

# Foundation Maths Support Pack

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materials to support *Foundation Maths Third Edition* by Anthony Croft & Robert Davison, ISBN 0 130 45426-5

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The *Foundation Maths Third Edition* Support Pack is a set of 46 freely-photocopiable leaflets which can be distributed by teachers, lecturers and others involved in providing mathematics support to students. Each leaflet provides a concise and student-friendly introduction to some important topic in *Foundation Maths*. The leaflets can be used alone or in conjunction with the textbook. Most of the examples and exercises on the leaflets are different from those in the text and so these can be regarded as an additional and complementary resource. Each leaflet provides a Chapter reference to guide students to further examples and exercises in the text.

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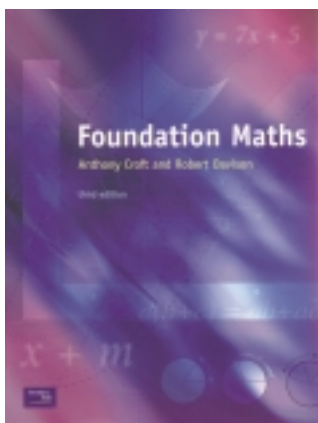
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(Please email any comments, corrections and suggestions for improvement to [a.c.croft@lboro.ac.uk](mailto:a.c.croft@lboro.ac.uk))



# Foundation Maths Support Pack

## 1.1 Fractions

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### Introduction

The ability to work confidently with fractions, both number fractions and algebraic fractions, is an essential skill which underpins all other algebraic processes. In this leaflet we remind you of how number fractions are simplified, added, subtracted, multiplied and divided.

### 1. Expressing a fraction in its simplest form

In any fraction  $\frac{p}{q}$ , say, the number  $p$  at the top is called the **numerator**. The number  $q$  at the bottom is called the **denominator**. The number  $q$  must never be zero. A fraction can always be expressed in different, yet **equivalent** forms. For example, the two fractions  $\frac{2}{6}$  and  $\frac{1}{3}$  are equivalent. They represent the same value. A fraction is expressed in its **simplest form** by cancelling any factors which are common to both the numerator and the denominator. You need to remember that factors are numbers which are multiplied together. We note that

$$\frac{2}{6} = \frac{1 \times 2}{2 \times 3}$$

and so there is a factor of 2 which is common to both the numerator and the denominator. This common factor can be cancelled to leave the equivalent fraction  $\frac{1}{3}$ . Cancelling is equivalent to dividing the top and the bottom by the common factor.

#### Example

$\frac{12}{20}$  is equivalent to  $\frac{3}{5}$  since

$$\frac{12}{20} = \frac{4 \times 3}{4 \times 5} = \frac{3}{5}$$

#### Exercises

1. Express each of the following fractions in its simplest form:

- a)  $\frac{12}{16}$     b)  $\frac{14}{21}$     c)  $\frac{3}{6}$     d)  $\frac{100}{45}$     e)  $\frac{7}{9}$     f)  $\frac{15}{55}$     g)  $\frac{3}{24}$     h)  $\frac{15}{42}$     i)  $\frac{36}{20}$ .

#### Answers

1. a)  $\frac{3}{4}$     b)  $\frac{2}{3}$     c)  $\frac{1}{2}$     d)  $\frac{20}{9}$     e)  $\frac{7}{9}$     f)  $\frac{3}{11}$     g)  $\frac{1}{8}$     h)  $\frac{5}{14}$     i)  $\frac{9}{5}$ .

### 2. Addition and subtraction of fractions

To add two fractions we first re-write each fraction so that they both have the same denominator. This denominator is chosen to be the **lowest common denominator**. This is the smallest

number which is a multiple of both denominators. Then, the numerators only are added, and the result is divided by the lowest common denominator.

### Example

Simplify a)  $\frac{7}{16} + \frac{5}{16}$       b)  $\frac{7}{16} + \frac{3}{8}$ .

### Solution

a) In this case the denominators of each fraction are already the same. The lowest common denominator is 16. We perform the addition by simply adding the numerators and dividing the result by the lowest common denominator. So,  $\frac{7}{16} + \frac{5}{16} = \frac{7+5}{16} = \frac{12}{16}$ . This answer can be expressed in the simpler form  $\frac{3}{4}$  by cancelling the common factor 4.

b) To add these fractions we must rewrite them so that they have the same denominator. The lowest common denominator is 16 because this is the smallest number which is a multiple of both denominators. Note that  $\frac{3}{8}$  is equivalent to  $\frac{6}{16}$  and so we write  $\frac{7}{16} + \frac{3}{8} = \frac{7}{16} + \frac{6}{16} = \frac{13}{16}$ .

### Example

Find  $\frac{1}{2} + \frac{2}{3} + \frac{4}{5}$ .

### Solution

The smallest number which is a multiple of the given denominators is 30. We express each fraction with a denominator of 30.

$$\frac{1}{2} + \frac{2}{3} + \frac{4}{5} = \frac{15}{30} + \frac{20}{30} + \frac{24}{30} = \frac{59}{30}$$

### Exercises

1. Evaluate each of the following:

a)  $\frac{2}{3} + \frac{5}{4}$       b)  $\frac{4}{9} - \frac{1}{2}$       c)  $\frac{3}{4} + \frac{5}{6}$       d)  $\frac{1}{4} + \frac{1}{3} + \frac{1}{2}$       e)  $\frac{2}{5} - \frac{1}{3} - \frac{1}{10}$       f)  $\frac{4}{5} + \frac{1}{3} - \frac{3}{4}$ .

### Answers

1. a)  $\frac{23}{12}$       b)  $-\frac{1}{18}$       c)  $\frac{19}{12}$       d)  $\frac{13}{12}$       e)  $-\frac{1}{30}$       f)  $\frac{23}{60}$ .

## 3. Multiplication and division of fractions

Multiplication of fractions is more straightforward. We simply multiply the numerators to give a new numerator, and multiply the denominators to give a new denominator. For example

$$\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4} = \frac{15}{28}$$

Division is performed by inverting the second fraction and then multiplying. So,

$$\frac{5}{7} \div \frac{3}{4} = \frac{5}{7} \times \frac{4}{3} = \frac{20}{21}$$

### Exercises

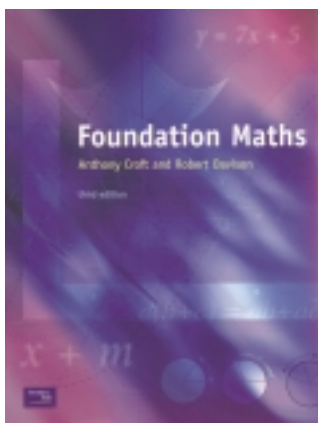
1. Find a)  $\frac{4}{26} \times \frac{13}{7}$       b)  $\frac{2}{11} \div \frac{3}{5}$       c)  $\frac{2}{1} \times \frac{1}{2}$       d)  $\frac{3}{7} \times \frac{2}{5}$       e)  $\frac{3}{11} \times \frac{22}{5}$       f)  $\frac{5}{6} \div \frac{4}{3}$ .

### Answers

1. a)  $\frac{2}{7}$       b)  $\frac{10}{33}$       c) 1      d)  $\frac{6}{35}$       e)  $\frac{6}{5}$       f)  $\frac{5}{8}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 2



# Foundation Maths Support Pack

## 1.2 Powers and roots

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### Introduction

Powers are used when we want to multiply a number by itself repeatedly.

### 1. Powers

When we wish to multiply a number by itself we use **powers**, or **indices** as they are also called. For example, the quantity  $7 \times 7 \times 7 \times 7$  is usually written as  $7^4$ . The number 4 tells us the number of sevens to be multiplied together. In this example, the power, or index, is 4. The number 7 is called the **base**.

#### Example

$6^2 = 6 \times 6 = 36$ . We say that '6 squared is 36', or '6 to the power 2 is 36'.

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ . We say that '2 to the power 5 is 32'.

Your calculator will be pre-programmed to evaluate powers. Most calculators have a button marked  $x^y$ , or simply  $\hat{}$ . Ensure that you are using your calculator correctly by verifying that  $3^{11} = 177147$ .

### 2. Square roots

When 5 is squared we obtain 25. That is  $5^2 = 25$ .

The reverse of this process is called **finding a square root**. The square root of 25 is 5. This is written as  $\sqrt[2]{25} = 5$ , or simply  $\sqrt{25} = 5$ .

Note also that when  $-5$  is squared we again obtain 25, that is  $(-5)^2 = 25$ . This means that 25 has another square root,  $-5$ .

In general, a square root of a number is a number which when squared gives the original number. There are always two square roots of any positive number, one positive and one negative. However, negative numbers do not possess any square roots.

Most calculators have a square root button, probably marked  $\sqrt{\quad}$ . Check that you can use your calculator correctly by verifying that  $\sqrt{79} = 8.8882$ , to four decimal places. Your calculator will only give the positive square root but you should be aware that the second, negative square root is  $-8.8882$ .

An important result is that the square root of a product of two numbers is equal to the product of the square roots of the two numbers. For example

$$\sqrt{16 \times 25} = \sqrt{16} \times \sqrt{25} = 4 \times 5 = 20$$

More generally,

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

However your attention is drawn to a common error which students make. It is not true that  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ . Substitute some simple values for yourself to see that this cannot be right.

### Exercises

1. Without using a calculator write down the value of  $\sqrt{9 \times 36}$ .
2. Find the square of the following: a)  $\sqrt{2}$ ,      b)  $\sqrt{12}$ .
3. Show that the square of  $5\sqrt{2}$  is 50.

### Answers

1. 18, (and also  $-18$ ).    2. a) 2,    b) 12.

## 3. Cube roots and higher roots

The cube root of a number, is the number which when cubed gives the original number. For example, because  $4^3 = 64$  we know that the cube root of 64 is 4, written  $\sqrt[3]{64} = 4$ . All numbers, both positive and negative, possess a single cube root.

Higher roots are defined in a similar way: because  $2^5 = 32$ , the fifth root of 32 is 2, written  $\sqrt[5]{32} = 2$ .

### Exercises

1. Without using a calculator find a)  $\sqrt[3]{27}$ ,      b)  $\sqrt[3]{125}$ .

### Answers

1. a) 3,    b) 5.

## 4. Surds

Expressions involving roots, for example  $\sqrt{2}$  and  $5\sqrt{2}$  are also known as **surds**. It is usually quite acceptable to leave an answer in surd form rather than calculating its decimal approximation with a calculator.

It is often possible to write surds in equivalent forms. For example,  $\sqrt{48}$  can be written as  $\sqrt{3 \times 16}$ , that is  $\sqrt{3} \times \sqrt{16} = 4\sqrt{3}$ .

### Exercises

1. Write the following in their simplest surd form: a)  $\sqrt{180}$ ,      b)  $\sqrt{63}$ .
2. By multiplying numerator and denominator by  $\sqrt{2} + 1$ , show that

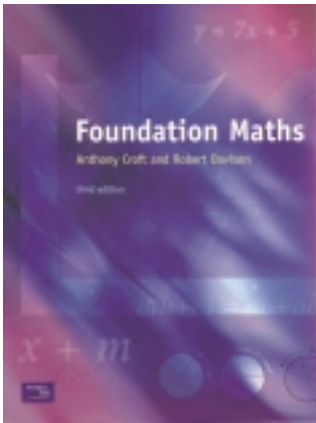
$$\frac{1}{\sqrt{2} - 1} \quad \text{is equivalent to} \quad \sqrt{2} + 1$$

### Answers

1. a)  $6\sqrt{5}$ ,    b)  $3\sqrt{7}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 6



# Foundation Maths Support Pack

## 1.3 Scientific notation

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### Introduction

In engineering calculations numbers are often very small or very large, for example 0.00000345 and 870,000,000. To avoid writing lengthy strings of numbers a notation has been developed, known as **scientific notation** which enables us to write numbers much more concisely.

### 1. Scientific notation

In scientific notation each number is written in the form

$$a \times 10^n$$

where  $a$  is a number between 1 and 10 and  $n$  is a positive or negative whole number.

Some numbers in scientific notation are

$$5 \times 10^3, \quad 2.67 \times 10^4, \quad 7.90 \times 10^{-3}$$

To understand scientific notation you need to be aware that

$$10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10000, \quad \text{and so on,}$$

and also that

$$10^{-1} = \frac{1}{10} = 0.1, \quad 10^{-2} = \frac{1}{100} = 0.01, \quad 10^{-3} = \frac{1}{1000} = 0.001, \quad \text{and so on.}$$

You also need to remember how simple it is to multiply a number by powers of 10. For example to multiply 3.45 by 10, the decimal point is moved one place to the right to give 34.5. To multiply 29.65 by 100, the decimal point is moved two places to the right to give 2965. In general, to multiply a number by  $10^n$  the decimal point is moved  $n$  places to the right if  $n$  is a positive whole number and  $n$  places to the left if  $n$  is a negative whole number. It may be necessary to insert additional zeros to make up the required number of digits.

#### Example

The following numbers are given in scientific notation. Write them out fully.

a)  $5 \times 10^3$ ,    b)  $2.67 \times 10^4$ ,    c)  $7.90 \times 10^{-3}$ .

#### Solution

a)  $5 \times 10^3 = 5 \times 1000 = 5000$ .

- b)  $2.67 \times 10^4 = 26700$ .  
c)  $7.90 \times 10^{-3} = 0.00790$ .

### Example

Express each of the following numbers in scientific notation.

- a) 5670000,      b) 0.0098.

### Solution

- a)  $5670000 = 5.67 \times 10^6$ .  
b)  $0.0098 = 9.8 \times 10^{-3}$ .

### Exercises

1. Express each of the following in scientific notation.

- a) 0.00254,      b) 82,      c)  $-0.342$ ,      d) 1000000.

### Answers

1. a)  $2.54 \times 10^{-3}$ ,      b)  $8.2 \times 10$ ,      c)  $-3.42 \times 10^{-1}$ ,      d)  $1 \times 10^6$  or simply  $10^6$ .

## 2. Using a calculator

Students often have difficulty using a calculator to deal with scientific notation. You may need to refer to your calculator manual to ensure that you are entering numbers correctly. You should also be aware that your calculator can display a number in lots of different forms including scientific notation. Usually a MODE button is used to select the appropriate format.

Commonly the EXP button is used to enter numbers in scientific notation. (EXP stands for exponent which is another name for a power). A number like  $3.45 \times 10^7$  is entered as 3.45EXP 7 and might appear in the calculator window as  $3.45^{07}$ . Alternatively your calculator may require you to enter the number as 3.45E7 and it may be displayed in the same way. You should seek help if in doubt.

Computer programming languages use similar notation. For example

$$8.25 \times 10^7 \quad \text{may be programmed as} \quad 8.25\text{E}7$$

and

$$9.1 \times 10^{-3} \quad \text{may be programmed as} \quad 9.1\text{E} - 3$$

Again, you need to take care and check the required syntax carefully.

A common error is to enter incorrectly numbers which are simply powers of 10. For example, the number  $10^7$  is erroneously entered as 10E7 which means  $10 \times 10^7$ , that is  $10^8$ . The number  $10^7$ , meaning  $1 \times 10^7$ , should be entered as 1E7.

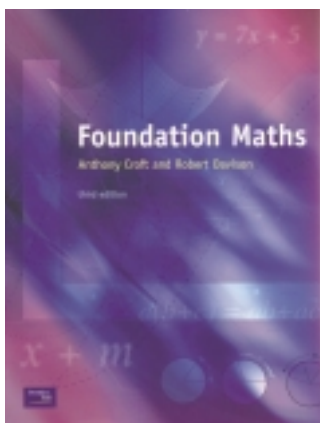
Check that you are using your calculator correctly by verifying that

$$(3 \times 10^7) \times (2.76 \times 10^{-4}) \times (10^5) = 8.28 \times 10^8$$

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 6





# Foundation Maths Support Pack

## 2.1 The laws of indices

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### Introduction

A **power**, or an **index**, is used to write a product of numbers very compactly. The plural of index is **indices**. In this leaflet we remind you of how this is done, and state a number of rules, or laws, which can be used to simplify expressions involving indices.

### 1. Powers, or indices

We write the expression

$$3 \times 3 \times 3 \times 3 \quad \text{as} \quad 3^4$$

We read this as 'three to the power four'.

Similarly

$$z \times z \times z = z^3$$

We read this as 'z to the power three' or 'z cubed'.

In the expression  $b^c$ , the **index** is  $c$  and the number  $b$  is called the **base**. Your calculator will probably have a button to evaluate powers of numbers. It may be marked  $x^y$ . Check this, and then use your calculator to verify that

$$7^4 = 2401 \quad \text{and} \quad 25^5 = 9765625$$

### Exercises

1. Without using a calculator work out the value of

a)  $4^2$ ,    b)  $5^3$ ,    c)  $2^5$ ,    d)  $\left(\frac{1}{2}\right)^2$ ,    e)  $\left(\frac{1}{3}\right)^2$ ,    f)  $\left(\frac{2}{5}\right)^3$ .

2. Write the following expressions more concisely by using an index.

a)  $a \times a \times a \times a$ ,    b)  $(yz) \times (yz) \times (yz)$ ,    c)  $\left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right)$ .

### Answers

1. a) 16,    b) 125,    c) 32,    d)  $\frac{1}{4}$ ,    e)  $\frac{1}{9}$ ,    f)  $\frac{8}{125}$ .

2. a)  $a^4$ ,    b)  $(yz)^3$ ,    c)  $\left(\frac{a}{b}\right)^3$ .

### 2. The laws of indices

To manipulate expressions involving indices we use rules known as the **laws of indices**. The laws should be used precisely as they are stated - do not be tempted to make up variations of your own! The three most important laws are given here:

**First law**

$$a^m \times a^n = a^{m+n}$$

When expressions with the same base are multiplied, the indices are added.

**Example**

We can write

$$7^6 \times 7^4 = 7^{6+4} = 7^{10}$$

You could verify this by evaluating both sides separately.

**Example**

$$z^4 \times z^3 = z^{4+3} = z^7$$

**Second Law**

$$\frac{a^m}{a^n} = a^{m-n}$$

When expressions with the same base are divided, the indices are subtracted.

**Example**

We can write

$$\frac{8^5}{8^3} = 8^{5-3} = 8^2 \quad \text{and similarly} \quad \frac{z^7}{z^4} = z^{7-4} = z^3$$

**Third law**

$$(a^m)^n = a^{mn}$$

Note that  $m$  and  $n$  have been multiplied to yield the new index  $mn$ .

**Example**

$$(6^4)^2 = 6^{4 \times 2} = 6^8 \quad \text{and} \quad (e^x)^y = e^{xy}$$

It will also be useful to note the following important results:

$$a^0 = 1, \quad a^1 = a$$

**Exercises**

1. In each case choose an appropriate law to simplify the expression:

a)  $5^3 \times 5^{13}$ ,    b)  $8^{13} \div 8^5$ ,    c)  $x^6 \times x^5$ ,    d)  $(a^3)^4$ ,    e)  $\frac{y^7}{y^3}$ ,    f)  $\frac{x^8}{x^7}$ .

2. Use one of the laws to simplify, if possible,  $a^6 \times b^5$ .

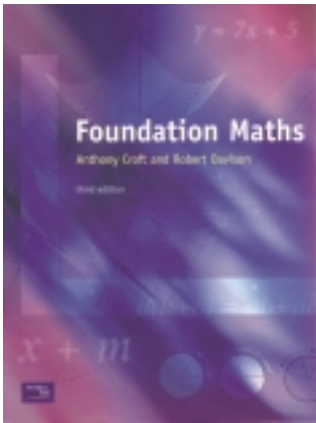
**Answers**

1. a)  $5^{16}$ ,    b)  $8^8$ ,    c)  $x^{11}$ ,    d)  $a^{12}$ ,    e)  $y^4$ ,    f)  $x^1 = x$ .

2. This cannot be simplified because the bases are not the same.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 6



# Foundation Maths Support Pack

## 2.2 Negative and fractional powers

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### Introduction

Sometimes it is useful to use negative and fractional powers. These are explained on this leaflet.

### 1. Negative powers

Sometimes you will meet a number raised to a negative power. This is interpreted as follows:

$$a^{-m} = \frac{1}{a^m}$$

This can be rearranged into the alternative form:

$$a^m = \frac{1}{a^{-m}}$$

### Example

$$3^{-2} = \frac{1}{3^2}, \quad \frac{1}{5^{-2}} = 5^2, \quad x^{-1} = \frac{1}{x^1} = \frac{1}{x}, \quad x^{-2} = \frac{1}{x^2}, \quad 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

### Exercises

1. Write the following using only positive powers:

a)  $\frac{1}{x^{-6}}$ ,   b)  $x^{-12}$ ,   c)  $t^{-3}$ ,   d)  $\frac{1}{4^{-3}}$ ,   e)  $5^{-17}$ .

2. Without using a calculator evaluate a)  $2^{-3}$ ,   b)  $3^{-2}$ ,   c)  $\frac{1}{4^{-2}}$ ,   d)  $\frac{1}{2^{-5}}$ ,   e)  $\frac{1}{4^{-3}}$ .

### Answers

1. a)  $x^6$ ,   b)  $\frac{1}{x^{12}}$ ,   c)  $\frac{1}{t^3}$ ,   d)  $4^3$ ,   e)  $\frac{1}{5^{17}}$ .

2. a)  $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ ,   b)  $\frac{1}{9}$ ,   c) 16,   d) 32,   e) 64.

### 2. Fractional powers

To understand fractional powers you first need to have an understanding of roots, and in particular square roots and cube roots. If necessary you should consult leaflet *1.2 Powers and Roots*.

When a number is raised to a fractional power this is interpreted as follows:

$$a^{1/n} = \sqrt[n]{a}$$

So,

$a^{1/2}$  is a square root of  $a$

$a^{1/3}$  is the cube root of  $a$

$a^{1/4}$  is a fourth root of  $a$

### Example

$$3^{1/2} = \sqrt{3}, \quad 27^{1/3} = \sqrt[3]{27} \text{ or } 3, \quad 32^{1/5} = \sqrt[5]{32} = 2, \\ 64^{1/3} = \sqrt[3]{64} = 4, \quad 81^{1/4} = \sqrt[4]{81} = 3$$

Fractional powers are useful when we need to calculate roots using a scientific calculator. For example to find  $\sqrt[7]{38}$  we rewrite this as  $38^{1/7}$  which can be evaluated using a scientific calculator. You may need to check your calculator manual to find the precise way of doing this, probably with the buttons  $x^y$  or  $x^{1/y}$ .

Check that you are using your calculator correctly by confirming that

$$38^{1/7} = 1.6814 \quad (4 \text{ dp})$$

More generally  $a^{m/n}$  means  $\sqrt[n]{a^m}$ , or equivalently  $(\sqrt[n]{a})^m$ .

$$a^{m/n} = \sqrt[n]{a^m} \text{ or equivalently } (\sqrt[n]{a})^m$$

### Example

$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4, \quad \text{and} \quad 32^{3/5} = (\sqrt[5]{32})^3 = 2^3 = 8$$

### Exercises

1. Use a calculator to find a)  $\sqrt[5]{96}$ , b)  $\sqrt[4]{32}$ .
2. Without using a calculator, evaluate a)  $4^{3/2}$ , b)  $27^{2/3}$ .
3. Use the third law of indices to show that

$$a^{m/n} = \sqrt[n]{a^m}$$

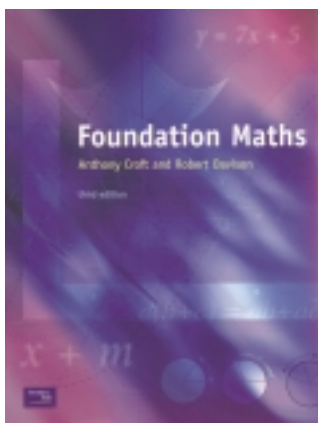
and equivalently

$$a^{m/n} = (\sqrt[n]{a})^m$$

### Answers

1. a) 2.4915, b) 2.3784. 2. a)  $4^{3/2} = 8$ , b)  $27^{2/3} = 9$ .

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 6



# Foundation Maths Support Pack

## 2.3 Removing brackets 1

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### Introduction

In order to simplify mathematical expressions it is frequently necessary to ‘remove brackets’. This means to rewrite an expression which includes bracketed terms in an equivalent form, but without any brackets. This operation must be carried out according to certain rules which are described in this leaflet.

### 1. The associativity and commutativity of multiplication

Multiplication is said to be a **commutative** operation. This means, for example, that  $4 \times 5$  has the same value as  $5 \times 4$ . Eitherway the result is 20. In symbols,  $xy$  is the same as  $yx$ , and so we can interchange the order as we wish.

Multiplication is also an **associative** operation. This means that when we want to multiply three numbers together such as  $4 \times 3 \times 5$  it doesn't matter whether we evaluate  $4 \times 3$  first and then multiply by 5, or evaluate  $3 \times 5$  first and then multiply by 4. That is

$$(4 \times 3) \times 5 \quad \text{is the same as} \quad 4 \times (3 \times 5)$$

where we have used brackets to indicate which terms are multiplied first. Eitherway, the result is the same, 60. In symbols, we have

$$(x \times y) \times z \quad \text{is the same as} \quad x \times (y \times z)$$

and since the result is the same eitherway, the brackets make no difference at all and we can write simply  $x \times y \times z$  or simply  $xyz$ . When mixing numbers and symbols we usually write the numbers first. So

$$\begin{aligned} 7 \times a \times 2 &= 7 \times 2 \times a && \text{through commutativity} \\ &= 14a \end{aligned}$$

#### Example

Remove the brackets from a)  $4(2x)$ ,      b)  $a(5b)$ .

#### Solution

a)  $4(2x)$  means  $4 \times (2 \times x)$ . Because of associativity of multiplication the brackets are unnecessary and we can write  $4 \times 2 \times x$  which equals  $8x$ .

b)  $a(5b)$  means  $a \times (5b)$ . Because of commutativity this is the same as  $(5b) \times a$ , that is  $(5 \times b) \times a$ . Because of associativity the brackets are unnecessary and we write simply  $5 \times b \times a$  which equals  $5ba$ . Note that this is also equal to  $5ab$  because of commutativity.

### Exercises

1. Simplify

a)  $9(3y)$ ,    b)  $(5x) \times (5y)$ ,    c)  $3(-2a)$ ,    d)  $-7(-9x)$ ,    e)  $12(3m)$ ,    f)  $5x(y)$ .

### Answers

1. a)  $27y$ ,    b)  $25xy$ ,    c)  $-6a$ ,    d)  $63x$ ,    e)  $36m$ ,    f)  $5xy$ .

## 2. Expressions of the form $a(b + c)$ and $a(b - c)$

Study the expression  $4 \times (2 + 3)$ . By working out the bracketed term first we obtain  $4 \times 5$  which equals 20. Note that this is the same as multiplying both the 2 and 3 separately by 4, and then adding the results. That is

$$4 \times (2 + 3) = 4 \times 2 + 4 \times 3 = 8 + 12 = 20$$

Note the way in which the '4' multiplies both bracketed numbers, '2' and '3'. We say that the '4' distributes itself over both terms in the brackets - *multiplication is distributive over addition*.

Now study the expression  $6 \times (8 - 3)$ . By working out the bracketed term first we obtain  $6 \times 5$  which equals 30. Note that this is the same as multiplying both the 8 and the 3 by 6 before carrying out the subtraction:

$$6 \times (8 - 3) = 6 \times 8 - 6 \times 3 = 48 - 18 = 30$$

Note the way in which the '6' multiplies both the bracketed numbers. We say that the '6' distributes itself over both the terms in the brackets - *multiplication is distributive over subtraction*. Exactly the same property holds when we deal with symbols.

$$a(b + c) = ab + ac \qquad a(b - c) = ab - ac$$

### Example

$4(5 + x)$  is equivalent to  $4 \times 5 + 4 \times x$  which equals  $20 + 4x$ .

$5(a - b)$  is equivalent to  $5 \times a - 5 \times b$  which equals  $5a - 5b$ .

$7(x - 2y)$  is equivalent to  $7 \times x - 7 \times 2y$  which equals  $7x - 14y$ .

$-4(5 + x)$  is equivalent to  $-4 \times 5 + -4 \times x$  which equals  $-20 - 4x$ .

$-5(a - b)$  is equivalent to  $-5 \times a - -5 \times b$  which equals  $-5a + 5b$ .

$-(a + b)$  is equivalent to  $-a - b$ .

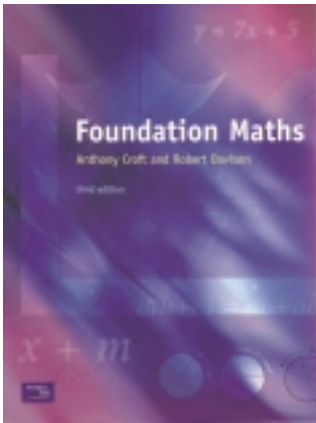
### Exercises

Remove the brackets from each of the following, simplifying your answers as appropriate.

1.  $8(3 + 2y)$ ,    2.  $7(-x + y)$ ,    3.  $-7(-x + y)$ ,    4.  $-(3 + 2x)$ ,    5.  $-(3 - 2x)$ ,  
 6.  $-(-3 - 2x)$ ,    7.  $x(x + 1)$ ,    8.  $15(x + y)$ ,    9.  $15(xy)$ ,    10.  $11(m + 3n)$ .

### Answers

1.  $24 + 16y$ ,    2.  $-7x + 7y$ ,    3.  $7x - 7y$ ,    4.  $-3 - 2x$ ,    5.  $-3 + 2x$ ,    6.  $3 + 2x$ ,    7.  $x^2 + x$ ,  
 8.  $15x + 15y$ ,    9.  $15xy$     10.  $11m + 33n$ .



# Foundation Maths Support Pack

## 2.4 Removing brackets 2

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### Introduction

In this leaflet we show the correct procedure for writing expressions of the form  $(a + b)(c + d)$  in an alternative form without brackets.

### 1. Expressions of the form $(a + b)(c + d)$

In the expression  $(a + b)(c + d)$  it is intended that each term in the first bracket multiplies each term in the second.

$$(a + b)(c + d) = ac + bc + ad + bd$$

#### Example

Removing the brackets from  $(5 + a)(2 + b)$  gives

$$5 \times 2 + a \times 2 + 5 \times b + a \times b$$

which simplifies to

$$10 + 2a + 5b + ab$$

#### Example

Removing the brackets from  $(x + 6)(x + 2)$  gives

$$x \times x + 6 \times x + x \times 2 + 6 \times 2$$

which equals

$$x^2 + 6x + 2x + 12$$

which simplifies to

$$x^2 + 8x + 12$$

#### Example

Removing the brackets from  $(x + 7)(x - 3)$  gives

$$x \times x + 7 \times x + x \times -3 + 7 \times -3$$

which equals

$$x^2 + 7x - 3x - 21$$

which simplifies to

$$x^2 + 4x - 21$$

### Example

Removing the brackets from  $(2x + 3)(x + 4)$  gives

$$2x \times x + 3 \times x + 2x \times 4 + 3 \times 4$$

which equals

$$2x^2 + 3x + 8x + 12$$

which simplifies to

$$2x^2 + 11x + 12$$

Occasionally you will need to square a bracketed expression. This can lead to errors. Study the following example.

### Example

Remove the brackets from  $(x + 1)^2$ .

### Solution

You need to be clear that when a quantity is squared it is multiplied by itself. So

$$(x + 1)^2 \quad \text{means} \quad (x + 1)(x + 1)$$

Then removing the brackets gives

$$x \times x + 1 \times x + x \times 1 + 1 \times 1$$

which equals

$$x^2 + x + x + 1$$

which simplifies to

$$x^2 + 2x + 1$$

Note that  $(x + 1)^2$  is not equal to  $x^2 + 1$ , and more generally  $(x + y)^2$  is not equal to  $x^2 + y^2$ .

### Exercises

Remove the brackets from each of the following expressions simplifying your answers where appropriate.

- a)  $(x + 2)(x + 3)$     b)  $(x - 4)(x + 1)$     c)  $(x - 1)^2$     d)  $(3x + 1)(2x - 4)$ .
- a)  $(2x - 7)(x - 1)$     b)  $(x + 5)(3x - 1)$     c)  $(2x + 1)^2$     d)  $(x - 3)^2$ .

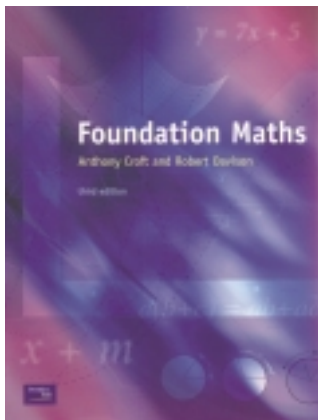
### Answers

- a)  $x^2 + 5x + 6$     b)  $x^2 - 3x - 4$     c)  $x^2 - 2x + 1$     d)  $6x^2 - 10x - 4$ .
- a)  $2x^2 - 9x + 7$     b)  $3x^2 + 14x - 5$     c)  $4x^2 + 4x + 1$     d)  $x^2 - 6x + 9$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 7





# Foundation Maths Support Pack

## 2.5 Factorising simple expressions

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### Introduction

Before studying this material you must be familiar with the process of ‘removing brackets’ as outlined on leaflets 2.3 & 2.4. This is because factorising can be thought of as reversing the process of removing brackets. When we factorise an expression it is written as a product of two or more terms, and these will normally involve brackets.

### 1. Products and Factors

To obtain the **product** of two numbers they are multiplied together. For example the product of 3 and 4 is  $3 \times 4$  which equals 12. The numbers which are multiplied together are called factors. We say that 3 and 4 are both factors of 12.

#### Example

The product of  $x$  and  $y$  is  $xy$ .

The product of  $5x$  and  $3y$  is  $15xy$ .

#### Example

$2x$  and  $5y$  are factors of  $10xy$  since when we multiply  $2x$  by  $5y$  we obtain  $10xy$ .

$(x + 1)$  and  $(x + 2)$  are factors of  $x^2 + 3x + 2$  because when we multiply  $(x + 1)$  by  $(x + 2)$  we obtain  $x^2 + 3x + 2$ .

3 and  $x - 5$  are factors of  $3x - 15$  because

$$3(x - 5) = 3x - 15$$

### 2. Common Factors

Sometimes, if we study two expressions to find their factors, we might note that some of the factors are the same. These factors are called **common factors**.

#### Example

Consider the numbers 18 and 12.

Both 6 and 3 are factors of 18 because  $6 \times 3 = 18$ .

Both 6 and 2 are factors of 12 because  $6 \times 2 = 12$ .

So, 18 and 12 share a common factor, namely 6.

In fact 18 and 12 share other common factors. Can you find them ?

### Example

The number 10 and the expression  $15x$  share a common factor of 5.

Note that  $10 = 5 \times 2$ , and  $15x = 5 \times 3x$ . Hence 5 is a common factor.

### Example

$3a^2$  and  $5a$  share a common factor of  $a$  since

$3a^2 = 3a \times a$  and  $5a = 5 \times a$ . Hence  $a$  is a common factor.

### Example

$8x^2$  and  $12x$  share a common factor of  $4x$  since

$8x^2 = 4x \times 2x$  and  $12x = 3 \times 4x$ . Hence  $4x$  is a common factor.

## 3. Factorising

To factorise an expression containing two or more terms it is necessary to look for factors which are common to the different terms. Once found, these common factors are written outside a bracketed term. It is ALWAYS possible to check your answers when you factorise by simply removing the brackets again, so you shouldn't get them wrong.

### Example

Factorise  $15x + 10$ .

### Solution

First we look for any factors which are common to both  $15x$  and 10. The common factor here is 5. So the original expression can be written

$$15x + 10 = 5(3x) + 5(2)$$

which shows clearly the common factor. This common factor is written outside a bracketed term, the remaining quantities being placed inside the bracket:

$$15x + 10 = 5(3x + 2)$$

and the expression has been factorised. We say that the factors of  $15x + 10$  are 5 and  $3x + 2$ . Your answer can be checked by showing

$$5(3x + 2) = 5(3x) + 5(2) = 15x + 10$$

### Exercises

Factorise each of the following:

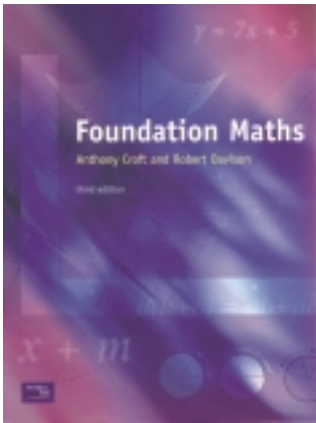
1.  $10x + 5y$ ,
2.  $21 + 7x$ ,
3.  $xy - 8x$ ,
4.  $4x - 8xy$

### Answers

1.  $5(2x + y)$ ,
2.  $7(3 + x)$ ,
3.  $x(y - 8)$ ,
4.  $4x(1 - 2y)$ .

---

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 8



# Foundation Maths Support Pack

## 2.6 Factorising quadratics

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### Introduction

On this leaflet we explain the procedure for factorising quadratic expressions such as  $x^2 + 5x + 6$ .

### 1. Factorising quadratics

You will find that you are expected to be able to factorise expressions such as  $x^2 + 5x + 6$ .

First of all note that by removing the brackets from

$$(x + 2)(x + 3)$$

we find

$$(x + 2)(x + 3) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

When we **factorise**  $x^2 + 5x + 6$  we are looking for the answer  $(x + 2)(x + 3)$ .

It is often convenient to do this by a process of educated guesswork and trial and error.

#### Example

Factorise  $x^2 + 6x + 5$ .

#### Solution

We would like to write  $x^2 + 6x + 5$  in the form

$$( \quad + \quad )( \quad + \quad )$$

First note that we can achieve the  $x^2$  term by placing an  $x$  in each bracket:

$$(x + \quad )(x + \quad )$$

The next place to look is the constant term in  $x^2 + 6x + 5$ , that is, 5. By removing the brackets you will see that this is calculated by multiplying the two numbers in the brackets together. We seek two numbers which multiply together to give 5. Clearly 5 and 1 have this property, although there are others. So

$$x^2 + 6x + 5 = (x + 5)(x + 1)$$

At this stage you should always remove the brackets again to check.

The factors of  $x^2 + 6x + 5$  are  $(x + 5)$  and  $(x + 1)$ .

### Example

Factorise  $x^2 - 6x + 5$ .

### Solution

Again we try to write the expression in the form

$$x^2 - 6x + 5 = (x + \quad)(x + \quad)$$

And again we seek two numbers which multiply to give 5. However this time 5 and 1 will not do, because using these we would obtain a middle term of  $+6x$  as we saw in the last example. Trying  $-5$  and  $-1$  will do the trick.

$$x^2 - 6x + 5 = (x - 5)(x - 1)$$

You see that some thought and perhaps a little experimentation is required.

You will need even more thought and care if the coefficient of  $x^2$ , that is the number in front of the  $x^2$ , is anything other than 1. Consider the following example.

### Example

Factorise  $2x^2 + 11x + 12$ .

### Solution

Always start by trying to obtain the correct  $x^2$  term:

We write

$$2x^2 + 11x + 12 = (2x + \quad)(x + \quad)$$

Then study the constant term 12. It has a number of pairs of factors, for example 3 and 4, 6 and 2 and so on. By trial and error you will find that the correct factorisation is

$$2x^2 + 11x + 12 = (2x + 3)(x + 4)$$

but you will only realise this by removing the brackets again.

### Exercises

Factorise each of the following:

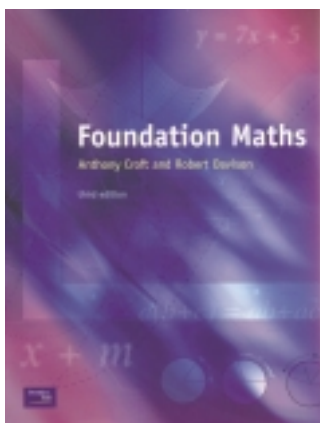
- a)  $x^2 + 5x + 4$ ,    b)  $x^2 - 5x + 4$ ,    c)  $x^2 + 3x - 4$ ,    d)  $x^2 - 3x - 4$ ,  
e)  $2x^2 - 13x - 7$ ,    f)  $2x^2 + 13x - 7$ ,    g)  $3x^2 - 2x - 1$ ,    h)  $3x^2 + 2x - 1$ ,  
i)  $6x^2 + 13x + 6$ .

### Answers

- a)  $(x + 1)(x + 4)$ ,    b)  $(x - 1)(x - 4)$ ,    c)  $(x - 1)(x + 4)$ ,    d)  $(x + 1)(x - 4)$ ,  
e)  $(2x + 1)(x - 7)$ ,    f)  $(2x - 1)(x + 7)$ ,    g)  $(3x + 1)(x - 1)$ ,    h)  $(3x - 1)(x + 1)$ ,  
i)  $(3x + 2)(2x + 3)$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 8



# Foundation Maths Support Pack

## 2.7 Simplifying fractions

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### Introduction

Fractions involving symbols occur very frequently in mathematics. It is necessary to be able to simplify these and rewrite them in different but equivalent forms. On this leaflet we revise how these processes are carried out. It will be helpful if you have already seen leaflet *1.1 Fractions*.

### 1. Expressing a fraction in its simplest form

An algebraic fraction can always be expressed in different, yet **equivalent** forms. A fraction is expressed in its **simplest form** by cancelling any **factors** which are common to both the numerator and the denominator. You need to remember that factors are multiplied together.

For example, the two fractions

$$\frac{7a}{ab} \quad \text{and} \quad \frac{7}{b}$$

are equivalent. Note that there is a common factor of  $a$  in the numerator and the denominator of  $\frac{7a}{ab}$  which can be cancelled to give  $\frac{7}{b}$ .

To express a fraction in its simplest form, any factors which are common to both the numerator and the denominator are cancelled

Notice that cancelling is equivalent to dividing the top and the bottom by the common factor.

It is also important to note that  $\frac{7}{b}$  can be converted back to the equivalent fraction  $\frac{7a}{ab}$  by multiplying both the numerator and denominator of  $\frac{7}{b}$  by  $a$ .

A fraction is expressed in an equivalent form by multiplying both top and bottom by the same quantity, or dividing top and bottom by the same quantity

#### Example

The two fractions

$$\frac{10y^2}{15y^5} \quad \text{and} \quad \frac{2}{3y^3}$$

are equivalent. Note that

$$\frac{10y^2}{15y^5} = \frac{2 \times 5 \times y \times y}{3 \times 5 \times y \times y \times y \times y}$$

and so there are common factors of 5 and  $y \times y$ . These can be cancelled to leave  $\frac{2}{3y^3}$ .

### Example

The fractions

$$\frac{(x-1)(x+3)}{(x+3)(x+5)} \quad \text{and} \quad \frac{(x-1)}{(x+5)}$$

are equivalent. In the first fraction, the common factor  $(x+3)$  can be cancelled.

### Example

The fractions

$$\frac{2a(3a-b)}{7a(a+b)} \quad \text{and} \quad \frac{2(3a-b)}{7(a+b)}$$

are equivalent. In the first fraction, the common factor  $a$  can be cancelled. Nothing else can be cancelled.

### Example

In the fraction

$$\frac{a-b}{a+b}$$

there are no common factors which can be cancelled. Neither  $a$  nor  $b$  is a factor of the numerator. Neither  $a$  nor  $b$  is a factor of the denominator.

### Example

Express  $\frac{5x}{2x+1}$  as an equivalent fraction with denominator  $(2x+1)(x-7)$ .

### Solution

To achieve the required denominator we must multiply both top and bottom by  $(x-7)$ . That is

$$\frac{5x}{2x+1} = \frac{(5x)(x-7)}{(2x+1)(x-7)}$$

If we wished, the brackets could now be removed to write the fraction as  $\frac{5x^2 - 35x}{2x^2 - 13x - 7}$ .

### Exercises

1. Express each of the following fractions in its simplest form:

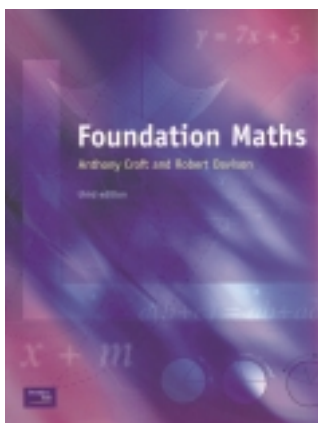
a)  $\frac{12xy}{16x}$ ,    b)  $\frac{14ab}{21a^2b^2}$ ,    c)  $\frac{3x^2y}{6x}$ ,    d)  $\frac{3(x+1)}{(x+1)^2}$ ,    e)  $\frac{(x+3)(x+1)}{(x+2)(x+3)}$ ,    f)  $\frac{100x}{45}$ ,    g)  $\frac{a+b}{ab}$ .

### Answers

1. a)  $\frac{3y}{4}$ ,    b)  $\frac{2}{3ab}$ ,    c)  $\frac{xy}{2}$ ,    d)  $\frac{3}{x+1}$ ,    e)  $\frac{x+1}{x+2}$ ,    f)  $\frac{20x}{9}$ ,    g) cannot be simplified. Whilst both  $a$  and  $b$  are factors of the denominator, neither  $a$  nor  $b$  is a factor of the numerator.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 9



# Foundation Maths Support Pack

## 2.8 Addition and subtraction

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### Introduction

Fractions involving symbols occur frequently. It is necessary to be able to add and subtract them. On this leaflet we revise how these processes are carried out. An understanding of writing fractions in equivalent forms is necessary. (See leaflet 2.7 *Simplifying fractions*.)

### 1. Addition and subtraction of fractions

To add two fractions we must first re-write each fraction so that they both have the same denominator. The denominator is called the **lowest common denominator**. It is the simplest expression which is a multiple of both of the original denominators. Then, the numerators only are added, and the result is divided by the lowest common denominator.

#### Example

Express as a single fraction

$$\frac{7}{a} + \frac{9}{b}$$

#### Solution

Both fractions must be written with the same denominator. To achieve this, note that if the numerator and denominator of the first are both multiplied by  $b$  we obtain  $\frac{7b}{ab}$ . This is equivalent to the original fraction - it is merely written in a different form. If the numerator and denominator of the second are both multiplied by  $a$  we obtain  $\frac{9a}{ab}$ . Then the problem becomes

$$\frac{7b}{ab} + \frac{9a}{ab}$$

In this form, both fractions have the same denominator. The lowest common denominator is  $ab$ .

Finally we add the numerators and divide the result by the lowest common denominator:

$$\frac{7b}{ab} + \frac{9a}{ab} = \frac{7b + 9a}{ab}$$

#### Example

Express as a single fraction

$$\frac{2}{x+3} + \frac{5}{x-1}$$

### Solution

Both fractions can be written with the same denominator if both the numerator and denominator of the first are multiplied by  $x - 1$  and if both the numerator and denominator of the second are multiplied by  $x + 3$ . This gives

$$\frac{2}{x+3} + \frac{5}{x-1} = \frac{2(x-1)}{(x+3)(x-1)} + \frac{5(x+3)}{(x+3)(x-1)}$$

Then, adding the numerators gives

$$\frac{2(x-1) + 5(x+3)}{(x+3)(x-1)}$$

which, by simplifying the numerator, gives

$$\frac{7x+13}{(x+3)(x-1)}$$

### Example

Find  $\frac{3}{x+1} + \frac{2}{(x+1)^2}$

### Solution

The simplest expression which is a multiple of the original denominators is  $(x+1)^2$ . This is the lowest common denominator. Both fractions must be written with this denominator.

$$\frac{3}{x+1} + \frac{2}{(x+1)^2} = \frac{3(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}$$

Adding the numerators and simplifying we find

$$\frac{3(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2} = \frac{3x+3+2}{(x+1)^2} = \frac{3x+5}{(x+1)^2}$$

### Exercises

1. Express each of the following as a single fraction:

a)  $\frac{3}{4} + \frac{1}{x}$ ,    b)  $\frac{1}{a} - \frac{2}{5b}$ ,    c)  $\frac{2}{x^2} + \frac{1}{x}$ ,    d)  $2 + \frac{1}{3x}$ .

2. Express as a single fraction:

a)  $\frac{2}{x+1} + \frac{3}{x+2}$ ,    b)  $\frac{2}{x+3} + \frac{5}{(x+3)^2}$ ,    c)  $\frac{3x}{x-1} + \frac{1}{x}$ ,    d)  $\frac{1}{x-5} - \frac{3}{x+2}$ ,    e)  $\frac{1}{2x+1} - \frac{7}{x+3}$ .

### Answers

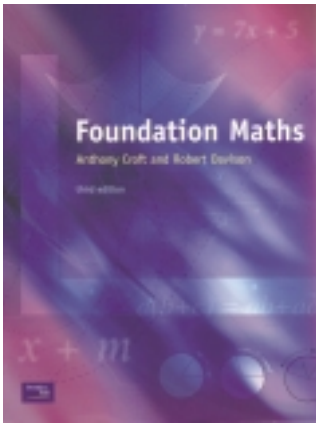
1. a)  $\frac{3x+4}{4x}$ ,    b)  $\frac{5b-2a}{5ab}$ ,    c)  $\frac{2+x}{x^2}$ ,    d)  $\frac{6x+1}{3x}$ .

2. a)  $\frac{5x+7}{(x+1)(x+2)}$ ,    b)  $\frac{2x+11}{(x+3)^2}$ ,    c)  $\frac{3x^2+x-1}{x(x-1)}$ ,    d)  $\frac{17-2x}{(x+2)(x-5)}$ ,    e)  $-\frac{13x+4}{(x+3)(2x+1)}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 9





# Foundation Maths Support Pack

## 2.9 Multiplication and division

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### Introduction

Fractions involving symbols occur frequently. It is necessary to be able to multiply and divide them. On this leaflet we revise how these processes are carried out. It will be helpful if you have already seen leaflet *1.1 Fractions*.

### 1. Multiplication and division of fractions

Multiplication of fractions is straightforward. We simply multiply the numerators to give a new numerator, and multiply the denominators to give a new denominator.

#### Example

Find

$$\frac{4}{7} \times \frac{a}{b}$$

#### Solution

Simply multiply the two numerators together, and multiply the two denominators together.

$$\frac{4}{7} \times \frac{a}{b} = \frac{4a}{7b}$$

#### Example

Find

$$\frac{3ab}{5} \times \frac{7}{6a}$$

#### Solution

$$\frac{3ab}{5} \times \frac{7}{6a} = \frac{21ab}{30a}$$

which, by cancelling common factors, can be simplified to  $\frac{7b}{10}$ .

Division is performed by inverting the second fraction and then multiplying.

#### Example

Find  $\frac{3}{2x} \div \frac{6}{5y}$ .

## Solution

$$\begin{aligned}\frac{3}{2x} \div \frac{6}{5y} &= \frac{3}{2x} \times \frac{5y}{6} \\ &= \frac{15y}{12x} \\ &= \frac{5y}{4x}\end{aligned}$$

## Example

Find  $\frac{3}{x+1} \div \frac{x}{(x+1)^2}$ .

## Solution

$$\begin{aligned}\frac{3}{x+1} \div \frac{x}{(x+1)^2} &= \frac{3}{x+1} \times \frac{(x+1)^2}{x} \\ &= \frac{3(x+1)^2}{x(x+1)} \\ &= \frac{3(x+1)}{x}\end{aligned}$$

## Exercises

1. Find a)  $\frac{1}{3} \times \frac{x}{2}$ ,    b)  $\frac{2}{x+1} \times \frac{x}{x-3}$ ,    c)  $-\frac{1}{4} \times \frac{3}{5}$ ,    d)  $\left(-\frac{1}{x}\right) \times \left(\frac{2}{5y}\right)$ ,    e)  $\frac{x+1}{2(x+3)} \times \frac{8}{x+1}$ .

2. Simplify

$$\frac{3}{x+2} \div \frac{x}{2x+4}$$

3. Simplify

$$\frac{x+2}{(x+5)(x+4)} \times \frac{x+5}{x+2}$$

4. Simplify

$$\frac{3}{x} \times \frac{3}{y} \times \frac{1}{z}$$

5. Find  $\frac{4}{3} \div \frac{16}{x}$ .

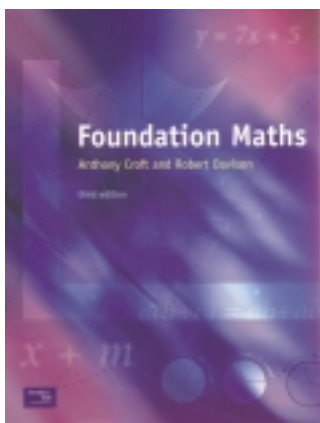
## Answers

1. a)  $\frac{x}{6}$ ,    b)  $\frac{2x}{(x+1)(x-3)}$ ,    c)  $-\frac{3}{20}$ ,    d)  $-\frac{2}{5xy}$ ,    e)  $\frac{4}{x+3}$ .

2.  $\frac{6}{x}$ ,    3.  $\frac{1}{x+4}$ ,    4.  $\frac{9}{xyz}$ ,    5.  $\frac{x}{12}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 9



# Foundation Maths Support Pack

## 2.10 Rearranging formulas 1

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### Introduction

The ability to rearrange formulas or rewrite them in different ways is an important mathematical skill. This leaflet will explain how to rearrange some simple formulas. Leaflet 2.11 deals with more complicated examples.

### 1. The subject of a formula

Many science students will be familiar with Ohm's law which states that  $V = IR$ . Here,  $V$  is a voltage drop,  $R$  is a resistance and  $I$  is a current. If the values of  $R$  and  $I$  are known then the formula  $V = IR$  enables us to calculate the value of  $V$ . In the form  $V = IR$ , we say that the **subject** of the formula is  $V$ . Usually the subject of a formula is on its own on the left-hand side. You may also be familiar with Ohm's law written in either of the forms

$$I = \frac{V}{R} \quad \text{and} \quad R = \frac{V}{I}$$

In the first case  $I$  is the subject of the formula whilst in the second case  $R$  is the subject. If we know values of  $V$  and  $R$  we can use  $I = \frac{V}{R}$  to find  $I$ . On the other hand, if we know values of  $V$  and  $I$  we can use  $R = \frac{V}{I}$  to find  $R$ . So you see, it is important to be able to write formulas in different ways, so that we can make a particular variable the subject.

### 2. Rules for rearranging, or transposing, a formula

You can think of a formula as a pair of balanced scales. The quantity on the left is equal to the quantity on the right. If we add an amount to one side of the scale pans, say the left one, then to keep balance we must add the same amount to the pan on the right. Similarly if we take away an amount from the left, we must take the same amount away from the pan on the right. The same applies to formulas. If we add an amount to one side, we must add the same to the other to keep the formula valid. If we subtract an amount from one side we must subtract the same amount from the other. Furthermore, if we multiply the left by any amount, we must multiply the right by the same amount. If we divide the left by any amount we must divide the right by the same amount. When you are trying to rearrange, or **transpose**, a formula, keep these operations clearly in mind.

To transpose or rearrange a formula you may

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity

Later, we shall see that a further group of operations is allowed, but first get some practice with these Examples and Exercises.

**Example**

Rearrange the formula  $y = x + 8$  in order to make  $x$  the subject instead of  $y$ .

**Solution**

To make  $x$  the subject we must remove the 8 from the right. So, we subtract 8 from the right, but we remember that we must do the same to the left. So

$$\begin{aligned} \text{if } y &= x + 8, && \text{subtracting 8 yields} \\ y - 8 &= x + 8 - 8 \\ y - 8 &= x \end{aligned}$$

We have  $x$  on its own, although it is on the right. This is no problem since if  $y - 8$  equals  $x$ , then  $x$  equals  $y - 8$ , that  $x = y - 8$ . We have succeeded in making  $x$  the subject of the formula.

**Example**

Rearrange the formula  $y = 3x$  to make  $x$  the subject.

**Solution**

The reason why  $x$  does not appear on its own is that it is multiplied by 3. If we divide  $3x$  by 3 we obtain  $\frac{3x}{3} = x$ . So, we can obtain  $x$  on its own by dividing both sides of the formula by 3.

$$\begin{aligned} y &= 3x \\ \frac{y}{3} &= \frac{3x}{3} \\ &= x \end{aligned}$$

Finally  $x = \frac{y}{3}$  and we have succeeded in making  $x$  the subject of the formula.

**Example**

Rearrange  $y = 11 + 7x$  to make  $x$  the subject.

**Solution**

Starting from  $y = 11 + 7x$  we subtract 11 from each side to give  $y - 11 = 7x$ . Then, dividing both sides by 7 gives  $\frac{y-11}{7} = x$ . Finally  $x = \frac{y-11}{7}$ .

**Exercises**

1. Transpose each of the following formulas to make  $x$  the subject.

a)  $y = x - 7$ ,      b)  $y = 2x - 7$ ,      c)  $y = 2x + 7$ ,      d)  $y = 7 - 2x$ ,      e)  $y = \frac{x}{5}$ .

2. Transpose each of the following formulas to make  $v$  the subject.

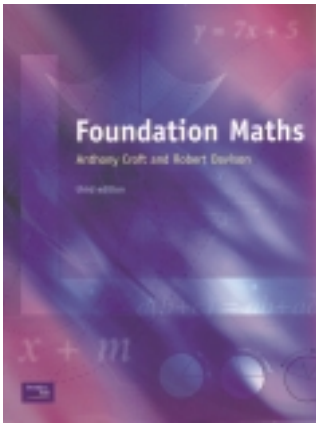
a)  $w = 3v$ ,      b)  $w = \frac{1}{3}v$ ,      c)  $w = \frac{v}{3}$ ,      d)  $w = \frac{2v}{3}$ ,      e)  $w = \frac{2}{3}v$ .

**Answers**

1. a)  $x = y + 7$ ,      b)  $x = \frac{y+7}{2}$ ,      c)  $x = \frac{y-7}{2}$ ,      d)  $x = \frac{7-y}{2}$ ,      e)  $x = 5y$ .

2. a)  $v = \frac{w}{3}$ ,      b)  $v = 3w$ ,      c) same as b),      d)  $v = \frac{3w}{2}$ ,      e) same as d).

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 10



# Foundation Maths Support Pack

## 2.11 Rearranging formulas 2

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### Introduction

This leaflet develops the work started on leaflet 2.11, and shows how more complicated formulas can be rearranged.

### 1. Further transposition

Remember that when you are trying to rearrange, or **transpose**, a formula, the following operations are allowed.

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity

A further group of operations is also permissible.

A formula remains balanced if we perform the same operation to both sides of it. For example, we can square both sides, we can square-root both sides. We can find the logarithm of both sides. Study the following examples.

#### Example

Transpose the formula  $p = \sqrt{q}$  to make  $q$  the subject.

#### Solution

Here we need to obtain  $q$  on its own. To do this we must find a way of removing the square root sign. This can be achieved by squaring both sides since

$$(\sqrt{q})^2 = q$$

So,

$$\begin{aligned} p &= \sqrt{q} \\ p^2 &= q \quad \text{by squaring both sides} \end{aligned}$$

Finally,  $q = p^2$ , and we have succeeded in making  $q$  the subject of the formula.

#### Example

Transpose  $p = \sqrt{a + b}$  to make  $b$  the subject.

**Solution**

$$\begin{aligned}
 p &= \sqrt{a+b} \\
 p^2 &= a+b && \text{by squaring both sides} \\
 p^2 - a &= b
 \end{aligned}$$

Finally,  $b = p^2 - a$ , and we have succeeded in making  $b$  the subject of the formula.

**Example**

Make  $x$  the subject of the formula  $v = \frac{k}{\sqrt{x}}$ .

**Solution**

$$\begin{aligned}
 v &= \frac{k}{\sqrt{x}} \\
 v^2 &= \frac{k^2}{x} && \text{by squaring both sides} \\
 xv^2 &= k^2 && \text{by multiplying both sides by } x \\
 x &= \frac{k^2}{v^2} && \text{by dividing both sides by } v^2
 \end{aligned}$$

and we have succeeded in making  $x$  the subject of the formula.

**Example**

Transpose the formula  $T = 2\pi\sqrt{\frac{\ell}{g}}$  for  $\ell$ .

**Solution**

This must be carried out carefully, in stages, until we obtain  $\ell$  on its own.

$$\begin{aligned}
 T &= 2\pi\sqrt{\frac{\ell}{g}} \\
 \frac{T}{2\pi} &= \sqrt{\frac{\ell}{g}} && \text{by dividing both sides by } 2\pi \\
 \left(\frac{T}{2\pi}\right)^2 &= \frac{\ell}{g} && \text{by squaring both sides} \\
 \ell &= g\left(\frac{T}{2\pi}\right)^2
 \end{aligned}$$

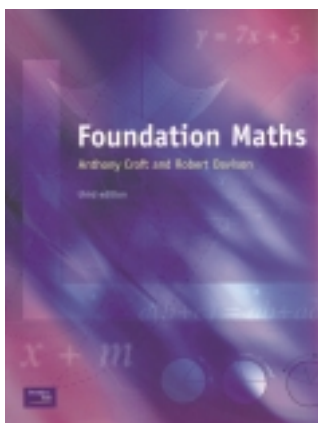
**Exercises**

1. Make  $r$  the subject of the formula  $V = \frac{4}{3}\pi r^3$ .
2. Make  $x$  the subject of the formula  $y = 4 - x^2$ .
3. Make  $s$  the subject of the formula  $v^2 = u^2 + 2as$

**Answers**

1.  $r = \sqrt[3]{\frac{3V}{4\pi}}$ .
2.  $x = \sqrt{4 - y}$ .
3.  $s = \frac{v^2 - u^2}{2a}$ .

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 10



# Foundation Maths Support Pack

## 2.12 Solving linear equations

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### Introduction

**Equations** always involve one or more unknown quantities which we try to find when we **solve** the equation. The simplest equations to deal with are **linear equations**. On this leaflet we describe how these are solved.

### 1. A linear equation

Linear equations are those which can be written in the form

$$ax + b = 0$$

where  $x$  is the unknown value, and  $a$  and  $b$  are known numbers. The following are all examples of linear equations.

$$3x + 2 = 0, \quad -5x + 11 = 0, \quad 3x - 11 = 0$$

The unknown does not have to have the symbol  $x$ , other letters can be used.

$$3t - 2 = 0, \quad 7z + 11 = 0, \quad 3w = 0$$

are all linear equations.

Sometimes you will come across a linear equation which at first sight might not appear to have the form  $ax + b = 0$ . The following are all linear equations. If you have some experience of solving linear equations, or of transposing formulas, you will be able to check that they can all be written in the standard form.

$$\frac{x - 7}{2} + 11 = 0, \quad \frac{2}{x} = 8, \quad 6x - 2 = 9$$

### 2. Solving a linear equation

To solve a linear equation it will be helpful if you know already how to transpose or rearrange formulas. (See leaflets 2.10 & 2.11 *Rearranging formulas*, for information about this if necessary).

When solving a linear equation we try to make the unknown quantity the subject of the equation. To do this we may

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity

**Example**

Solve the equation  $x + 7 = 18$ .

**Solution**

We try to obtain  $x$  on its own on the left hand side.

$$\begin{aligned}x + 7 &= 18 \\x &= 18 - 7 && \text{by subtracting 7 from both sides} \\x &= 11\end{aligned}$$

We have solved the equation and found the solution:  $x = 11$ . The solution is that value of  $x$  which can be substituted into the original equation to make both sides the same. You can, and should, check this. Substituting  $x = 11$  in the left-hand side of the equation  $x + 7 = 18$  we find  $11 + 7$  which equals 18, the same as the right-hand side.

**Example**

Solve the equation  $5x + 11 = 22$ .

**Solution**

$$\begin{aligned}5x + 11 &= 22 \\5x &= 22 - 11 && \text{by subtracting 11 from both sides} \\x &= \frac{11}{5} && \text{by dividing both sides by 5}\end{aligned}$$

**Example**

Solve the equation  $13x - 2 = 11x + 17$ .

**Solution**

$$\begin{aligned}13x - 2 &= 11x + 17 \\13x - 11x - 2 &= 17 && \text{by subtracting } 11x \text{ from both sides} \\2x - 2 &= 17 \\2x &= 17 + 2 && \text{by adding 2 to both sides} \\2x &= 19 \\x &= \frac{19}{2}\end{aligned}$$

**Exercises**

1. Solve the following linear equations.

a)  $4x + 8 = 0$ ,   b)  $3x - 11 = 2$ ,   c)  $8(x + 3) = 64$ ,   d)  $7(x - 5) = -56$ ,   e)  $3c - 5 = 14c - 27$ .

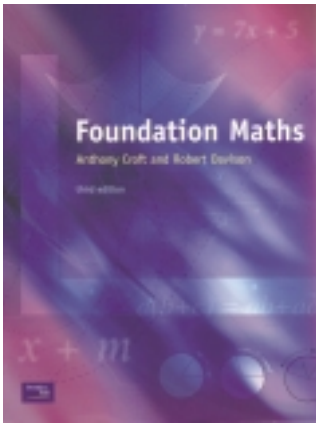
**Answers**

1. a)  $x = -2$ ,   b)  $x = \frac{13}{3}$ ,   c)  $x = 5$ ,   d)  $x = -3$ ,   e)  $c = 2$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 11





# Foundation Maths Support Pack

## 2.13 Simultaneous equations

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### Introduction

On occasions you will come across two or more unknown quantities, and two or more equations relating them. These are called **simultaneous equations** and when asked to solve them you must find values of the unknowns which satisfy all the given equations at the same time. On this leaflet we illustrate one way in which this can be done.

### 1. The solution of a pair of simultaneous equations

The solution of the pair of simultaneous equations

$$3x + 2y = 36, \quad \text{and} \quad 5x + 4y = 64$$

is  $x = 8$  and  $y = 6$ . This is easily verified by substituting these values into the left-hand sides to obtain the values on the right. So  $x = 8$ ,  $y = 6$  **satisfy** the simultaneous equations.

### 2. Solving a pair of simultaneous equations

There are many ways of solving simultaneous equations. Perhaps the simplest way is **elimination**. This is a process which involves removing or eliminating one of the unknowns to leave a single equation which involves the other unknown. The method is best illustrated by example.

#### Example

Solve the simultaneous equations

$$\begin{array}{rcl} 3x + 2y & = & 36 \quad (1) \\ 5x + 4y & = & 64 \quad (2) \end{array} \cdot$$

#### Solution

Notice that if we multiply both sides of the first equation by 2 we obtain an equivalent equation

$$6x + 4y = 72 \quad (3)$$

Now, if equation (2) is subtracted from equation (3) the terms involving  $y$  will be eliminated:

$$\begin{array}{rcl} 6x + 4y & = & 72 \quad (3) \\ 5x + 4y & = & 64 \quad (2) \\ \hline x + 0y & = & 8 \end{array}$$

So,  $x = 8$  is part of the solution. Taking equation (1) (or if you wish, equation (2)) we substitute this value for  $x$ , which will enable us to find  $y$ :

$$\begin{aligned} 3(8) + 2y &= 36 \\ 24 + 2y &= 36 \\ 2y &= 36 - 24 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

Hence the full solution is  $x = 8, y = 6$ .

You will notice that the idea behind this method is to multiply one (or both) equations by a suitable number so that either the number of  $y$ 's or the number of  $x$ 's are the same, so that subtraction eliminates that unknown. It may also be possible to eliminate an unknown by addition, as shown in the next example.

### Example

Solve the simultaneous equations 
$$\begin{aligned} 5x - 3y &= 26 & (1) \\ 4x + 2y &= 34 & (2) \end{aligned}$$

### Solution

There are many ways that the elimination can be carried out. Suppose we choose to eliminate  $y$ . The number of  $y$ 's in both equations can be made the same by multiplying equation (1) by 2 and equation (2) by 3. This gives

$$\begin{aligned} 10x - 6y &= 52 & (3) \\ 12x + 6y &= 102 & (4) \end{aligned}$$

If these equations are now added we find

$$\begin{array}{r} 10x - 6y = 52 \quad + \quad (3) \\ 12x + 6y = 102 \quad (4) \\ \hline 22x + 0y = 154 \end{array}$$

so that  $x = \frac{154}{22} = 7$ . Substituting this value for  $x$  in equation (1) gives

$$\begin{aligned} 5(7) - 3y &= 26 \\ 35 - 3y &= 26 \\ -3y &= 26 - 35 \\ -3y &= -9 \\ y &= 3 \end{aligned}$$

Hence the full solution is  $x = 7, y = 3$ .

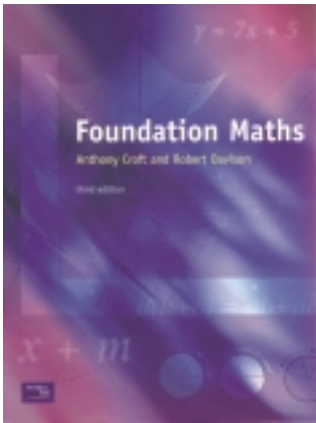
### Exercises

Solve the following pairs of simultaneous equations:

a)  $\begin{aligned} 7x + y &= 25 \\ 5x - y &= 11 \end{aligned}$ ,    b)  $\begin{aligned} 8x + 9y &= 3 \\ x + y &= 0 \end{aligned}$ ,    c)  $\begin{aligned} 2x + 13y &= 36 \\ 13x + 2y &= 69 \end{aligned}$     d)  $\begin{aligned} 7x - y &= 15 \\ 3x - 2y &= 19 \end{aligned}$

### Answers

a)  $x = 3, y = 4$ .    b)  $x = -3, y = 3$ .    c)  $x = 5, y = 2$ .    d)  $x = 1, y = -8$ .



# Foundation Maths Support Pack

## 2.14 Quadratic equations 1

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### Introduction

This leaflet will explain how many quadratic equations can be solved by **factorisation**.

### 1. Quadratic equations

A **quadratic equation** is an equation of the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$  and  $c$  are constants. For example  $3x^2 + 2x - 9 = 0$  is a quadratic equation with  $a = 3$ ,  $b = 2$  and  $c = -9$ . The constants  $b$  and  $c$  can have any value including 0. The constant  $a$  can have any value except 0. This is to ensure that the equation has an  $x^2$  term. We often refer to  $a$  as the coefficient of  $x^2$ , to  $b$  as the coefficient of  $x$  and to  $c$  as the constant term. Usually,  $a$ ,  $b$  and  $c$  are known numbers, whilst  $x$  represents an unknown quantity which we will be trying to find.

### 2. The solutions of a quadratic equation

To **solve** a quadratic equation we must find values for  $x$  which when substituted into the equation make the left-hand and right-hand sides equal. These values are also called **roots**. For example the value  $x = 4$  is a solution of the equation  $x^2 - 3x - 4 = 0$  because substituting 4 for  $x$  we find

$$4^2 - 3(4) - 4 = 16 - 12 - 4$$

which simplifies to zero, the same as the right-hand side of the equation. There are several techniques which can be used to solve quadratic equations. One of these, *factorisation*, is discussed in this leaflet. You should be aware that not all quadratic equations can be solved by this method. An alternative method which uses a formula is described on leaflet 2.15.

### 3. Solving a quadratic equation by factorisation.

Sometimes, but not always, it is possible to solve a quadratic equation using factorisation. If you need to revise factorisation you should see leaflet *2.6 Factorising quadratics*.

#### Example

Solve the equation  $x^2 + 7x + 12 = 0$  by factorisation.

#### Solution

We first factorise  $x^2 + 7x + 12$  as  $(x + 3)(x + 4)$ . Then the equation becomes  $(x + 3)(x + 4) = 0$ . It is important that you realise that if the product of two quantities is zero, then one or both of the quantities must be zero. It follows that either

$$x + 3 = 0, \text{ that is } x = -3 \quad \text{or} \quad x + 4 = 0, \text{ that is } x = -4$$

The roots of  $x^2 + 7x + 12 = 0$  are  $x = -3$  and  $x = -4$ .

### Example

Solve the quadratic equation  $x^2 + 4x - 21 = 0$ .

### Solution

$x^2 + 4x - 21$  can be factorised as  $(x + 7)(x - 3)$ . Then

$$\begin{aligned}x^2 + 4x - 21 &= 0 \\(x + 7)(x - 3) &= 0\end{aligned}$$

Then either

$$x + 7 = 0, \text{ that is } x = -7 \quad \text{or} \quad x - 3 = 0, \text{ that is } x = 3$$

The roots of  $x^2 + 4x - 21 = 0$  are  $x = -7$  and  $x = 3$ .

### Example

Find the roots of the quadratic equation  $x^2 - 10x + 25 = 0$ .

### Solution

$$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$$

Then

$$\begin{aligned}x^2 - 10x + 25 &= 0 \\(x - 5)^2 &= 0 \\x &= 5\end{aligned}$$

There is one root,  $x = 5$ . Such a root is called a **repeated root**.

### Example

Solve the quadratic equation  $2x^2 + 3x - 2 = 0$ .

### Solution

The equation is factorised to give

$$(2x - 1)(x + 2) = 0$$

so, from  $2x - 1 = 0$  we find  $2x = 1$ , that is  $x = \frac{1}{2}$ . From  $x + 2 = 0$  we find  $x = -2$ . The two solutions are therefore  $x = \frac{1}{2}$  and  $x = -2$ .

### Exercises

1. Solve the following quadratic equations by factorization.

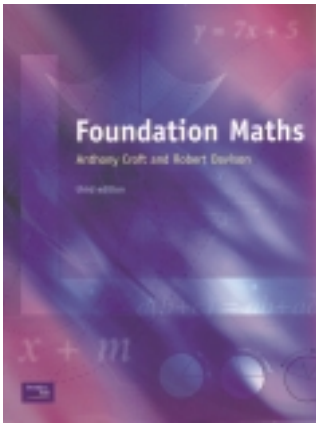
- a)  $x^2 + 7x + 6 = 0$       b)  $x^2 - 8x + 15 = 0$       c)  $x^2 - 9x + 14 = 0$       d)  $2x^2 - 5x - 3 = 0$   
e)  $6x^2 - 11x - 10 = 0$       f)  $6x^2 + 13x + 6 = 0$ .

### Answers

- a)  $-1, -6$ ,    b)  $3, 5$ ,    c)  $2, 7$ ,    d)  $3, -\frac{1}{2}$ ,    e)  $\frac{5}{2}, -\frac{2}{3}$ ,    f)  $x = -\frac{3}{2}, x = -\frac{2}{3}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 11



# Foundation Maths Support Pack

## 2.15 Quadratic equations 2

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### Introduction

This leaflet will explain how quadratic equations can be solved using a formula.

### 1. Solving a quadratic equation using a formula.

Any quadratic equation can be solved using the **quadratic formula**.

If

$$ax^2 + bx + c = 0$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A quadratic equation has two solutions; one obtained using the positive square root in the formula, and the other obtained using the negative square root. The answers are often referred to as **roots** of the equation.

#### Example.

Solve the quadratic equation

$$3x^2 + 9x + 4 = 0$$

#### Solution

Here  $a = 3$ ,  $b = 9$  and  $c = 4$ . Putting these values into the quadratic formula gives

$$\begin{aligned}x &= \frac{-9 \pm \sqrt{9^2 - 4(3)(4)}}{2(3)} \\&= \frac{-9 \pm \sqrt{81 - 48}}{6} \\&= \frac{-9 \pm \sqrt{33}}{6} \\&= \frac{-9 - \sqrt{33}}{6}, \frac{-9 + \sqrt{33}}{6} \\&= -2.4574, -0.5426 \quad (4\text{dp})\end{aligned}$$

The roots of  $3x^2 + 9x + 4 = 0$  are  $x = -2.4574$  and  $x = -0.5426$ .

### Example

Solve the equation  $8x^2 + 3x - 4 = 0$ .

### Solution

Care is needed here because the value of  $c$  is negative, that is  $c = -4$ .

$$\begin{aligned}x &= \frac{-3 \pm \sqrt{3^2 - 4(8)(-4)}}{(2)(8)} \\&= \frac{-3 \pm \sqrt{137}}{16} \\&= 0.5440, -0.9190 \quad (4\text{dp})\end{aligned}$$

### Example

Find the roots of the quadratic equation  $9x^2 + 6x + 1 = 0$ .

### Solution

Here  $a = 9$ ,  $b = 6$  and  $c = 1$ . Using the quadratic formula we have

$$\begin{aligned}x &= \frac{-6 \pm \sqrt{6^2 - 4(9)(1)}}{2(9)} \\&= \frac{-6 \pm \sqrt{36 - 36}}{18} \\&= \frac{-6 \pm \sqrt{0}}{18} \\&= -\frac{6}{18} \\&= -\frac{1}{3}\end{aligned}$$

In this example there is only one root:  $x = -\frac{1}{3}$ .

The quantity  $b^2 - 4ac$  is called the **discriminant** of the equation. When the discriminant is 0, as in the previous Example, the equation has only one root. If the discriminant is negative we are faced with the problem of finding the square root of a negative number. Such equations require special treatment using what are called *complex numbers*.

### Exercises

1. Find the roots of the following quadratic equations:

a)  $x^2 + 6x - 8 = 0$ ,      b)  $2x^2 - 8x - 3 = 0$ ,      c)  $-3x^2 + x + 1 = 0$ .

### Answers

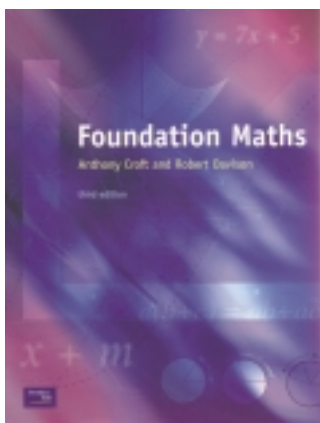
a)  $x = -3 \pm \sqrt{17} = 1.123, -7.123$  (3dp).

b)  $x = 2 \pm \frac{\sqrt{22}}{2} = 4.345, -0.345$  (3dp).

c)  $x = \frac{1}{6} \pm \frac{\sqrt{13}}{6} = 0.768, -0.434$  (3dp).

---

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 11



# Foundation Maths Support Pack

## 2.16 What is a logarithm ?

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materials to support *Foundation Maths Third Edition* by Anthony Croft & Robert Davison, ISBN 0 130 45426-5

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### Introduction

We use logarithms to write expressions involving powers in a different form. If you can work confidently with powers you should have no problems handling logarithms.

### 1. Logarithms

Consider the statement  $100 = 10^2$ . In this statement we say that 10 is the **base** and 2 is the **power** or **index**.

**Logarithms** are simply an alternative way of writing a statement such as this. We rewrite it as

$$\log_{10} 100 = 2$$

This is read as 'log to the base 10 of 100 is 2'.

As another example, since

$$2^5 = 32$$

we can write

$$\log_2 32 = 5$$

More generally,

if $a = b^c$ , then $\log_b a = c$
------------------------------------

The only restriction that is placed on the value of the base is that it is a positive real number excluding the number 1. In practice logarithms are calculated using only a few common bases. Most frequently you will meet bases 10 and e. The letter e stands for the number 2.718... and is used because it is found to occur in the mathematical description of many physical phenomena. Your calculator will be able to calculate logarithms to bases 10 and e. Usually the 'log' button is used for base 10, and the 'ln' button is used for base e. ('ln' stands for 'natural logarithm'). Check that you can use your calculator correctly by verifying that

$$\log_{10} 73 = 1.8633, \quad \text{and} \quad \log_e 5.64 = 1.7299$$

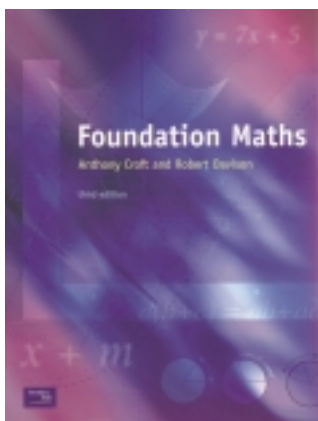
You may also like to verify the alternative forms

$$10^{1.8633} = 73 \quad \text{and} \quad e^{1.7299} = 5.64$$

Occasionally we need to find logs to other bases. For example, logs to the base 2 are used in communications engineering. Your calculator can still be used but we need to apply a formula for changing the base. This is dealt with on the leaflet *2.18 Bases other than 10 and e*.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 17



# Foundation Maths Support Pack

## 2.17 The laws of logarithms

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### Introduction

There are a number of rules known as the **laws of logarithms**. These allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base but the same base must be used throughout a calculation.

### 1. The laws of logarithms

The three main laws are stated here:

#### First Law

$$\log A + \log B = \log AB$$

This law tells us how to add two logarithms together. Adding  $\log A$  and  $\log B$  results in the logarithm of the product of  $A$  and  $B$ , that is  $\log AB$ .

For example, we can write

$$\log_{10} 5 + \log_{10} 4 = \log_{10}(5 \times 4) = \log_{10} 20$$

The same base, in this case 10, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

#### Second Law

$$\log A - \log B = \log \frac{A}{B}$$

So, subtracting  $\log B$  from  $\log A$  results in  $\log \frac{A}{B}$ .

For example, we can write

$$\log_e 12 - \log_e 2 = \log_e \frac{12}{2} = \log_e 6$$

The same base, in this case  $e$ , is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

#### Third Law

$$\log A^n = n \log A$$

So, for example

$$\log_{10} 5^3 = 3 \log_{10} 5$$

You should verify this by evaluating both sides separately on your calculator.

Two other important results are



$$\log 1 = 0, \quad \log_m m = 1$$

The logarithm of 1 to any base is always 0, and the logarithm of a number to the same base is always 1. In particular,

$$\log_{10} 10 = 1, \quad \text{and} \quad \log_e e = 1$$

### Exercises

1. Use the first law to simplify the following.

- a)  $\log_{10} 6 + \log_{10} 3$ ,
- b)  $\log x + \log y$ ,
- c)  $\log 4x + \log x$ ,
- d)  $\log a + \log b^2 + \log c^3$ .

2. Use the second law to simplify the following.

- a)  $\log_{10} 6 - \log_{10} 3$ ,
- b)  $\log x - \log y$ ,
- c)  $\log 4x - \log x$ .

3. Use the third law to write each of the following in an alternative form.

- a)  $3 \log_{10} 5$ ,
- b)  $2 \log x$ ,
- c)  $\log(4x)^2$ ,
- d)  $5 \ln x^4$ ,
- e)  $\ln 1000$ .

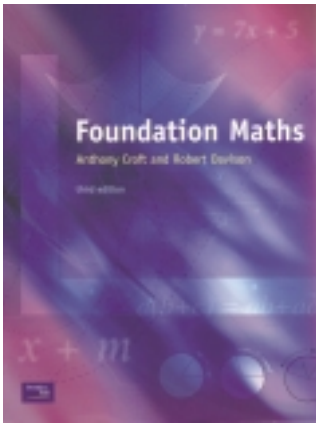
4. Simplify  $3 \log x - \log x^2$ .

### Answers

- 1. a)  $\log_{10} 18$ , b)  $\log xy$ , c)  $\log 4x^2$ , d)  $\log ab^2c^3$ .
- 2. a)  $\log_{10} 2$ , b)  $\log \frac{x}{y}$ , c)  $\log 4$ .
- 3. a)  $\log_{10} 5^3$  or  $\log_{10} 125$ , b)  $\log x^2$ , c)  $2 \log(4x)$ , d)  $20 \ln x$  or  $\ln x^{20}$ , e)  $1000 = 10^3$  so  $\ln 1000 = 3 \ln 10$ .
- 4.  $\log x$ .

---

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 17



# Foundation Maths Support Pack

## 2.18 Bases other than 10 and e

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### Introduction

Occasionally you may need to find logarithms to bases other than 10 and e. For example, logarithms to the base 2 are used in communications engineering and information technology. Your calculator can still be used but we need to apply a formula for changing the base. This leaflet gives this formula and shows how to use it.

### 1. A formula for change of base

Suppose we want to calculate a logarithm to base 2. The formula states

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2}$$

So we can calculate base 2 logarithms using base 10 logarithms obtained using a calculator. For example

$$\log_2 36 = \frac{\log_{10} 36}{\log_{10} 2} = \frac{1.5563}{0.3010} = 5.1704$$

Check this for yourself.

More generally, for any bases  $a$  and  $b$ ,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, by choosing  $b = 10$  we find

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

Use this formula to check that  $\log_{20} 100 = 1.5372$ .

### Exercises

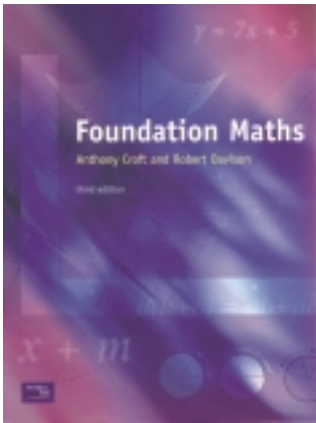
1. Find a)  $\log_2 15$ ,    b)  $\log_2 56.25$ ,    c)  $\log_3 16$ .

### Answers

1. a) 3.907 (3dp),    b) 5.814 (3dp),    c) 2.524 (3dp).

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 17



# Foundation Maths Support Pack

## 2.19 Sigma notation

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### Introduction

**Sigma notation**,  $\Sigma$ , provides a concise and convenient way of writing long sums. This leaflet explains how.

### 1. Sigma notation

The sum

$$1 + 2 + 3 + 4 + 5 + \dots + 10 + 11 + 12$$

can be written very concisely using the capital Greek letter  $\Sigma$  as

$$\sum_{k=1}^{k=12} k$$

The  $\Sigma$  stands for a sum, in this case the sum of all the values of  $k$  as  $k$  ranges through all whole numbers from 1 to 12. Note that the lower-most and upper-most values of  $k$  are written at the bottom and top of the sigma sign respectively. You may also see this written as  $\sum_{k=1}^{k=12} k$ , or even as  $\sum_{k=1}^{12} k$ .

#### Example

Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

#### Solution

We must let  $k$  range from 1 to 5, cube each value of  $k$ , and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

#### Example

Express  $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$  concisely using sigma notation.

#### Solution

Each term takes the form  $\frac{1}{k}$  where  $k$  varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

### Example

The sum

$$x_1 + x_2 + x_3 + x_4 + \dots + x_{19} + x_{20}$$

can be written

$$\sum_{k=1}^{k=20} x_k$$

There is nothing special about using the letter  $k$ . For example

$$\sum_{n=1}^{n=7} n^2 \quad \text{stands for} \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

We can also use a little trick to alternate the signs of the numbers between  $+$  and  $-$ . Note that  $(-1)^2 = 1$ ,  $(-1)^3 = -1$  and so on.

### Example

Write out fully what is meant by

$$\sum_{i=0}^5 \frac{(-1)^{i+1}}{2i+1}$$

### Solution

$$\sum_{i=0}^5 \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

### Exercises

1. Write out fully what is meant by

- a)  $\sum_{i=1}^{i=5} i^2$
- b)  $\sum_{k=1}^4 (2k+1)^2$
- c)  $\sum_{k=0}^4 (2k+1)^2$

2. Write out fully what is meant by

$$\sum_{k=1}^{k=3} (\bar{x} - x_k)$$

3. Sigma notation is often used in statistical calculations. For example the **mean**,  $\bar{x}$ , of the  $n$  quantities  $x_1, x_2, \dots$  and  $x_n$ , is found by adding them up and dividing the result by  $n$ . Show that the mean can be written as

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

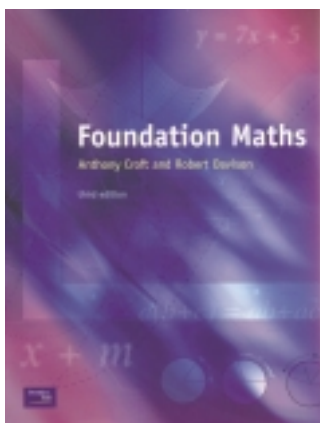
4. Write out fully what is meant by  $\sum_{i=1}^4 \frac{i}{i+1}$ .

5. Write out fully what is meant by  $\sum_{k=1}^3 \frac{(-1)^k}{k}$ .

### Answers

- 1. a)  $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ ,      b)  $3^2 + 5^2 + 7^2 + 9^2$ ,      c)  $1^2 + 3^2 + 5^2 + 7^2 + 9^2$ .
- 2.  $(\bar{x} - x_1) + (\bar{x} - x_2) + (\bar{x} - x_3)$ ,      4.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$ ,      5.  $\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3}$  which equals  $-1 + \frac{1}{2} - \frac{1}{3}$ .

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 12



# Foundation Maths Support Pack

## 3.1 What is a function ?

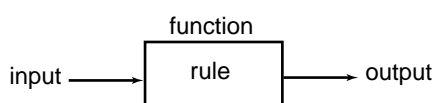
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### Introduction

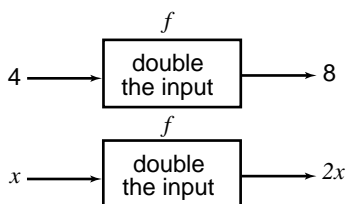
A quantity whose value can change is known as a **variable**. **Functions** are used to describe the rules which define the ways in which such a change can occur. The purpose of this leaflet is to explain functions and their notation.

### 1. The function rule

A function is a rule which operates on an **input** and produces an **output**. This can be illustrated using a **block diagram** such as that shown below. We can think of the function as a mathematical machine which processes the input, using a given rule, in order to produce an output. We often write the rule inside the box.



In order for a rule to be a function it must produce only a single output for any given input. The function with the rule 'double the input' is shown below.



Note that with an input of 4 the function would produce an output of 8. With a more general input,  $x$  say, the output will be  $2x$ . It is usual to assign a letter or other symbol to a function in order to label it. The doubling function pictured in the example above has been given the symbol  $f$ .

**A function is a rule which operates on an input and produces a single output from that input.**

For the doubling function it is common to use the notation

$$f(x) = 2x$$

This indicates that with an input  $x$ , the function,  $f$ , produces an output of  $2x$ . The input to the function is placed in the brackets after the function label ' $f$ '.  $f(x)$  is read as ' $f$  is a function of  $x$ ', or just ' $f$  of  $x$ ', meaning that the output from the function depends upon the value of  $x$ .

### Example

State the rule of each of the following functions:

a)  $f(x) = 7x + 9$ ,   b)  $h(t) = t^3 + 2$ ,   c)  $p(x) = x^3 + 2$ .

### Solution

a) The rule for  $f$  is ‘multiply the input by 7 and then add 9’.

b) The rule for  $h$  is ‘cube the input and add 2’.

c) The rule for  $p$  is ‘cube the input and add 2’.

Note from parts b) and c) that it is the rule that is important when describing a function and not the letters being used. Both  $h(t)$  and  $p(x)$  instruct us to ‘cube the input and add 2’.

The input to a function is called its **argument**. We can obtain the output from a function if we are given its argument. For example, given the function  $f(x) = 3x + 2$  we may require the value of the output when the argument is 5. We write this as  $f(5)$ . Here,  $f(5) = 3 \times 5 + 2 = 17$ .

### Example

Given the function  $f(x) = 4x + 3$  find a)  $f(-1)$ ,      b)  $f(6)$

### Solution

a) Here the argument is  $-1$ . We find  $f(-1) = 4 \times (-1) + 3 = -1$ .

b)  $f(6) = 4(6) + 3 = 27$ .

Sometimes the argument will be an algebraic expression, as in the following example.

### Example

Given the function  $y(x) = 5x - 3$  find

a)  $y(t)$ ,      b)  $y(7t)$ ,      c)  $y(z + 2)$ .

### Solution

The rule is multiply the input by 5, and subtract 3. We can apply this rule whatever the argument.

a) To find  $y(t)$  multiply the argument,  $t$ , by 5 and subtract 3 to give  $y(t) = 5t - 3$ .

b) Now the argument is  $7t$ . So  $y(7t) = 5(7t) - 3 = 35t - 3$ .

c) In this case the argument is  $z + 2$ . We find  $y(z + 2) = 5(z + 2) - 3 = 5z + 10 - 3 = 5z + 7$ .

### Exercises

1. Write down a function which can be used to describe the following rules:

a) ‘cube the input and divide the result by 2’,      b) ‘divide the input by 5 and then add 7’

2. Given the function  $f(x) = 7x - 3$  find a)  $f(3)$ ,   b)  $f(6)$ ,   c)  $f(-2)$ .

3. If  $g(t) = t^2$  write down expressions for a)  $g(x)$ ,      b)  $g(3t)$ ,      c)  $g(x + 4)$ .

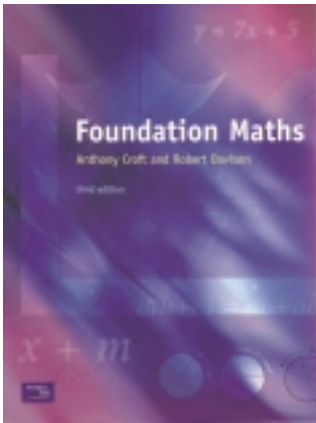
### Answers

1. a)  $f(x) = \frac{x^3}{2}$ ,      b)  $f(x) = \frac{x}{5} + 7$ .      2. a) 18,   b) 39,   c)  $-17$

3. a)  $g(x) = x^2$ ,      b)  $g(3t) = (3t)^2 = 9t^2$ ,      c)  $g(x + 4) = (x + 4)^2 = x^2 + 8x + 16$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 13



# Foundation Maths Support Pack

## 3.2 The graph of a function

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### Introduction

A very useful pictorial representation of a function is the **graph**. On this leaflet we remind you of important conventions when graph plotting.

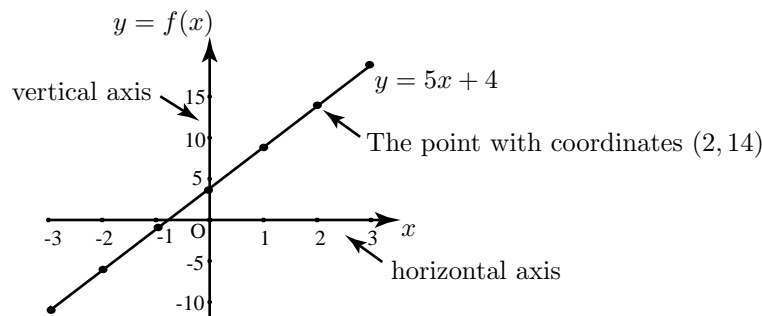
### 1. The graph of a function

Consider the function  $f(x) = 5x + 4$ .

We can choose several values for the input and calculate the corresponding outputs. We have done this for integer values of  $x$  between  $-3$  and  $3$  and the results are shown in the table.

$x$	$-3$	$-2$	$-1$	$0$	$1$	$2$	$3$
$f(x)$	$-11$	$-6$	$-1$	$4$	$9$	$14$	$19$

To plot the graph we first draw a pair of **axes** - a vertical axis and a horizontal axis. These are drawn at right-angles to each other and intersect at the **origin**  $O$  as shown below.



Each pair of input and output values can be represented on a graph by a single point. The input values are measured along the horizontal axis and the output values along the vertical axis. A uniform scale is drawn on each axis sufficient to accommodate all the required points. The points plotted in this way are then joined together, in this case by a straight line. This is the graph of the function. Each point on the graph can be represented by a pair of **coordinates** in the form  $(x, f(x))$ . Each axis should be labelled to show its variable.

### 2. Dependent and independent variables

The horizontal axis is often called the  $x$  axis. The vertical axis is commonly referred to as the  $y$  axis. So, we often write the function above, not as  $f(x) = 5x + 4$ , but rather as

$$y = 5x + 4$$

Since  $x$  and  $y$  can have a number of different values they are variables. Here  $x$  is called the **independent variable** and  $y$  is called the **dependent variable**. Knowing or choosing a value of the independent variable,  $x$ , the function allows us to calculate the corresponding value of the dependent variable,  $y$ . To show this dependence we often write  $y(x)$ . This notation simply means that  $y$  depends upon  $x$ . Note that it is the independent variable which is the input to the function and the dependent variable which is the output.

### Example

Consider the function given by  $y = 2t^2 + 1$ , for values of  $t$  between  $-2$  and  $2$ .

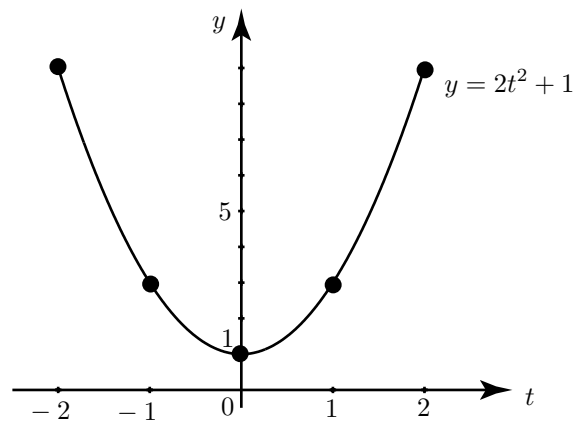
- State the independent variable.
- State the dependent variable.
- Plot a graph of the function.

### Solution

- The independent variable is  $t$ .
- The dependent variable is  $y$ .
- A table of input and output values should be constructed first. Such a table is shown below.

$t$	$-2$	$-1$	$0$	$1$	$2$
$y$	$9$	$3$	$1$	$3$	$9$

Each pair of  $t$  and  $y$  values in the table is plotted as a single point. The points are then joined with a smooth curve to produce the required graph as shown below.



### Exercises

1. Plot a graph of each of the following functions. In each case state the dependent and independent variables.

- $y = f(x) = 3x + 2$ , for  $x$  between  $-2$  and  $5$ ,
- $y = f(t) = 6 - t^2$ , for  $t$  between  $1$  and  $5$ .

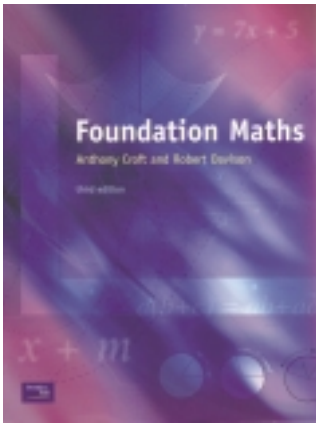
### Answers

- dependent variable is  $y$ , independent variable is  $x$ .
  - dependent variable is  $y$ , independent variable is  $t$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 14





# Foundation Maths Support Pack

## 3.3 The straight line

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### Introduction

Straight line graphs arise in many applications. This leaflet discusses the mathematical equation which describes a straight line and explains the terms 'gradient' and 'intercept'.

### 1. The equation of a straight line

Any equation of the form

$$y = mx + c$$

where  $m$  and  $c$  are fixed numbers, (i.e. constants), has a graph which is a straight line.

For example,  $y = 3x + 5$ ,  $y = \frac{2}{3}x + 8$  and  $y = -3x - 7$

all have graphs which are straight lines, but

$$y = 3x^2 + 4, \quad y = \frac{2}{3x} - 7, \quad \text{and} \quad y = -14\sqrt{x}$$

have graphs which are not straight lines. The essential feature of a straight line equation is that  $x$  and  $y$  occur only to the power 1.

### 2. The straight line graph

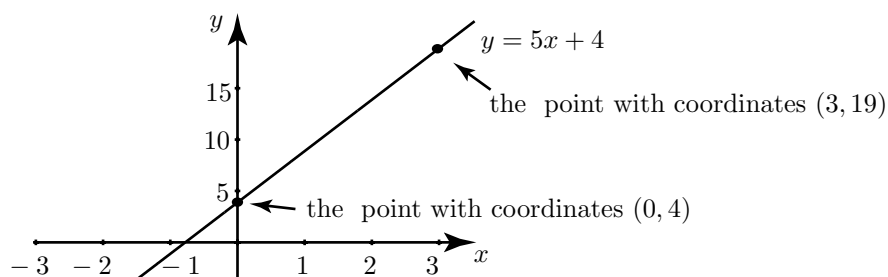
Any straight line graph can be plotted very simply by finding just two points which lie on the line and joining them. It is a good idea to find a third point just as a check.

#### Example

Plot a graph of the straight line with equation  $y = 5x + 4$ .

#### Solution

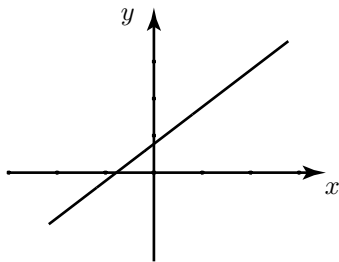
From the equation, note that when  $x = 0$ , the value of  $y$  is 4. Similarly when  $x = 3$ ,  $y = 19$ . So the points  $(0, 4)$  and  $(3, 19)$  lie on the graph. These points are plotted and joined together to form the straight line graph.



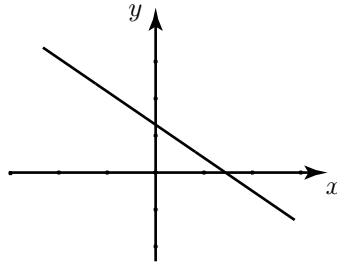
### 3. The gradient and intercept of a straight line

In the equation  $y = mx + c$  the value of  $m$  is called the **gradient** of the line. It can be positive, negative or zero. Lines with a positive gradient slope upwards, from left to right. Lines with a negative gradient slope downwards from left to right. Lines with a zero gradient are horizontal.

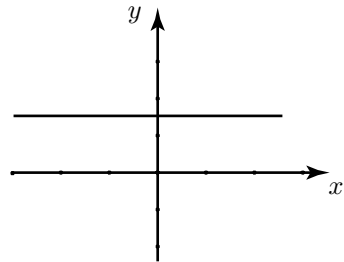
this line has a positive gradient



this line has a negative gradient

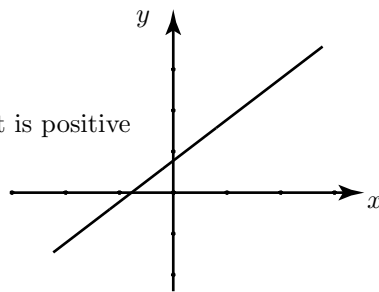


the gradient of this line is zero

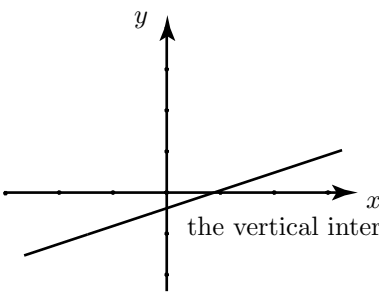


The value of  $c$  is called the **vertical intercept** of the line. It is the value of  $y$  when  $x = 0$ . When drawing a line,  $c$  gives the position where the line cuts the vertical axis.

the vertical intercept is positive



the vertical intercept is negative



#### Example

Determine the gradient and vertical intercept of each line.

- a)  $y = 12x - 6$ ,   b)  $y = 5 - 2x$ ,   c)  $4x - y + 13 = 0$ ,   d)  $y = 8$ ,   e)  $y = 4x$ .

#### Solution

- a) Comparing  $y = 12x - 6$  with  $y = mx + c$  we see that  $m = 12$ , so the gradient of the line is 12. The fact that this is positive means that the line slopes upwards as we move from left to right. The vertical intercept is  $-6$ . This line cuts the vertical axis below the horizontal axis.
- b) Comparing  $y = 5 - 2x$  with  $y = mx + c$  we see that  $m = -2$ , so the gradient is  $-2$ . The line slopes downwards as we move from left to right. The vertical intercept is 5.
- c) We write  $4x - y + 13 = 0$  in standard form as  $y = 4x + 13$  and note that  $m = 4$ ,  $c = 13$ .
- d) Comparing  $y = 8$  with  $y = mx + c$  we see that  $m = 0$  and  $c = 8$ . This line is horizontal.
- e) Comparing  $y = 4x$  with  $y = mx + c$  we see that  $m = 4$  and  $c = 0$ .

#### Exercises

1. State the gradient and intercept of each of the following lines.

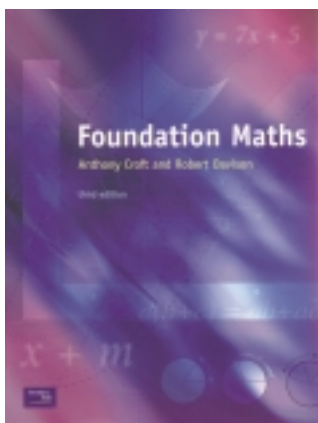
- a)  $y = 5x + 6$ ,   b)  $y = 3x - 11$ ,   c)  $y = -2x + 7$ ,   d)  $y = 9$ ,   e)  $y = 7 - x$

#### Answers

1. a) gradient 5, intercept 6   b) 3,  $-11$ ,   c)  $-2$ , 7,   d) 0, 9,   e)  $-1$ , 7.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 15



# Foundation Maths Support Pack

## 3.4 The exponential constant $e$

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materials to support *Foundation Maths Third Edition* by Anthony Croft & Robert Davison, ISBN 0 130 45426-5

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### Introduction

The letter  $e$  is used in many mathematical calculations to stand for a particular number known as the **exponential constant**. This leaflet provides information about this important constant, and the related **exponential function**.

### 1. The exponential constant

The exponential constant is an important mathematical constant and is given the symbol  $e$ . Its value is approximately 2.718. It has been found that this value occurs so frequently when mathematics is used to model physical and chemical phenomena that it is convenient to write simply  $e$ .

It is often necessary to work out powers of this constant, such as  $e^2$ ,  $e^3$  and so on. Your scientific calculator will be programmed to do this already. You should check that you can use your calculator to do this. Look for a button marked  $e^x$ , and check that

$$e^2 = 7.389, \quad \text{and} \quad e^3 = 20.086$$

In both cases we have quoted the answer to three decimal places although your calculator will give a more accurate answer than this.

You should also check that you can evaluate negative and fractional powers of  $e$  such as

$$e^{1/2} = 1.649 \quad \text{and} \quad e^{-2} = 0.135$$

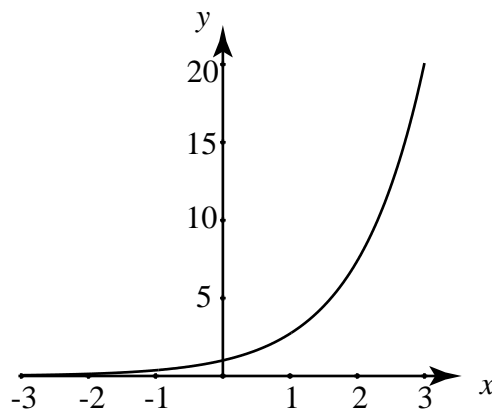
### 2. The exponential function

If we write  $y = e^x$  we can calculate the value of  $y$  as we vary  $x$ . Values obtained in this way can be placed in a table. For example:

$x$	-3	-2	-1	0	1	2	3
$y = e^x$	0.050	0.135	0.368	1	2.718	7.389	20.086

This is a table of values of the **exponential function**  $e^x$ . If pairs of  $x$  and  $y$  values are plotted we obtain a **graph** of the exponential function as shown overleaf. If you have never seen this

function before it will be a worthwhile exercise to plot it for yourself.



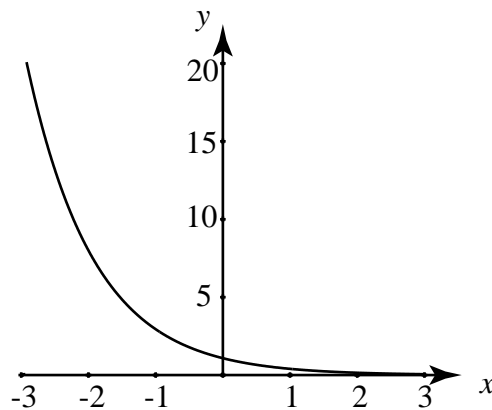
A graph of the exponential function  $y = e^x$

It is important to note that as  $x$  becomes larger, the value of  $e^x$  grows without bound. We write this mathematically as  $e^x \rightarrow \infty$  as  $x \rightarrow \infty$ . This behaviour is known as **exponential growth**.

### 3. The negative exponential function

A related function is the **negative exponential function**  $y = e^{-x}$ . A table of values of this function is shown below together with its graph.

$x$	-3	-2	-1	0	1	2	3
$y = e^{-x}$	20.086	7.389	2.718	1	0.368	0.135	0.050



A graph of the negative exponential function  $y = e^{-x}$

It is very important to note that as  $x$  becomes larger, the value of  $e^{-x}$  approaches zero. We write this mathematically as  $e^{-x} \rightarrow 0$  as  $x \rightarrow \infty$ . This behaviour is known as **exponential decay**.

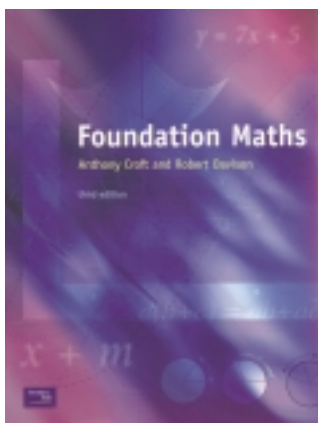
#### Exercises

A useful exercise would be to draw up tables of values and plot graphs of some related functions:

- a)  $y = e^{2x}$ ,    b)  $y = e^{0.5x}$ ,    c)  $y = -e^x$ ,    d)  $y = -e^{-x}$ ,    e)  $y = 1 - e^{-x}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 16



# Foundation Maths Support Pack

## 3.5 The logarithm function

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### Introduction

This leaflet provides a table of values and graph of the logarithm function  $y = \log_e x$ .

### 1. The logarithm function and its graph

Logarithms have been explained on leaflet 2.16 *What is a logarithm ?*. There we showed how logarithms provide alternative ways of writing expressions involving powers, and we showed how a calculator can be used to find logarithms.

The **natural logarithm function** is  $y = \log_e x$ , also written  $\ln x$ .

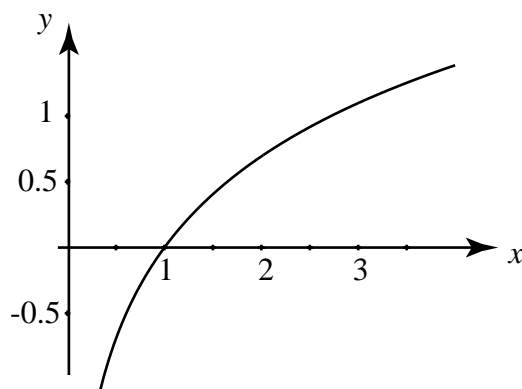
Note that we have chosen to use logarithms to base e as this is the most common base.

Using a calculator it is possible to construct a table of values of  $y = \log_e x$  as follows:

$x$	0.5	1	1.5	2	2.5	3	3.5
$y = \log_e x$	-0.693	0	0.405	0.693	0.916	1.099	1.253

You should check these values for yourself to make sure that you can obtain them.

If pairs of  $x$  and  $y$  values are plotted we obtain a **graph** of the logarithm function as shown.



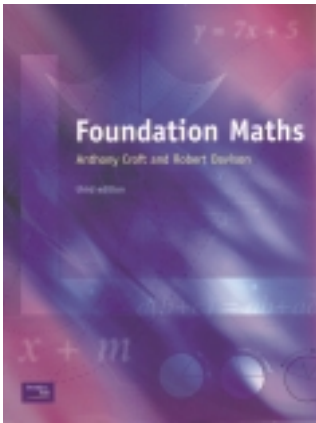
The graph of the natural logarithm  $y = \log_e x$

Note that the logarithm function is only defined for positive values of  $x$ . We cannot find the logarithm of 0, or the logarithm of a negative number.

As an exercise you should draw up a similar table for the function  $y = \log_{10} x$  and plot its graph. The graph should have the same general shape as the one above although most of the points on the graph are different.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 17



# Foundation Maths Support Pack

## 3.6 Solving equations involving logarithms and exponentials

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### Introduction

It is often necessary to solve an equation in which the unknown occurs as a power, or exponent. For example, you may need to find the value of  $x$  which satisfies  $2^x = 32$ . Very often the base will be the exponential constant  $e$ , as in the equation  $e^x = 20$ . To understand what follows you must be familiar with the exponential constant. See leaflet *3.4 The exponential constant* if necessary.

You will also come across equations involving logarithms. For example you may need to find the value of  $x$  which satisfies  $\log_{10} x = 34$ . You will need to understand what is meant by a logarithm, and the laws of logarithms (leaflets *2.16 What is a logarithm ?* and *2.17 The laws of logarithms*). On this leaflet we explain how such equations can be solved.

### 1. Revision of logarithms

Logarithms provide an alternative way of writing expressions involving powers. If

$$a = b^c \quad \text{then} \quad \log_b a = c$$

For example:  $100 = 10^2$  can be written as  $\log_{10} 100 = 2$ .

Similarly,  $e^3 = 20.086$  can be written as  $\log_e 20.086 = 3$ .

The third law of logarithms states that, for logarithms of any base,

$$\log A^n = n \log A$$

For example, we can write  $\log_{10} 5^2$  as  $2 \log_{10} 5$ , and  $\log_e 7^3$  as  $3 \log_e 7$ .

### 2. Solving equations involving powers

#### Example

Solve the equation  $e^x = 14$ .

#### Solution

Writing  $e^x = 14$  in its alternative form using logarithms we obtain  $x = \log_e 14$ , which can be evaluated directly using a calculator to give 2.639.

### Example

Solve the equation  $e^{3x} = 14$ .

### Solution

Writing  $e^{3x} = 14$  in its alternative form using logarithms we obtain  $3x = \log_e 14 = 2.639$ . Hence  $x = \frac{2.639}{3} = 0.880$ .

---

To solve an equation of the form  $2^x = 32$  it is necessary to take the logarithm of both sides of the equation. This is referred to as 'taking logs'. Usually we use logarithms to base 10 or base  $e$  because values of these logarithms can be obtained using a scientific calculator.

Starting with  $2^x = 32$ , then taking logs produces  $\log_{10} 2^x = \log_{10} 32$ . Using the third law of logarithms, we can rewrite the left-hand side to give  $x \log_{10} 2 = \log_{10} 32$ . Dividing both sides by  $\log_{10} 2$  gives

$$x = \frac{\log_{10} 32}{\log_{10} 2}$$

The right-hand side can now be evaluated using a calculator in order to find  $x$ :

$$x = \frac{\log_{10} 32}{\log_{10} 2} = \frac{1.5051}{0.3010} = 5$$

Hence  $2^5 = 32$ . Note that this answer can be checked by substitution into the original equation.

## 3. Solving equations involving logarithms

### Example

Solve the equation  $\log_{10} x = 0.98$

### Solution

Rewriting the equation in its alternative form using powers gives  $10^{0.98} = x$ . A calculator can be used to evaluate  $10^{0.98}$  to give  $x = 9.550$ .

### Example

Solve the equation  $\log_e 5x = 1.7$

### Solution

Rewriting the equation in its alternative form using powers gives  $e^{1.7} = 5x$ . A calculator can be used to evaluate  $e^{1.7}$  to give  $5x = 5.4739$  so that  $x = 1.095$  to 3dp.

### Exercises

1. Solve each of the following equations to find  $x$ .

a)  $3^x = 15$ ,    b)  $e^x = 15$ ,    c)  $3^{2x} = 9$ ,    d)  $e^{5x-1} = 17$ ,    e)  $10^{3x} = 4$ .

2. Solve the equations a)  $\log_e 2x = 1.36$ ,    b)  $\log_{10} 5x = 2$ ,    c)  $\log_{10}(5x + 3) = 1.2$ .

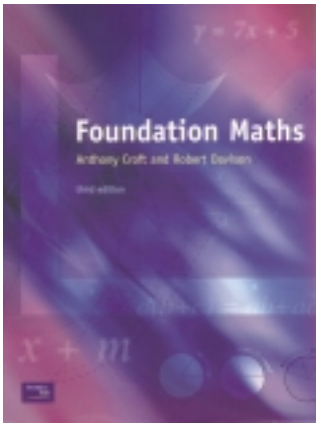
### Answers

1. a) 2.465,    b) 2.708,    c) 1,    d) 0.767,    e) 0.201.

2. a) 1.948, (3dp).    b) 20,    c) 2.570 (3dp).

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 17



# Foundation Maths Support Pack

## 4.1 Degrees and radians

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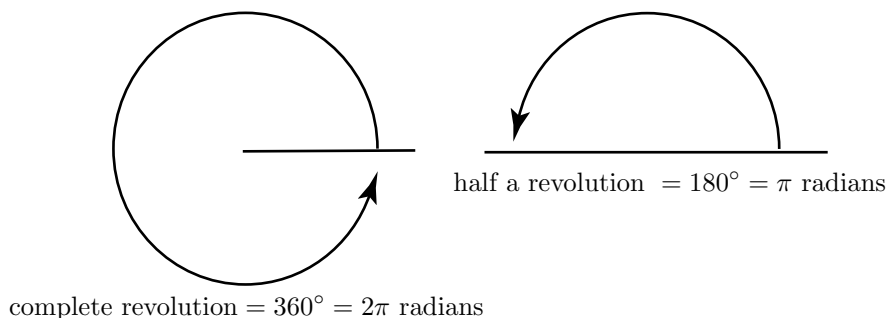
### Introduction

Angles can be measured in units of either degrees or radians. This leaflet explains these units and shows how it is possible to convert between them.

### 1. Degrees and radians

Angles can be measured in units of either degrees or radians. The symbol for degree is  $^\circ$ . Usually no symbol is used to denote radians.

A complete revolution is defined as  $360^\circ$  or  $2\pi$  radians.  $\pi$  stands for the number  $3.14159\dots$  and you can work with this if you prefer. However in many calculations you will find that you need to work directly with multiples of  $\pi$ .



It is easy to use the fact that  $360^\circ = 2\pi$  radians to convert between the two measures. We have

$$\begin{aligned} 360^\circ &= 2\pi \text{ radians} \\ 1^\circ &= \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians} \\ 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ \end{aligned}$$

### Example

- a) Convert  $65^\circ$  to radians.      b) Convert 1.75 radians to degrees.



## Solution

a)

$$\begin{aligned}1^\circ &= \frac{\pi}{180} \text{ radians} \\65^\circ &= 65 \times \frac{\pi}{180} \\&= 1.134 \text{ radians}\end{aligned}$$

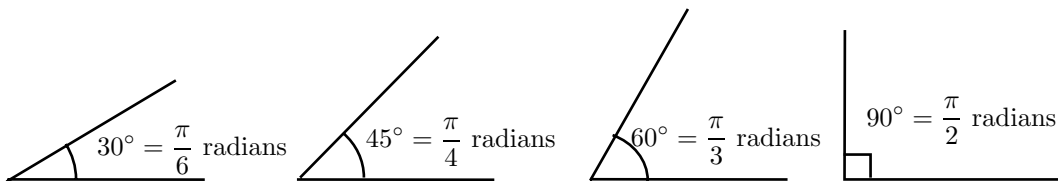
b)

$$\begin{aligned}1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \\1.75 \text{ radians} &= 1.75 \times \frac{180}{\pi} \\&= 100.268^\circ\end{aligned}$$

---

Note the following commonly met angles:

$$\begin{aligned}30^\circ &= \frac{\pi}{6} \text{ radians} & 45^\circ &= \frac{\pi}{4} \text{ radians} & 60^\circ &= \frac{\pi}{3} \text{ radians} \\90^\circ &= \frac{\pi}{2} \text{ radians} & 135^\circ &= \frac{3\pi}{4} \text{ radians} & 180^\circ &= \pi \text{ radians}\end{aligned}$$



Your calculator should be able to work with angles measured in both radians and degrees. Usually the MODE button allows you to select the appropriate measure. When calculations involve calculus you should always work with radians and not degrees.

## Exercises

1. Convert each of the following angles given in degrees, to radians. Give your answers correct to 2 decimal places.

a)  $32^\circ$ ,    b)  $95^\circ$ ,    c)  $217^\circ$ .

2. Convert each of the following angles given in radians, to degrees. Give your answers correct to 2 decimal places.

a) 3 radians,    b) 2.4 radians,    c) 1 radian.

3. Convert each of the following angles given in radians, to degrees. Do not use a calculator.

a)  $\frac{\pi}{15}$ ,    b)  $\frac{\pi}{5}$ .

4. Convert the following angles given in degrees, to radians. Do not use a calculator and give your answers as multiples of  $\pi$ .

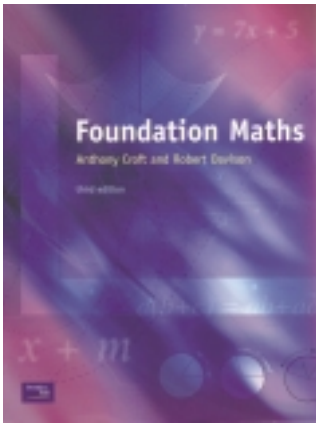
a)  $90^\circ$ ,    b)  $72^\circ$ ,    c)  $-45^\circ$ .

## Answers

1. a) 0.56 radians, b) 1.66 radians, c) 3.79 radians.    2. a)  $171.89^\circ$ ,    b)  $137.51^\circ$ ,    c)  $57.30^\circ$ .  
3. a)  $12^\circ$ ,    b)  $36^\circ$ .    4. a)  $\frac{\pi}{2}$  radians,    b)  $\frac{2\pi}{5}$  radians,    c)  $-\frac{\pi}{4}$  radians.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 18



# Foundation Maths Support Pack

## 4.2 The Trigonometrical Ratios

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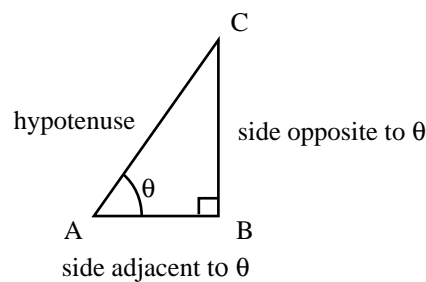
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### Introduction

The trigonometrical ratios sine, cosine and tangent appear frequently in many mathematical problems. This leaflet revises the meaning of these terms.

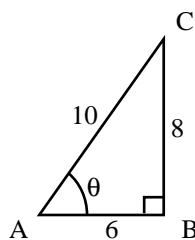
### 1. Sine, cosine and tangent ratios

Study the right-angled triangle  $ABC$  shown below.



The side opposite the right-angle is called the **hypotenuse**. The side **opposite** to  $\theta$  is  $BC$ . The remaining side,  $AB$ , is said to be **adjacent** to  $\theta$ .

Suppose we know the lengths of each of the sides as in the figure below.



We can then divide the length of one side by the length of one of the other sides.

The ratio  $\frac{BC}{AC}$  is known as the **sine** of angle  $\theta$ . This is abbreviated to  $\sin \theta$ . In the triangle shown we see that

$$\sin \theta = \frac{8}{10} = 0.8$$

The ratio  $\frac{AB}{AC}$  is known as the **cosine** of angle  $\theta$ . This is abbreviated to  $\cos \theta$ . In the triangle shown we see that

$$\cos \theta = \frac{6}{10} = 0.6$$

The ratio  $\frac{BC}{AB}$  is known as the **tangent** of angle  $\theta$ . This is abbreviated to  $\tan \theta$ . In the triangle shown we see that

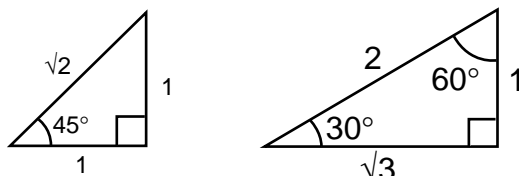
$$\tan \theta = \frac{8}{6} = 1.3333$$

In any right-angled triangle we define the trigonometrical ratios as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC} \qquad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB}$$

## 2. Some standard, or common, triangles



$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = 1$$

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3}$$

## 3. Using a calculator

If we know the angles in a right-angled triangle the trigonometrical ratios can be found using a scientific calculator. Look for the sine, cosine and tangent buttons on your calculator and make sure that you can use them by verifying that

$$\sin 50^\circ = 0.7660, \quad \cos 32^\circ = 0.8480$$

Your calculator will be able to handle angles measured in either radians or degrees. It will be necessary for you to choose the appropriate units. Study your calculator manual to learn how to do this. Check that

$$\sin 0.56 \text{ radians} = 0.5312, \quad \tan 1.4 \text{ radians} = 5.7979$$

## 4. Finding an angle when a trigonometrical ratio is known

If we are given, or know, a value for  $\sin \theta$ ,  $\cos \theta$  or  $\tan \theta$  we may want to work out the corresponding angle  $\theta$ . This process is known as finding the inverse sine, inverse cosine or inverse tangent. Your calculator will be pre-programmed for doing this. The buttons will be labelled  $\text{invsin}$ , or  $\sin^{-1}$ , and so on.

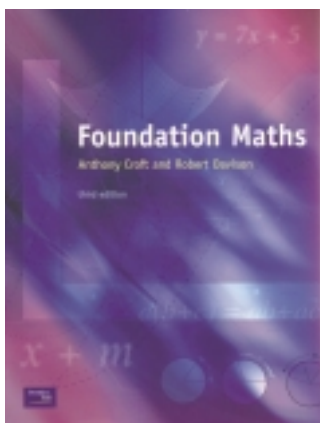
Check that you can use your calculator to show that if  $\sin \theta = 0.75$  then  $\theta = 48.59^\circ$ .

Mathematically we write this as follows:

$$\text{if } \sin \theta = 0.75, \text{ then } \theta = \sin^{-1} 0.75 = 48.59^\circ$$

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 19



# Foundation Maths Support Pack

## 4.3 Graphs of the trig functions

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### Introduction

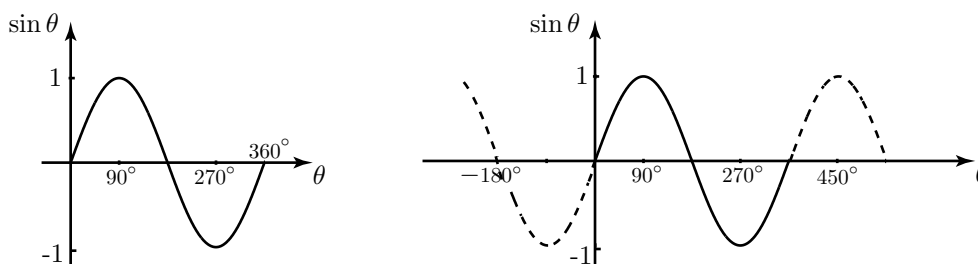
The trigonometric functions play a very important role in mathematics. Familiarity with the graphs of these functions is essential. Graphs of the trigonometric functions sine, cosine and tangent, together with tables of values are shown here for reference.

### 1. The sine function

Using a scientific calculator a table of values of  $\sin \theta$  can be drawn up as  $\theta$  varies from  $0$  to  $360^\circ$ .

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$
$\sin \theta$	0	0.5000	0.8660	1	0.8660	0.5000
$\theta$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$360^\circ$
$\sin \theta$	0	-0.5000	-0.8660	-1	-0.8660	0

Using the table, a graph of the function  $y = \sin \theta$  can be plotted and is shown below on the left.



If further values, outside the range  $0^\circ$  to  $360^\circ$ , are calculated we find that the wavy pattern repeats itself as shown on the right. We say that the sine function is **periodic** with period  $360^\circ$ . Some values are particularly important and should be remembered:

$$\sin 0^\circ = 0, \quad \sin 90^\circ = 1, \quad \sin 180^\circ = 0, \quad \sin 270^\circ = -1$$

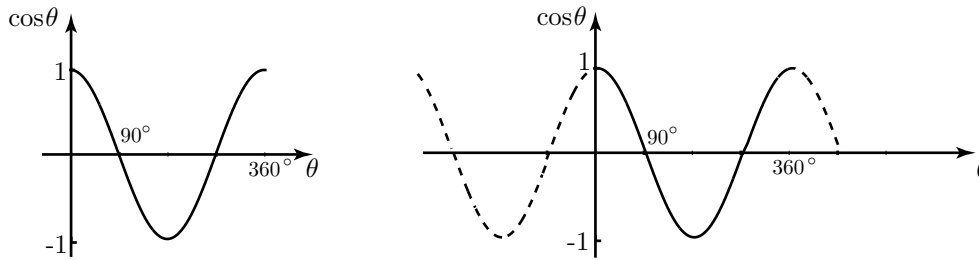
The maximum value of  $\sin \theta$  is 1, and the minimum value is  $-1$ .

### 2. The cosine function

Using a scientific calculator a table of values of  $\cos \theta$  can be drawn up as  $\theta$  varies from  $0$  to  $360^\circ$ .

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$
$\cos \theta$	1	0.8660	0.5000	0	-0.5000	-0.8660
$\theta$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$360^\circ$
$\cos \theta$	-1	-0.8660	-0.5000	0	0.5000	1

Using the table, a graph of the function  $y = \cos \theta$  can be plotted as shown on the left.



If further values are calculated outside the range  $0 \leq \theta \leq 360^\circ$  we find that the wavy pattern repeats itself as shown on the right. We say that the cosine function is periodic with period  $360^\circ$ . Some values are particularly important and should be remembered:

$$\cos 0^\circ = 1, \quad \cos 90^\circ = 0, \quad \cos 180^\circ = -1, \quad \cos 270^\circ = 0$$

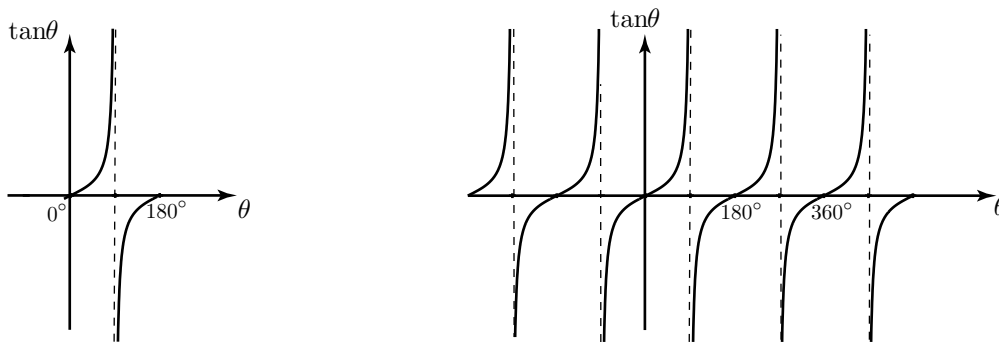
The maximum value of  $\cos \theta$  is 1, and the minimum value is  $-1$ .

### 3. The tangent function

Using a scientific calculator a table of values of  $\tan \theta$  can be drawn up as  $\theta$  varies from 0 to  $180^\circ$  although when  $\theta = 90^\circ$  you will find that this function is not defined.

$\theta$	0	$45^\circ$	$90^\circ$	$135^\circ$	$180^\circ$
$\tan \theta$	0	1	$\infty$	-1	0

Using the table, a graph of the function  $y = \tan \theta$  can be plotted and is shown below on the left.



If further values are calculated outside the range  $0 \leq \theta \leq 180^\circ$  we find that the pattern repeats itself as shown on the right. We say that the tangent function is periodic with period  $180^\circ$ .

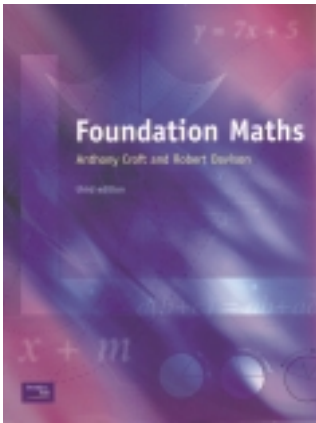
Some values are particularly important and should be remembered:

$$\tan 0^\circ = 0, \quad \tan 45^\circ = 1$$

There is no maximum value of  $\tan \theta$  because it increases without bound. There is no minimum value. However there are certain values where  $\tan \theta$  is not defined, including  $-90^\circ$ ,  $90^\circ$ ,  $270^\circ$  and so on. Here the graph shoots off to infinity.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 20



# Foundation Maths Support Pack

## 4.4 Trigonometrical Identities

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### Introduction

Very often it is necessary to rewrite expressions involving sines, cosines and tangents in alternative forms. To do this we use formulas known as **trigonometric identities**. A number of commonly used identities are listed here:

### 1. The identities

$$\tan A = \frac{\sin A}{\cos A} \quad \sec A = \frac{1}{\cos A} \quad \operatorname{cosec} A = \frac{1}{\sin A} \quad \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}, \quad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A + \sin B = 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)$$

$$\sin A - \sin B = 2 \cos \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)$$

$$\cos A + \cos B = 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)$$

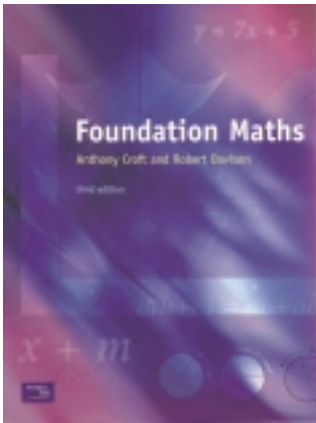
$$\cos A - \cos B = 2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{B - A}{2} \right)$$

---

Note:  $\sin^2 A$  is the notation used for  $(\sin A)^2$ . Similarly  $\cos^2 A$  means  $(\cos A)^2$  and so on.

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For examples and practice exercises see *Foundation Maths Third Edition*, Chapter 21



# Foundation Maths Support Pack

## 4.5 Pythagoras' theorem

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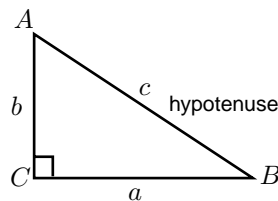
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### Introduction

Pythagoras' theorem relates the lengths of the sides of a right-angled triangle. This leaflet reminds you of the theorem and provides some revision examples and exercises.

### 1. Pythagoras' theorem

Study the right-angled triangle shown.

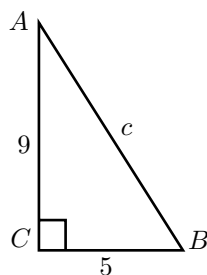


In any right-angled triangle,  $ABC$ , the side opposite the right-angle is called the **hypotenuse**. Here we use the convention that the side opposite angle  $A$  is labelled  $a$ . The side opposite  $B$  is labelled  $b$  and the side opposite  $C$  is labelled  $c$ .

**Pythagoras' theorem** states that the square of the hypotenuse, ( $c^2$ ), is equal to the sum of the squares of the other two sides, ( $a^2 + b^2$ ).

Pythagoras' theorem: $c^2 = a^2 + b^2$
--

### Example



Suppose  $AC = 9\text{cm}$  and  $BC = 5\text{cm}$  as shown. Find the length of the hypotenuse,  $AB$ .

### Solution

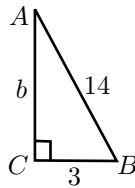
Here,  $a = BC = 5$ , and  $b = AC = 9$ . Using the theorem

$$\begin{aligned}c^2 &= a^2 + b^2 \\ &= 5^2 + 9^2 \\ &= 25 + 81 \\ &= 106 \\ c &= \sqrt{106} = 10.30 \quad (2\text{dp.})\end{aligned}$$

The hypotenuse has length 10.30cm.

### Example

In triangle  $ABC$  shown, suppose that the length of the hypotenuse is 14cm and that  $a = BC = 3$ cm. Find the length of  $AC$ .



### Solution

Here  $a = BC = 3$ , and  $c = AB = 14$ . Using the theorem

$$\begin{aligned}c^2 &= a^2 + b^2 \\ 14^2 &= 3^2 + b^2 \\ 196 &= 9 + b^2 \\ b^2 &= 196 - 9 \\ &= 187 \\ b &= \sqrt{187} = 13.67 \quad (2\text{dp.})\end{aligned}$$

The length of  $AC$  is 13.67cm.

### Exercises

1. In triangle  $ABC$  in which  $C = 90^\circ$ ,  $AB = 25$  cm and  $AC = 17$  cm. Find the length  $BC$ .
2. In triangle  $ABC$ , the angle at  $B$  is the right-angle. If  $AB = BC = 5$  cm find  $AC$ .
3. In triangle  $CDE$  the right-angle is  $E$ . If  $CD = 55$ cm and  $DE = 37$ cm find  $EC$ .

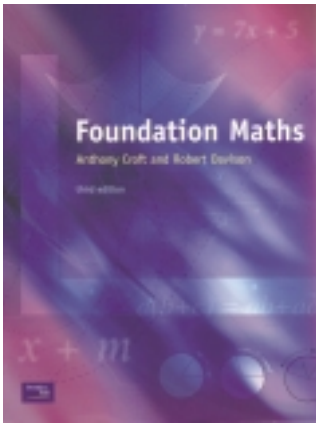
### Answers

1. 18.33 cm. (2dp.)
2.  $AC = \sqrt{50} = 7.07$  cm. (2dp.)
3.  $EC = \sqrt{1656} = 40.69$  cm. (2dp.)

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 22





# Foundation Maths Support Pack

## 4.6 The sine rule and cosine rule

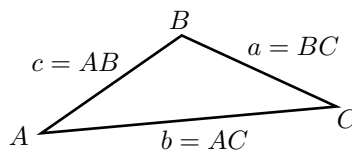
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### Introduction

To **solve** a triangle is to find the lengths of each of its sides and all its angles. The **sine rule** is used when we are given either a) two angles and one side, or b) two sides and a non-included angle. The **cosine rule** is used when we are given either a) three sides or b) two sides and the included angle.

### 1. The sine rule

Study the triangle  $ABC$  shown below. Let  $B$  stand for the angle at  $B$ . Let  $C$  stand for the angle at  $C$  and so on. Also, let  $b = AC$ ,  $a = BC$  and  $c = AB$ .



The sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

#### Example

In triangle  $ABC$ ,  $B = 21^\circ$ ,  $C = 46^\circ$  and  $AB = 9\text{cm}$ . Solve this triangle.

#### Solution

We are given two angles and one side and so the sine rule can be used. Furthermore, since the angles in any triangle must add up to  $180^\circ$  then angle  $A$  must be  $113^\circ$ . We know that  $c = AB = 9$ . Using the sine rule

$$\frac{a}{\sin 113^\circ} = \frac{b}{\sin 21^\circ} = \frac{9}{\sin 46^\circ}$$

So,

$$\frac{b}{\sin 21^\circ} = \frac{9}{\sin 46^\circ}$$

from which

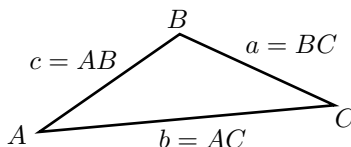
$$b = \sin 21^\circ \times \frac{9}{\sin 46^\circ} = 4.484\text{cm.} \quad (3\text{dp})$$

Similarly

$$a = \sin 113^\circ \times \frac{9}{\sin 46^\circ} = 11.517\text{cm.} \quad (3\text{dp})$$

## 2. The cosine rule

Refer to the triangle shown below.



The cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A, \quad b^2 = a^2 + c^2 - 2ac \cos B, \quad c^2 = a^2 + b^2 - 2ab \cos C$$

### Example

In triangle  $ABC$ ,  $AB = 42\text{cm}$ ,  $BC = 37\text{cm}$  and  $AC = 26\text{cm}$ . Solve this triangle.

### Solution

We are given three sides of the triangle and so the cosine rule can be used. Writing  $a = 37$ ,  $b = 26$  and  $c = 42$  we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

from which

$$37^2 = 26^2 + 42^2 - 2(26)(42) \cos A$$
$$\cos A = \frac{26^2 + 42^2 - 37^2}{(2)(26)(42)} = \frac{1071}{2184} = 0.4904$$

and so

$$A = \cos^{-1} 0.4904 = 60.63^\circ$$

You should apply the same technique to verify that  $B = 37.76^\circ$  and  $C = 81.61^\circ$ . You should also check that the angles you obtain add up to  $180^\circ$ .

### Exercises

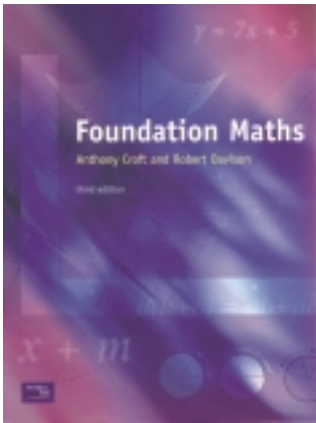
1. Solve the triangle  $ABC$  in which  $AC = 105\text{cm}$ ,  $AB = 76\text{cm}$  and  $A = 29^\circ$ .
2. Solve the triangle  $ABC$  given  $C = 40^\circ$ ,  $b = 23\text{cm}$  and  $c = 19\text{cm}$ .

### Answers

1.  $a = 53.31\text{cm}$ ,  $B = 107.28^\circ$ ,  $C = 43.72^\circ$ .
2.  $A = 11.09^\circ$ ,  $B = 128.91^\circ$ ,  $a = 5.69\text{cm}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 22



# Foundation Maths Support Pack

## 5.1 Determinants

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### Introduction

Determinants are mathematical objects which have applications in numerous branches of mathematics. For example, they can be used in the solution of simultaneous equations. This leaflet will show you how to calculate the value of a determinant.

### 1. Evaluating a determinant

The symbol  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  represents the expression  $ad - bc$  and is called a **determinant**.

For example

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \quad \text{means} \quad 3 \times 4 - 2 \times 1 = 12 - 2 = 10$$

Because  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  has two rows and two columns we describe it as a '2 by 2' or second-order determinant. Its value is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If we are given values for  $a$ ,  $b$ ,  $c$  and  $d$  we can use this to calculate the value of the determinant. Note that, once we have worked it out, a determinant is a single number.

#### Exercises

Evaluate the following determinants:

$$\text{a) } \begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix}, \quad \text{b) } \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix}, \quad \text{c) } \begin{vmatrix} 8 & 5 \\ -2 & 4 \end{vmatrix}, \quad \text{d) } \begin{vmatrix} 6 & 10 \\ -3 & -5 \end{vmatrix}, \quad \text{e) } \begin{vmatrix} x & 5 \\ y & 2 \end{vmatrix}.$$

#### Answers

$$\text{a) } 15 - 24 = -9, \quad \text{b) } 8 - (-2) = 10, \quad \text{c) } 32 - (-10) = 42, \quad \text{d) } -30 - (-30) = 0, \quad \text{e) } 2x - 5y.$$

### 2. Third-order determinants

A third-order, or '3 by 3' determinant can be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

One way in which it can be evaluated is to use second-order determinants as follows:

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Note in particular the way that the signs alternate between + and -.

For example

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} \\ &= 1(2) - 2(-22) + 1(-16) \\ &= 2 + 44 - 16 \\ &= 30 \end{aligned}$$

### Exercises.

1. Evaluate each of the following determinants.

$$\text{a) } \begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix} \quad \text{b) } \begin{vmatrix} 0 & -3 & 2 \\ -9 & 4 & 1 \\ 6 & 0 & 2 \end{vmatrix} \quad \text{c) } \begin{vmatrix} 7 & -2 & 3 \\ -1 & -4 & -4 \\ 6 & -2 & 12 \end{vmatrix} \quad \text{d) } \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

2. Evaluate each of the following determinants.

$$\text{a) } \begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix}, \quad \text{b) } \begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix}, \quad \text{c) } \begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix}, \quad \text{d) } \begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix}$$

### Answers.

1. a) 75, b) -120, c) -290, d)  $abc$ . 2. a) 40, b) 146, c) 116, d) 198.

## 3. Fourth-order determinants

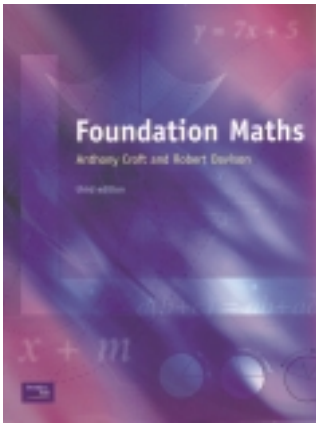
These are evaluated using third-order determinants. Once again note the alternating plus and minus sign.

### Example

$$\begin{aligned} \begin{vmatrix} 5 & 2 & 6 & 3 \\ 3 & 9 & 12 & 1 \\ -3 & 1 & 4 & 1 \\ 4 & 1 & 5 & 3 \end{vmatrix} &= 5 \begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix} + 6 \begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix} \\ &= 5(40) - 2(146) + 6(116) - 3(198) \\ &= 200 - 292 + 696 - 594 \\ &= 10 \end{aligned}$$

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 23



# Foundation Maths Support Pack

## 5.2 Multiplying matrices

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### Introduction

One of the most important operations carried out with matrices is **matrix multiplication**. At first sight this is done in a rather strange way. The reason for this only becomes apparent when matrices are used to solve equations.

### 1. Some simple examples

To multiply  $\begin{pmatrix} 3 & 7 \end{pmatrix}$  by  $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$  perform the following calculation.

$$\begin{pmatrix} 3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = 3 \times 2 + 7 \times 9 = 6 + 63 = 69$$

Note that we have paired elements in the row of the first matrix with elements in the column of the second matrix, multiplied the paired elements together and added the results.

Another, larger example:

$$\begin{pmatrix} 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} = 4 \times 3 + 2 \times 6 + 5 \times 8 = 12 + 12 + 40 = 64$$

### Exercises

1. Evaluate the following:

a)  $\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ ,   b)  $\begin{pmatrix} -3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ ,   c)  $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$ ,   d)  $\begin{pmatrix} -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ -8 \end{pmatrix}$ .

### Answers

1. a) 53,   b) 57,   c) 64,   d) -40.

### 2. More general matrix multiplication

When we multiplied matrices in the previous section the answers were always single numbers. Usually however, the result of multiplying two matrices is another matrix. Two matrices can only be multiplied together if the number of columns in the first matrix is the same as the number of rows in the second. So, if the first matrix has size  $p \times q$ , that is, it has  $p$  rows and  $q$  columns, and the second has size  $r \times s$ , that is, it has  $r$  rows and  $s$  columns, we can only multiply them together if  $q = r$ . When this is so, the result of multiplying them together is a  $p \times s$  matrix.

Two matrices can only ever be multiplied together if the number of columns in the first is the same as the number of rows in the second.

### Example

Find  $\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ .

### Solution

The first matrix has size  $2 \times 2$ . The second has size  $2 \times 1$ . Clearly the number of columns in the first is the same as the number of rows in the second. So, multiplication is possible and the result will be a  $2 \times 1$  matrix. The calculation is performed using the same operations as in the examples in the previous section.

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

To obtain the first entry in the solution, ignore the second row of the first matrix. You have already seen the required calculations.

$$\begin{pmatrix} 3 & 7 \\ & \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ \end{pmatrix}$$

To obtain the second entry in the solution, ignore the first row of the first matrix.

$$\begin{pmatrix} & \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} \\ 53 \end{pmatrix}$$

Putting it all together

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ 53 \end{pmatrix}$$

### Example

Find  $\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix}$ .

### Solution

The first matrix has size  $2 \times 2$ . The second matrix has size  $2 \times 2$ . Clearly the number of columns in the first is the same as the number of rows in the second. The multiplication can be performed and the result will be a  $2 \times 2$  matrix.

$$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 4 \times (-1) & 2 \times 6 + 4 \times 9 \\ 5 \times 3 + 3 \times (-1) & 5 \times 6 + 3 \times 9 \end{pmatrix} = \begin{pmatrix} 2 & 48 \\ 12 & 57 \end{pmatrix}$$

### Exercises

1. Evaluate the following.

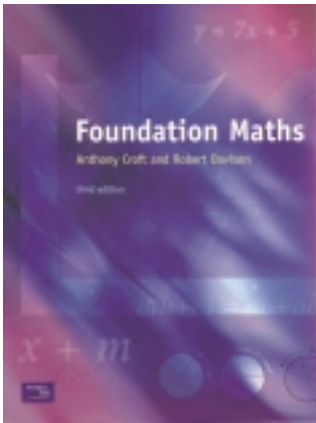
a)  $\begin{pmatrix} -3 & 2 \\ 3 & 11 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ,    b)  $\begin{pmatrix} 4 & 2 \\ 5 & 11 \end{pmatrix} \begin{pmatrix} 3 & 10 \\ -1 & 9 \end{pmatrix}$ ,    c)  $\begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 5 & 13 & 1 \end{pmatrix}$ .

### Answers

1. a)  $\begin{pmatrix} -11 \\ -2 \end{pmatrix}$ ,    b)  $\begin{pmatrix} 10 & 58 \\ 4 & 149 \end{pmatrix}$ ,    c)  $\begin{pmatrix} 9 & 13 & 5 \\ 47 & 117 & 11 \end{pmatrix}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 23



# Foundation Maths Support Pack

## 5.3 The inverse of a $2 \times 2$ matrix

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### Introduction

Once you know how to multiply matrices it is natural to ask whether they can be divided. The answer is no. However, by defining another matrix called the **inverse matrix** it is possible to work with an operation which plays a similar role to division. In this leaflet we explain what is meant by an inverse matrix and how the inverse of a  $2 \times 2$  matrix is calculated.

### 1. The inverse of a $2 \times 2$ matrix

The **inverse** of a  $2 \times 2$  matrix  $A$ , is another  $2 \times 2$  matrix denoted by  $A^{-1}$  with the property that

$$AA^{-1} = A^{-1}A = I$$

where  $I$  is the  $2 \times 2$  identity matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . That is, multiplying a matrix by its inverse produces an identity matrix. Note that in this context  $A^{-1}$  does not mean  $\frac{1}{A}$ .

Not all  $2 \times 2$  matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

### 2. A simple formula for the inverse

In the case of a  $2 \times 2$  matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  a simple formula exists to find its inverse:

$$\text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

#### Example

Find the inverse of the matrix  $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ .

#### Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(2) - (1)(4)} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

You should check that this answer is correct by performing the matrix multiplication  $AA^{-1}$ .

The result should be the identity matrix  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

### Example

Find the inverse of the matrix  $A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$ .

### Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1) - (4)(-3)} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

This can be written

$$A^{-1} = \begin{pmatrix} 1/14 & -4/14 \\ 3/14 & 2/14 \end{pmatrix} = \begin{pmatrix} 1/14 & -2/7 \\ 3/14 & 1/7 \end{pmatrix}$$

although it is quite permissible to leave the factor  $\frac{1}{14}$  at the front of the matrix.

### Exercises

1. Find the inverse of  $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$ .
2. Explain why the inverse of the matrix  $\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$  cannot be calculated.
3. Show that  $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$  is the inverse of  $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ .

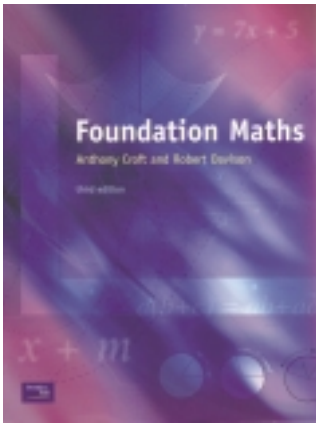
### Answers

1.  $A^{-1} = \frac{1}{-13} \begin{pmatrix} 2 & -5 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{1}{13} \end{pmatrix}$ .
2. The determinant of the matrix is zero, that is, it is singular and so has no inverse.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 23





# Foundation Maths Support Pack

## 5.4 Using the inverse matrix to solve equations

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### Introduction

One of the most important applications of matrices is to the solution of linear simultaneous equations. On this leaflet we explain how this can be done.

### 1. Writing simultaneous equations in matrix form

Consider the simultaneous equations

$$\begin{aligned}x + 2y &= 4 \\ 3x - 5y &= 1\end{aligned}$$

Provided you understand how matrices are multiplied together you will realise that these can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

we have

$$AX = B$$

This is the **matrix form** of the simultaneous equations. Here the unknown is the matrix  $X$ , since  $A$  and  $B$  are already known.  $A$  is called the **matrix of coefficients**.

### 2. Solving the simultaneous equations

Given

$$AX = B$$

we can multiply both sides by the inverse of  $A$ , provided this exists, to give

$$A^{-1}AX = A^{-1}B$$

But  $A^{-1}A = I$ , the identity matrix. Furthermore,  $IX = X$ , because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered. So

$$X = A^{-1}B$$

$$\text{if } AX = B, \quad \text{then } X = A^{-1}B$$

This result gives us a method for solving simultaneous equations. All we need do is write them in matrix form, calculate the inverse of the matrix of coefficients, and finally perform a matrix multiplication.

### Example

Solve the simultaneous equations

$$\begin{aligned}x + 2y &= 4 \\3x - 5y &= 1\end{aligned}$$

### Solution

We have already seen these equations in matrix form:

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of  $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$ .

$$\begin{aligned}A^{-1} &= \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}\end{aligned}$$

Then  $X$  is given by

$$\begin{aligned}X = A^{-1}B &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -22 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}\end{aligned}$$

Hence  $x = 2, y = 1$  is the solution of the simultaneous equations.

### Exercises

1. Solve the following sets of simultaneous equations using the inverse matrix method.

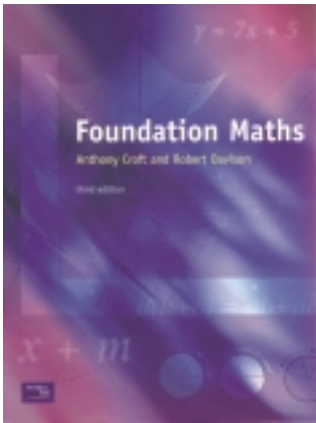
$$\begin{array}{ll} \text{a)} & \begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned} \\ \text{b)} & \begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned} \end{array}$$

### Answers

1. a)  $x = 3, y = -2$ ,    b)  $x = -2, y = 2$ .

---

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 23



# Foundation Maths Support Pack

## 6.1 Introduction to differentiation

materials to support *Foundation Maths Third Edition* by Anthony Croft & Robert Davison, ISBN 0 130 45426-5

### Introduction

This leaflet provides a rough and ready introduction to **differentiation**. This is a technique used to calculate the gradient, or slope, of a graph at different points.

### 1. The gradient function

Given a function, for example,  $y = x^2$ , it is possible to derive a formula for the gradient of its graph. We can think of this formula as the **gradient function**, precisely because it tells us the gradient of the graph. For example,

when  $y = x^2$  the gradient function is  $2x$

So, the gradient of the graph of  $y = x^2$  at any point is twice the  $x$  value there. To understand how this formula is actually found you would need to refer to a textbook on calculus. The important point is that using this formula we can calculate the gradient of  $y = x^2$  at different points on the graph. For example,

when  $x = 3$ , the gradient is  $2 \times 3 = 6$ .

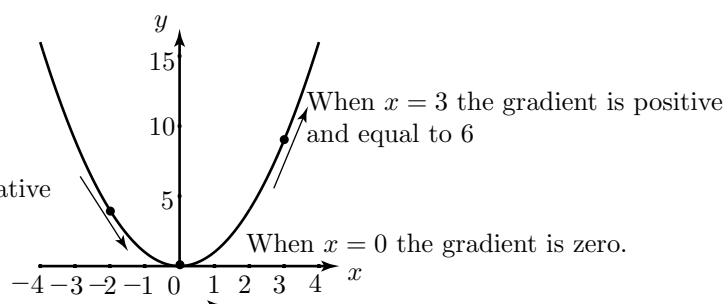
when  $x = -2$ , the gradient is  $2 \times (-2) = -4$ .

How do we interpret these numbers? A gradient of 6 means that values of  $y$  are increasing at the rate of 6 units for every 1 unit increase in  $x$ . A gradient of  $-4$  means that values of  $y$  are decreasing at a rate of 4 units for every 1 unit increase in  $x$ .

Note that when  $x = 0$ , the gradient is  $2 \times 0 = 0$ .

Below is a graph of the function  $y = x^2$ . Study the graph and you will note that when  $x = 3$  the graph has a positive gradient. When  $x = -2$  the graph has a negative gradient. When  $x = 0$  the gradient of the graph is zero. Note how these properties of the graph can be predicted from knowledge of the gradient function,  $2x$ .

When  $x = -2$  the gradient is negative and equal to  $-4$ .



### Example

When  $y = x^3$ , its gradient function is  $3x^2$ . Calculate the gradient of the graph of  $y = x^3$  when  
a)  $x = 2$ , b)  $x = -1$ , c)  $x = 0$ .

### Solution

a) when  $x = 2$  the gradient function is  $3(2)^2 = 12$ .

b) when  $x = -1$  the gradient function is  $3(-1)^2 = 3$ .

c) when  $x = 0$  the gradient function is  $3(0)^2 = 0$ .

## 2. Notation for the gradient function

You will need to use a notation for the gradient function which is in widespread use.

If  $y$  is a function of  $x$ , that is  $y = f(x)$ , we write its gradient function as  $\frac{dy}{dx}$ .

$\frac{dy}{dx}$ , pronounced 'dee  $y$  by dee  $x$ ', is not a fraction even though it might look like one! This notation can be confusing. Think of  $\frac{dy}{dx}$  as the 'symbol' for the gradient function of  $y = f(x)$ . The process of finding  $\frac{dy}{dx}$  is called **differentiation with respect to  $x$** .

### Example

For any value of  $n$ , the gradient function of  $x^n$  is  $nx^{n-1}$ . We write:

$$\text{if } y = x^n, \quad \text{then } \frac{dy}{dx} = nx^{n-1}$$

You have seen specific cases of this result earlier on. For example, if  $y = x^3$ ,  $\frac{dy}{dx} = 3x^2$ .

## 3. More notation and terminology

When  $y = f(x)$  alternative ways of writing the gradient function,  $\frac{dy}{dx}$ , are  $y'$ , pronounced 'y dash', or  $\frac{df}{dx}$ , or  $f'$ , pronounced 'f dash'. In practice you do not need to remember the formulas for the gradient functions of all the common functions. Engineers usually refer to a table known as a *Table of Derivatives*. A **derivative** is another name for a gradient function. Such a table is available on leaflet 6.2. The derivative is also known as the **rate of change** of a function.

### Exercises

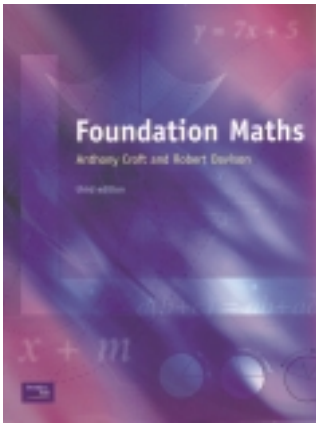
- Given that when  $y = x^2$ ,  $\frac{dy}{dx} = 2x$ , find the gradient of  $y = x^2$  when  $x = 7$ .
- Given that when  $y = x^n$ ,  $\frac{dy}{dx} = nx^{n-1}$ , find the gradient of  $y = x^4$  when a)  $x = 2$ , b)  $x = -1$ .
- Find the rate of change of  $y = x^3$  when a)  $x = -2$ , b)  $x = 6$ .
- Given that when  $y = 7x^2 + 5x$ ,  $\frac{dy}{dx} = 14x + 5$ , find the gradient of  $y = 7x^2 + 5x$  when  $x = 2$ .

### Answers

1. 14.    2. a) 32, b) -4.    3. a) 12, b) 108.    4. 33.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 25



# Foundation Maths Support Pack

## 6.2 Table of derivatives

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materials to support *Foundation Maths Third Edition* by Anthony Croft & Robert Davison, ISBN 0 130 45426-5

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This leaflet provides a table of common functions and their derivatives.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$k$ , any constant	0
$x$	1
$x^2$	$2x$
$x^3$	$3x^2$
$x^n$ , any constant $n$	$nx^{n-1}$
$e^x$	$e^x$
$e^{kx}$	$ke^{kx}$
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\operatorname{cosec} x = \frac{1}{\sin x}$	$-\operatorname{cosec} x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

## Exercises

1. In each case, use the table of derivatives to write down  $\frac{dy}{dx}$ .

a)  $y = 8$

b)  $y = -2$

c)  $y = 0$

d)  $y = x$

e)  $y = x^5$

f)  $y = x^7$

g)  $y = x^{-3}$

h)  $y = x^{1/2}$

i)  $y = x^{-1/2}$

j)  $y = \sin x$

k)  $y = \cos x$

l)  $y = \sin 4x$

m)  $y = \cos \frac{1}{2}x$

n)  $y = e^{4x}$

o)  $y = e^x$

p)  $y = e^{-2x}$

q)  $y = e^{-x}$

r)  $y = \ln x$

s)  $y = \log_e x$

t)  $y = \sqrt{x}$

u)  $y = \sqrt[3]{x}$

v)  $y = \frac{1}{\sqrt{x}}$

w)  $y = e^{x/2}$

2. You should be able to use the table when other variables are used. Find  $\frac{dy}{dt}$  if

a)  $y = e^{7t}$ ,    b)  $y = t^4$ ,    c)  $y = t^{-1}$ ,    d)  $y = \sin 3t$ .

## Answers

1.

a) 0,

b) 0,

c) 0,

d) 1,

e)  $5x^4$

f)  $7x^6$ ,

g)  $-3x^{-4}$ ,

h)  $\frac{1}{2}x^{-1/2}$ ,

i)  $-\frac{1}{2}x^{-3/2}$ ,

j)  $\cos x$

k)  $-\sin x$ ,

l)  $4 \cos 4x$ ,

m)  $-\frac{1}{2} \sin \frac{1}{2}x$ ,

n)  $4e^{4x}$ ,

o)  $e^x$

p)  $-2e^{-2x}$ ,

q)  $-e^{-x}$ ,

r)  $\frac{1}{x}$ ,

s)  $\frac{1}{x}$

t)  $\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

u)  $\frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$ ,

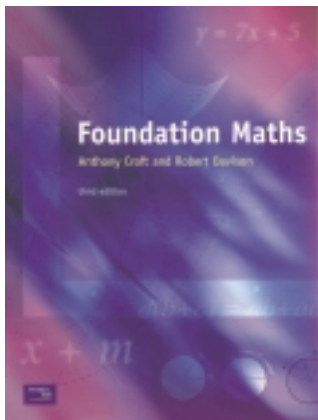
v)  $-\frac{1}{2}x^{-3/2}$ ,

w)  $\frac{1}{2}e^{x/2}$ .

2. a)  $7e^{7t}$ ,    b)  $4t^3$ ,    c)  $-\frac{1}{t^2}$ ,    d)  $3 \cos 3t$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 25



# Foundation Maths Support Pack

## 6.3 Linearity rules

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### Introduction

There are two rules known as **linearity rules** which, when used with a Table of Derivatives, enable us to differentiate a wider range of functions. These rules are summarised here.

#### 1. Some notation

Before we look at the rules, we need to be clear about the meaning of the notation  $\frac{d}{dx}$ .

When we are given a function  $y(x)$  and are asked to find  $\frac{dy}{dx}$  we are being instructed to carry out an operation on the function  $y(x)$ . The operation is that of differentiation. A notation for this operation is used widely:

$\frac{d}{dx}$  stands for the operation: 'differentiate with respect to  $x$ '

For example,  $\frac{d}{dx}(x^3) = 3x^2$  and  $\frac{d}{dx}(\sin x) = \cos x$ .

#### 2. Differentiation of a function multiplied by a constant

If  $k$  is a constant and  $f$  is a function of  $x$ , then

$$\frac{d}{dx}(kf) = k \frac{df}{dx}$$

This means that a constant factor can be brought outside the differentiation operation.

##### Example

Given that  $\frac{d}{dx}(x^3) = 3x^2$ , then it follows that

$$\frac{d}{dx}(7x^3) = 7 \times \frac{d}{dx}(x^3) = 7 \times 3x^2 = 21x^2$$

Given that  $\frac{d}{dx}(\sin x) = \cos x$ , then it follows that

$$\frac{d}{dx}(8 \sin x) = 8 \times \frac{d}{dx}(\sin x) = 8 \cos x$$

### 3. Differentiation of the sum or difference of two functions

If  $f$  and  $g$  are functions of  $x$ , then

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx} \qquad \frac{d}{dx}(f - g) = \frac{df}{dx} - \frac{dg}{dx}$$

This means that to differentiate a sum of two functions, simply differentiate each separately and then add the results. Similarly, to differentiate the difference of two functions, differentiate each separately and then find the difference of the results.

#### Example

Find  $\frac{dy}{dx}$  when  $y = x^2 + x$ .

#### Solution

We require  $\frac{d}{dx}(x^2 + x)$ . The sum rule tells us to differentiate each term separately. Thus

$$\frac{d}{dx}(x^2 + x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) = 2x + 1$$

So  $\frac{dy}{dx} = 2x + 1$ .

#### Example

Find  $\frac{dy}{dx}$  when  $y = e^{2x} - \sin 3x$ .

#### Solution

The difference rule tells us to differentiate each term separately.

$$\frac{d}{dx}(e^{2x} - \sin 3x) = \frac{d}{dx}(e^{2x}) - \frac{d}{dx}(\sin 3x) = 2e^{2x} - 3 \cos 3x$$

So

$$\frac{dy}{dx} = 2e^{2x} - 3 \cos 3x$$

#### Exercises

In each case use a Table of Derivatives and the rules on this leaflet to find  $\frac{dy}{dx}$ .

1.  $y = e^{5x} + \cos 2x$
2.  $y = x^2 - \sin x$
3.  $y = 3x^2 + 7x + 2$
4.  $y = 5$ .
5.  $y = 8e^{-9x}$ .

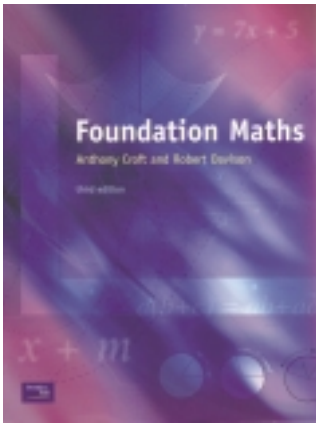
#### Answers

1.  $5e^{5x} - 2 \sin 2x$ , 2.  $2x - \cos x$ , 3.  $6x + 7$ , 4. 0, 5.  $-72e^{-9x}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 25





# Foundation Maths Support Pack

## 6.4 Product and quotient rules

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### Introduction

As their names suggest, the **product rule** and the **quotient rule** are used to differentiate products of functions and quotients of functions. This leaflet explains how.

### 1. The product rule

It is appropriate to use this rule when you want to differentiate two functions which are multiplied together. For example

$$y = e^x \sin x \quad \text{is a product of the functions } e^x \text{ and } \sin x$$

In the rule which follows we let  $u$  stand for the first of the functions and  $v$  stand for the second.

If  $u$  and  $v$  are functions of  $x$ , then

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

#### Example

If  $y = 7xe^{2x}$  find  $\frac{dy}{dx}$ .

#### Solution

Comparing the given function with the product rule we let

$$u = 7x, \quad v = e^{2x}$$

It follows that

$$\frac{du}{dx} = 7, \quad \text{and} \quad \frac{dv}{dx} = 2e^{2x}$$

Thus, using the product rule,

$$\frac{d}{dx}(7xe^{2x}) = 7x(2e^{2x}) + e^{2x}(7) = 7e^{2x}(2x + 1)$$

## 2. The quotient rule

It is appropriate to use this rule when you want to differentiate a quotient of two functions, that is, one function divided by another. For example

$$y = \frac{e^x}{\sin x} \quad \text{is a quotient of the functions } e^x \text{ and } \sin x$$

In the rule which follows we let  $u$  stand for the function in the numerator and  $v$  stand for the function in the denominator.

If  $u$  and  $v$  are functions of  $x$ , then

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

### Example

If  $y = \frac{\sin x}{3x^2}$  find  $\frac{dy}{dx}$ .

### Solution

Comparing the given function with the quotient rule we let

$$u = \sin x, \quad \text{and} \quad v = 3x^2$$

It follows that

$$\frac{du}{dx} = \cos x \quad \text{and} \quad \frac{dv}{dx} = 6x$$

Applying the quotient rule gives

$$\frac{dy}{dx} = \frac{3x^2 \cos x - \sin x (6x)}{9x^4} = \frac{3x(x \cos x - 2 \sin x)}{9x^4} = \frac{x \cos x - 2 \sin x}{3x^3}$$

### Exercises

Choose an appropriate rule in each case to find  $\frac{dy}{dx}$ .

1.  $y = x^2 \sin x$

2.  $y = e^x \cos x$

3.  $y = \frac{e^x}{x^2+1}$

4.  $y = \frac{x^2+1}{e^x}$

5.  $y = 7x \log_e x$

6.  $y = \frac{x-1}{\sin 2x}$

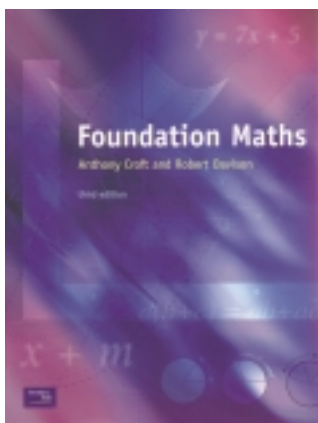
### Answers

1.  $x^2 \cos x + 2x \sin x$     2.  $-e^x \sin x + e^x \cos x = e^x(\cos x - \sin x)$     3.  $\frac{e^x(x^2-2x+1)}{(x^2+1)^2}$

4.  $\frac{2x-x^2-1}{e^x}$ ,    5.  $7(1 + \log_e x)$ ,    6.  $\frac{\sin 2x - 2(x-1) \cos 2x}{\sin^2 2x}$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 26



# Foundation Maths Support Pack

## 6.5 Integration as the reverse of differentiation

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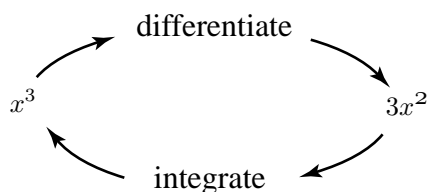
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### Introduction

**Integration** can be introduced in several different ways. One way is to think of it as differentiation in reverse. This approach is described in this leaflet.

### 1. Differentiation in reverse

Suppose we differentiate the function  $y = x^3$ . We obtain  $\frac{dy}{dx} = 3x^2$ . Integration reverses this process and we say that the integral of  $3x^2$  is  $x^3$ . Pictorially we can think of this as follows:



The situation is just a little more complicated because there are lots of functions we can differentiate to give  $3x^2$ . Here are some of them:

$$x^3 + 14, \quad x^3 + 7, \quad x^3 - 0.25, \quad x^3 - \frac{1}{2}$$

Each of these functions has the same derivative,  $3x^2$ , because when we differentiate the constant term we obtain zero. Consequently, when we try to reverse the process, we have no idea what the original constant term might have been. Because of this we include in our answer an unknown constant,  $c$  say, called the **constant of integration**. We state that the integral of  $3x^2$  is  $x^3 + c$ .

The symbol for integration is  $\int$ , known as an **integral sign**. Formally we write

$$\int 3x^2 dx = x^3 + c$$

Along with the integral sign there is a term 'dx', which must always be written, and which indicates the name of the variable involved, in this case  $x$ . Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with on a subsequent leaflet. When asked to find an indefinite integral your answer should always contain a constant of integration.

Common integrals are usually found in a ‘Table of Integrals’ such as that shown here. A more complete table is available on leaflet 6.6 *Table of integrals*.

**Table of integrals**

Function $f(x)$	Indefinite integral $\int f(x)dx$
constant, $k$	$kx + c$
$x$	$\frac{x^2}{2} + c$
$x^2$	$\frac{x^3}{3} + c$
$x^n$	$\frac{x^{n+1}}{n+1} + c \quad n \neq -1$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sin kx$	$-\frac{\cos kx}{k} + c$
$\cos kx$	$\frac{\sin kx}{k} + c$
$\tan kx$	$\frac{1}{k} \ln  \sec kx  + c$
$e^x$	$e^x + c$
$e^{-x}$	$-e^{-x} + c$
$e^{kx}$	$\frac{e^{kx}}{k} + c$
$x^{-1} = \frac{1}{x}$	$\ln  x  + c$

When dealing with the trigonometric functions the variable  $x$  must always be measured in radians.

**Example**

Use the table above to find a)  $\int x^8 dx$ ,      b)  $\int x^{-4} dx$ .

**Solution**

From the table note that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

a) With  $n = 8$  we find

$$\int x^8 dx = \frac{x^{8+1}}{8+1} + c = \frac{x^9}{9} + c$$

b) With  $n = -4$  we find

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c$$

Note that the final answer can be written in a variety of equivalent ways, for example

$$-\frac{1}{3}x^{-3} + c, \quad \text{or} \quad -\frac{1}{3} \cdot \frac{1}{x^3} + c, \quad \text{or} \quad -\frac{1}{3x^3} + c$$

**Exercises**

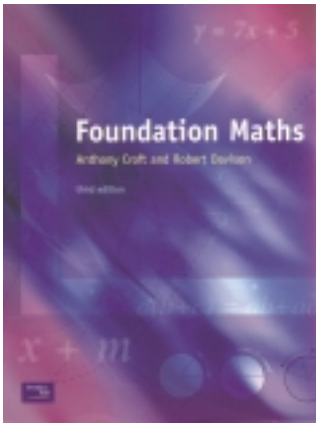
1. Integrate each of the following functions:

- a)  $x^9$ ,    b)  $x^{1/2}$ ,    c)  $x^{-3}$ ,    d)  $\frac{1}{x^4}$ ,    e) 4,    f)  $\sqrt{x}$ ,    g)  $e^{4x}$ ,    h) 17,    i)  $\cos 5x$ .

**Answers**

1. a)  $\frac{x^{10}}{10} + c$ ,    b)  $\frac{2x^{3/2}}{3} + c$ ,    c)  $-\frac{1}{2}x^{-2} + c$ ,    d)  $-\frac{1}{3}x^{-3} + c$ ,  
 e)  $4x + c$ ,    f) same as b),    g)  $\frac{e^{4x}}{4} + c$ ,    h)  $17x + c$ ,    i)  $\frac{\sin 5x}{5} + c$ .

For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 27



# Foundation Maths Support Pack

## 6.6 Table of Integrals

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Usually we refer to a table of integrals when performing calculations involving integration. This leaflet provides such a table. Sometimes restrictions need to be placed on the values of some of the variables. These restrictions are shown in the third column.

### 1. A table of integrals

$f(x)$	$\int f(x) dx$	
$k$ , any constant	$kx + c$	
$x$	$\frac{x^2}{2} + c$	
$x^2$	$\frac{x^3}{3} + c$	
$x^n$	$\frac{x^{n+1}}{n+1} + c$	$n \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln x  + c$	
$e^x$	$e^x + c$	
$e^{kx}$	$\frac{1}{k}e^{kx} + c$	
$\cos x$	$\sin x + c$	
$\cos kx$	$\frac{1}{k} \sin kx + c$	
$\sin x$	$-\cos x + c$	
$\sin kx$	$-\frac{1}{k} \cos kx + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln \cosh x + c$	
$\operatorname{coth} x$	$\ln \sinh x + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a} + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln \frac{x-a}{x+a} + c$	$ x  > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \frac{a+x}{a-x} + c$	$ x  < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1} \frac{x}{a} + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1} \frac{x}{a} + c$	$x \geq a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln(x + \sqrt{x^2+k}) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a} + c$	$-a \leq x \leq a$

## Exercises

1. In each case, use the Table of Integrals to integrate the given function with respect to  $x$ .

- a)  $x$
- b)  $x^6$
- c)  $x^{-2}$
- d)  $x^{-3}$
- e)  $x^{-1}$ , (be careful!)
- f)  $x^{1/2}$
- g)  $x^{-1/2}$
- h)  $e^{3x}$
- i)  $e^{7x}$
- j)  $e^{-2x}$
- k)  $e^{0.5x}$
- l)  $e^x$
- m)  $e^{-x}$
- n)  $\cos x$
- o)  $\sin x$
- p)  $\sin 3x$
- q)  $\cos 2x$
- r) 5

2. You should be able to use the table when variables other than  $x$  are involved. Use the table to integrate each of the following functions with respect to  $t$ .

- a)  $e^t$ ,    b)  $e^{5t}$ ,    c)  $t^7$ ,    d)  $\sqrt{t}$ ,    e)  $\cos 5t$ ,    f)  $e^{-t}$ .

## Answers

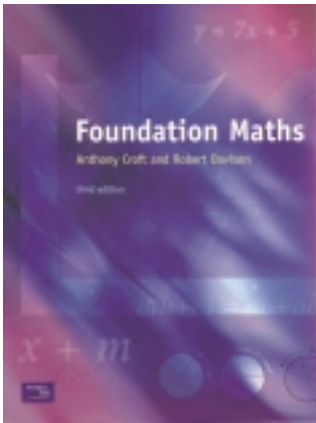
1.

- |  |                               |  |
|--|-------------------------------|--|
| a) $\frac{x^2}{2} + c$ ,   | b) $\frac{x^7}{7} + c$ ,      | c) $\frac{x^{-1}}{-1} + c = -x^{-1} + c$ , or $-\frac{1}{x} + c$ , |
| d) $\frac{x^{-2}}{-2} + c = -\frac{1}{2}x^{-2} + c$ , or $-\frac{1}{2x^2} + c$ , | e) $\ln x  + c$               | f) $\frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$ ,            |
| g) $\frac{x^{1/2}}{1/2} + c = 2x^{1/2} + c$ ,                                    | h) $\frac{1}{3}e^{3x} + c$ ,  | i) $\frac{1}{7}e^{7x} + c$   |
| j) $-\frac{1}{2}e^{-2x} + c$ ,   | k) $2e^{0.5x} + c$ ,          | l) $e^x + c$   |
| m) $-e^{-x} + c$ ,   | n) $\sin x + c$ ,             | o) $-\cos x + c$   |
| p) $-\frac{1}{3}\cos 3x + c$ ,   | q) $\frac{1}{2}\sin 2x + c$ , | r) $5x + c$ .  |

2. a)  $e^t + c$ ,    b)  $\frac{e^{5t}}{5} + c$ ,    c)  $\frac{t^8}{8} + c$ ,    d)  $\frac{2t^{3/2}}{3} + c$ ,    e)  $\frac{\sin 5t}{5} + c$ ,    f)  $-e^{-t} + c$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 27



# Foundation Maths Support Pack

## 6.7 Linearity rules of integration

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### Introduction

To enable us to find integrals of a wider range of functions than those normally given in a Table of Integrals we can make use of two rules known as **linearity rules**.

### 1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

$$\int k f(x) dx = k \int f(x) dx$$

This is only possible when  $k$  is a constant, and it multiplies some function of  $x$ .

#### Example

Find  $\int 11x^2 dx$ .

#### Solution

We are integrating a multiple of  $x^2$ . The constant factor, 11, can be moved outside the integral sign.

$$\int 11x^2 dx = 11 \int x^2 dx = 11 \left( \frac{x^3}{3} + c \right) = \frac{11x^3}{3} + 11c$$

where  $c$  is the constant of integration. Because  $11c$  is a constant we would normally write the answer in the form  $\frac{11x^3}{3} + K$  where  $K$  is another constant.

#### Example

Find  $\int -5 \cos x dx$ .

#### Solution

We are integrating a multiple of  $\cos x$ . The constant factor,  $-5$ , can be moved outside the integral sign.

$$\int -5 \cos x dx = -5 \int \cos x dx = -5 (\sin x + c) = -5 \sin x + K$$

where  $K$  is a constant.

## 2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$$
$$\int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$$

### Example

Find  $\int (x^3 + \sin x) dx$ .

### Solution

$$\int (x^3 + \sin x) dx = \int x^3 dx + \int \sin x dx = \frac{x^4}{4} - \cos x + c$$

Note that only a single constant of integration is needed.

### Example

Find  $\int e^{3x} - x^7 dx$ .

### Solution

$$\int e^{3x} - x^7 dx = \int e^{3x} dx - \int x^7 dx = \frac{e^{3x}}{3} - \frac{x^8}{8} + c$$

### Exercises

1. a) Find  $\int 8x^5 + 3x^2 dx$ ,      b)  $\int \frac{2}{3}x dx$ .
2. Find  $\int 3 \cos x + 7x^3 dx$ .
3. Find  $\int 7x^{-2} dx$ .
4. Find  $\int \frac{5}{x} dx$ .
5. Find  $\int \frac{x + \cos 2x}{3} dx$ .
6. Find  $\int 5e^{4x} dx$ .
7. Find  $\int \frac{e^x - e^{-x}}{2} dx$ .

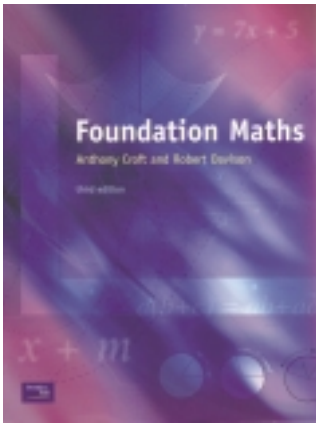
### Answers

1. a)  $\frac{4x^6}{3} + x^3 + c$ ,    b)  $\frac{1}{3}x^2 + c$ ,    2.  $3 \sin x + \frac{7x^4}{4} + c$ ,    3.  $-\frac{7}{x} + c$     4.  $5 \log_e |x| + c$ ,
5.  $\frac{x^2}{6} + \frac{\sin 2x}{6} + c$ ,    6.  $\frac{5e^{4x}}{4} + c$ ,    7.  $\frac{e^x + e^{-x}}{2} + c$ .

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 27





# Foundation Maths Support Pack

## 6.8 Evaluating definite integrals

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### Introduction

**Definite integrals** can be recognised by numbers written to the upper and lower right of the integral sign. This leaflet explains how to evaluate definite integrals.

### 1. Definite integrals

The quantity

$$\int_a^b f(x) dx$$

is called the **definite integral** of  $f(x)$  from  $a$  to  $b$ . The numbers  $a$  and  $b$  are known as the **lower** and **upper limits** of the integral. To see how to evaluate a definite integral consider the following example.

#### Example

Find  $\int_1^4 x^2 dx$ .

#### Solution

First of all the integration of  $x^2$  is performed in the normal way. However, to show we are dealing with a definite integral, the result is usually enclosed in square brackets and the limits of integration are written on the right bracket:

$$\int_1^4 x^2 dx = \left[ \frac{x^3}{3} + c \right]_1^4$$

Then, the quantity in the square brackets is evaluated, first by letting  $x$  take the value of the upper limit, then by letting  $x$  take the value of the lower limit. The difference between these two results gives the value of the definite integral:

$$\begin{aligned} \left[ \frac{x^3}{3} + c \right]_1^4 &= (\text{evaluate at upper limit}) - (\text{evaluate at lower limit}) \\ &= \left( \frac{4^3}{3} + c \right) - \left( \frac{1^3}{3} + c \right) \\ &= \frac{64}{3} - \frac{1}{3} \\ &= 21 \end{aligned}$$

Note that the constants of integration cancel out. This will always happen, and so in future we can ignore them when we are evaluating definite integrals.

### Example

Find  $\int_{-2}^3 x^3 dx$ .

### Solution

$$\begin{aligned}\int_{-2}^3 x^3 dx &= \left[ \frac{x^4}{4} \right]_{-2}^3 \\ &= \left( \frac{(3)^4}{4} \right) - \left( \frac{(-2)^4}{4} \right) \\ &= \frac{81}{4} - \frac{16}{4} \\ &= \frac{65}{4} \\ &= 16.25\end{aligned}$$

### Example

Find  $\int_0^{\pi/2} \cos x dx$ .

### Solution

$$\begin{aligned}\int_0^{\pi/2} \cos x dx &= [\sin x]_0^{\pi/2} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin 0 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

### Exercises

1. Evaluate

a)  $\int_0^1 x^2 dx$ ,    b)  $\int_2^3 \frac{1}{x^2} dx$ ,    c)  $\int_1^2 x^2 dx$ ,    d)  $\int_0^4 x^3 dx$ ,    e)  $\int_{-1}^1 x^3 dx$ .

2. Evaluate  $\int_3^4 x + 7x^2 dx$ .

3. Evaluate a)  $\int_0^1 e^{2x} dx$ ,    b)  $\int_0^2 e^{-x} dx$ ,    c)  $\int_{-1}^1 x^2 dx$ ,    d)  $\int_{-1}^1 5x^3 dx$ .

4. Find  $\int_0^{\pi/2} \sin x dx$ .

### Answers

1. a)  $\frac{1}{3}$ ,    b)  $\frac{1}{6}$ ,    c)  $\frac{7}{3}$ ,    d) 64,    e) 0.

2. 89.833 (3dp).

3. a)  $\frac{e^2}{2} - \frac{1}{2} = 3.195$ , (3dp),    b)  $1 - e^{-2} = 0.865$  (3dp),    c)  $\frac{2}{3}$ ,    d) 0.

4. 1.

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For plenty more examples and practice exercises see *Foundation Maths Third Edition*, Chapter 27