2 \tag{15}$$

is transversely unstable. The unit interval is still an attractor in the transverse direction but it has a locally riddled basin [8] with holes consisting of cuspoidal regions about the codimension one transverse unstable manifolds of each of the transversely unstable invariant sets and their preimages. In the language of [8], ε_a is a blowout bifurcation point.
- If $\varepsilon_a > \varepsilon > \varepsilon_\alpha$ then the synchronized state is no longer attracting, but there are some synchronized states which are stable in the transverse direction. In the language of [8], ε_α is a point of bifurcation to normal repulsion.

• If $\varepsilon_\alpha > \varepsilon > 0$ then all synchronized states are transversely unstable.

As ε decreases from ε_β to ε_a a succession of invariant sets loses transverse stability, starting with the fixed point with code 0^∞ if $a > 2$ or the fixed point with code 1^∞ if $a < 2$. From (9) and (10) we see that between ε_β and ε_a all invariant sets with $\rho \in (0,1-a^{-1})$ if $a > 2$, or all invariant sets with $\rho \in (1-a^{-1},1)$ if $a < 2$, lose transverse stability. For any given ρ in these intervals the stability boundary is

$$\varepsilon_\rho = \frac{a-(a-1)^\rho}{a}. \tag{16}$$

This means that although riddling starts with only one orbit losing transverse stability (the appropriate fixed point in this example) as soon as $\varepsilon < \varepsilon_\beta$ there is an infinite set of transversely unstable periodic orbits. What is more, these orbits are dense in the unit interval even before their preimages are taken into account (the preimage of any invariant set of the base map is also dense in the unit interval). This suggests that if $a > 2$ then the dominant hole will be close to the fixed point of f_a at $x = 0$ and its preimages, whilst if $a < 2$ it will be the non-trivial fixed point and its preimages which dominate.

The final question which needs to be addressed is the nature of the attractors if $\varepsilon < \varepsilon_\beta$. Here we shall consider briefly the case $a \in (1,2)$ and $M = 2$, we hope to report further details together with the case $a > 2$ elsewhere [28]. First note (see Figure 2) that numerical evidence suggests that if $\varepsilon \in (\varepsilon_a, \frac{1}{a})$ then almost all points are attracted to the synchronized state (the diagonal), after the blowout bifurcation at ε_a there is on-off intermittency and the attractor appears to have the form of a quadrilateral, and if ε is sufficiently small (less than ε_α) there is no longer evidence of on-off intermittency, although the attractor still appears to be a quadrilateral.

Since $M = 2$, (1) may be written as a two-dimensional difference equation in the plane:

$$\begin{aligned} x_{n+1} &= (1-\frac{1}{2}\varepsilon) f_a(x_n) + \frac{1}{2}\varepsilon f_a(y_n) \\ y_{n+1} &= \frac{1}{2}\varepsilon f_a(x_n) + (1-\frac{1}{2}\varepsilon) f_a(y_n) \end{aligned}$$

or in vector notation $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$. If $a \in (1,2)$ and $\varepsilon \in (\frac{a-1}{a}, \frac{1}{a})$ then some synchronized states are transversely stable whilst others are transversely unstable, and the synchronized state is a Milnor attractor if $\varepsilon \in (\varepsilon_a, \frac{1}{a})$. (Recall that $\mathcal{A}$ is a Milnor attractor if the set of orbits in any $\eta-$neighbourhood of $\mathcal{A}$ which converge to $\mathcal{A}$ has positive relative measure, m_η, and that $m_\eta \to 1$ as $\eta \to 0$, cf. [3,4,8,12,13].) However, as we will argue below, if $\varepsilon \in (\varepsilon_a, \frac{1}{a})$ and $a \in (\frac{1+\sqrt{5}}{2}, 2)$ there is a closed region $\mathcal{D}$ (which contains the synchronized state, the diagonal) on which the dynamics is transitive and periodic points are dense. We do this by showing that for any open set U in $\mathcal{D}$ there exists a finite $n > 0$ (depending on the choice of U) and $V \subseteq U$ such that $F^n(V) = \mathcal{D}$ and F^n restricted to V is affine. This means that if

$$(1+\sqrt{5})/2 < a < 2 \;\text{ and }\; \varepsilon_a < \varepsilon < 1/a$$

the map (17) gives an example of a dynamical system for which the attractor of open sets (the quadrilateral $\mathcal{D}$) is not equal to the measure-theoretic attractor (the synchronized state, which lies on the diagonal $x = y$), and such that the proof of these statements is almost elementary.

We only sketch the argument here, details will be given in [28]. The methods borrow from [18,19,20,21] in the construction of the absorbing region, together with the standard transitivity argument for one-dimensional maps, e.g. [29]. It is shown in [4,12] that the synchronized state is a Milnor attractor if $\varepsilon \in (\varepsilon_a, \frac{1}{a})$, so we do not need to worry about that part of the result below. It will be convenient to define $\omega = \frac{1}{2}\varepsilon$, so for the remainder of this note we consider (17) with

$$(1+\sqrt{5})/2 < a < 2 \text{ and } 0 < 2a\omega < 1 \ \ (\omega = \varepsilon/2). \tag{18}$$

Following [18] we consider images of the critical lines $x = a^{-1}$ and $y = a^{-1}$, and are therefore led to consider the closed absorbing region $\mathcal{D}$ with boundary $ORIR'$, where $O = (0,0)$, $I = (1,1)$, $R = (\frac{1-2\omega+2\omega^2}{1-\omega}, 2\omega)$ and $R' = (2\omega, \frac{1-2\omega+2\omega^2}{1-\omega})$ as shown in Figure 3. Pikovsky and Grassberger consider the same region [3]. Here, RI is a subset of the image of $y = a^{-1}$ and OR is the image of $R'I$. Henceforth, if $X = (x,y)$ then the point (y,x) will be denoted X', and given a set of points X_j, $1 \leq j \leq s$ $(s \geq 3)$ then $X_1X_2\ldots X_s$ will denote the closed region bounded by the straight lines X_1X_2, …, X_sX_1. If $a \in (1,2)$ and $2a\omega < 1$ then $2\omega < a^{-1}$ and $\frac{1-2\omega+2\omega^2}{1-\omega} > a^{-1}$ so the points R and R' are do lie in the quadrants as shown in Figure 3. It is not hard to show that $\mathcal{D}$ is invariant (cf. [18]) and that the one-dimensional base map is transitive [29]. By a slightly more involved argument than the expansion proof of [29], considering the way in which the area of a region is expanded or contracted when mapped across the critical lines, it can be shown that for any open region U in $\mathcal{D}$ there exists $k_1 > 0$ and $V_1 \subseteq U$ such that the interior of $F^{k_1}(V_1)$ contains a piece of the diagonal and $F|_{V_1}$ is affine; it is here that the slope condition $a > \frac{1+\sqrt{5}}{2}$ is used [28]. Hence (by the transitivity of the base map) there exists $k_2 \geq k_1$ and $V_2 \subseteq V_1$ such that the non-trivial synchronized fixed point, $G = (\frac{a}{2a-1}, \frac{a}{2a-1})$, is contained in the interior of $F^{k_2}(V_2)$ and $F|_{V_2}$ is affine. Now, if $2a\omega < 1$ then G is unstable in both the synchronized and transverse directions (it corresponds to the fixed point with code 1^∞) and the linear unstable eigenspace corresponding to the transversely unstable direction intersects the line $x = a^{-1}$ at H (which can be computed explicitly) and the line RI at L where $L = F(H')$ (cf. Figure 3). Let $A = (a^{-1}, a^{-1})$. In any open neighbourhood of G and so, in particular, in the interior of $F^{k_2}(V_2)$ we can choose triangular domains T_1 and T_2 such that there exists q with

$$F^q(T_1) = F^{q+1}(T_2) = HAH' \tag{19}$$

(T_2 is just the preimage of T_1 under the linear map in the quadrant with $x,y > a^{-1}$). Moreover, there exists $V_3 \subseteq V_2$ such that $F^{k_2}(V_3) = T_1 \cup T_2$ and $F|_{V_3}$ is affine. The image of the triangle HAH' is easily seen to be the triangle LIL' and so

$$F^{q+1}(T_1) = LIL'. \tag{20}$$

Let M be the intersection of RI with $x = a^{-1}$. Then

$$F^{k_2+q+1}(U) \supseteq F^{q+1}(T_1 \cup T_2) \supseteq (HAH') \cup (LIL')$$

$$\supseteq AMIM' \tag{21}$$

But $F(M) = R'$, $F(A) = I$, $F(M') = R$ and $F(I) = O$, so we can choose $V \subseteq V_3$ such that

$$F^{k_2+q+2}(V) = F(AMIM') = ORIR' = \mathcal{D} \tag{22}$$

and $F^{k_2+q+2}|_V$ is affine, as required. It is now straightforward to show from this property that the dynamics on $\mathcal{D}$ is transitive and periodic points are dense in $\mathcal{D}$ as conjectured in [3]. See [28] for details.

Conclusion: The example presented allows us to calculate blowout bifurcation curves explicitly. If $M = 2$ and $1 < a < 2$ then numerical evidence suggests that the blowout bifurcation is supercritical, with on-off intermittency immediately after the loss of transverse stability of the attractor at ε_a. In the region of parameter space between the loss of asymptotic stability and blowout of the synchronized state we have shown that if $a \in (\frac{1+\sqrt{5}}{2}, 2)$ there is a transitive topological attractor (the region $\mathcal{D}$) even though the synchronised state (the diagonal) is a Milnor attractor. After the blowout bifurcation the synchronized state is no longer a Milnor attractor and so the only attractor is the region $\mathcal{D}$. We conjecture (following [3]) that this slope condition can be relaxed further, and that this type of behaviour is typical in supercritical blowout bifurcations. The equations are sufficiently simple that it should be possible to derive equivalent statements in $a > 2$ (where the mechanism can be more complicated). Details of the proof will be given in [28].

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Figure Captions

Figure 1: Bifurcation curves in the $(a,\varepsilon)-$plane. S denotes the curves on which `typical' points on the synchronized attractor lose transverse stability.

Figure 2: (a) On-off intermittency: $(a,\varepsilon) = (1.5,0.4)$; (b) the attractor with $(a,\varepsilon) = (1.5,0.25)$. Note that if $a = 1.5$ then

$$(\varepsilon_\alpha,\varepsilon_a,\varepsilon_\beta) \approx (0.3333,0.470866,0.6666)$$

Figure 3: The geometry of critical curves for the map (17). $\mathcal{D}$ is the quadrilateral $ORIR'$. RI is part of the image of the critical line $y = a^{-1}$, and the image of RI is OR', with $F(M) = R'$. G denotes the non-trivial synchronized fixed point. For the significance of H, H', L and L' see the text. Primed points are obtained from unprimed points by reflection in the diagonal.

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