Analysis and Approximation of a Fractional Differential Equation

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Abstract

A differential equation is fractional if it involves an operator that can be considered to be between a $(k-1)$th and kth order differential operator, for some positive integer k, and it is said to be of fractional-order if this operator is the highest order operator in the equation. The diffusion equation is of order 2, because its highest order operator is the Laplacian, a 2nd order differential operator, but we can consider an analogous equation of order 2s, where $s \in (0, 1)$, involving the so-called fractional Laplacian operator. Such fractional-order equations appear in a surprising number of real world models. For example, a diffusion model used for cardiac tissue is what is known as anomalous, or non-Fickian, because the diffusion does not satisfy Fick's law of diffusion and is not modelled accurately by the diffusion equation, but actually by a differential equation of fractional order. The diffusion is also anisotropic (directionally dependent) because diffusion along fibers happens at a different rate to that across fibers in the tissue; the mathematical models of are harder to work with. This thesis covers some analysis for the study of fractional-order advection-diffusion equations relevant to this anisotropic cardiac tissue model.

The study of fractional-order equations is difficult: Firstly, fractional-order operators are nonlocal, i.e. the value of a fractional derivative of a function at a point in the domain depends on values of the function throughout the domain; and secondly, boundary conditions (traces) do not make sense in fractional Sobolev spaces of order $s \leq 1/2$, so constraints must be defined on a region of non-zero volume. We review and derive some relevant results on fractional Sobolev spaces, fractional-order operators and the nonlocal calculus developed by Du, Gunzburger, Lehoucq, Zhou (2011). We prove well-posedness of a general class of fractional-order elliptic problems and develop Galerkin approximations, focusing on the derivation of a-priori error bounds.

Preface

This thesis is a fourth year mathematics dissertation worth a whole unit towards the degree of Master of Mathematics and Computer Science.

The target audience is a fourth year undergraduate at Oxford who has taken the *C5.1a Methods of Functional Analysis for PDEs* and *C12.2b Finite Element Methods for PDEs* courses. In particular, we assume that the reader is familiar with the following concepts:

- For $k \in \mathbb{N}$, $1 \leq p \leq \infty$, and open subsets Ω of $\mathbb{R}^n$, basic properties of:
  - The Lebesgue spaces $L^p(\Omega)$
  - The Sobolev spaces $W^{k,p}(\Omega), H^k(\Omega)$
  - Continuous function spaces $C(\Omega), C^k(\Omega), C^\infty(\Omega)$
  - The space of infinitely differentiable functions with compact support $C_0^\infty(\Omega)$
- The finite element method for second-order elliptic PDEs
- A priori and a posteriori error analysis of these methods

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Chapter 1

Introduction

1.1 Introduction

A differential equation is *fractional* if it involves an operator that can be considered to be between a $(k-1)$th and kth order differential operator, for some positive integer k, and it is said to be a *fractional-order differential equation* if this operator is the highest order operator in the equation. What on Earth do we mean by such a vague statement?

Consider Poisson's equation on an open subset Ω of $\mathbb{R}^n$, with source function $f \in L^2(\Omega)$, for a function $u \in H^2(\Omega)$:

$$-\Delta u = f. \tag{1.1}$$

This equation is of order 2 because the Laplacian $-\Delta = -\sum_j \partial_j^2$ is a 2nd-order differential operator. We can consider an analogous equation of order $2s$, where s is a positive real number, involving the so-called fractional Laplacian operator $(-\Delta)^s$:

$$(-\Delta)^s u = f. \tag{1.2}$$

How can we define such an operator? What types of functions u can it operate on? The reader who is familiar with the Fourier transform (Definition A.1) will know that for functions $u \in H^2(\mathbb{R}^n)$ the Laplacian satisfies

$$-\Delta u = \mathcal{F}^{-1}\left[|\boldsymbol{\xi}|^2 \mathcal{F}u\right], \tag{1.3}$$

so a first guess at a definition for the fractional Laplacian could be:

$$(-\Delta)^s u = \mathcal{F}^{-1}\left[|\boldsymbol{\xi}|^{2s} \mathcal{F}u\right]. \tag{1.4}$$

This operator certainly makes sense for $u \in C_0^\infty(\mathbb{R}^n)$ (smooth, compactly supported), but can it be extended to some other function space? Some kind of fractional Sobolev space $H^{2s}(\mathbb{R}^n)$ perhaps (do these spaces even exist)? Does this operator have an explicit form? Can it be generalised to make sense within an open domain $\Omega \subset \mathbb{R}^n$? We hope to answer some of these questions.

For a positive integer k, partial derivatives of a function $u \in H^k(\mathbb{R}^n)$ satisfy:

$$\partial_j^k u = \mathcal{F}^{-1}\left[(i\xi_j)^k \mathcal{F}u\right], \tag{1.5}$$

so one could define a partial derivative of order $s > 0$ for functions in $C_0^\infty(\mathbb{R}^n)$ by:

$$\partial_j^s u = \mathcal{F}^{-1}\left[(i\xi_j)^s \mathcal{F}u\right]. \tag{1.6}$$

However, with this definition $(-\Delta)^s \neq \sum_j \partial_j^{2s}$, because in general $(\xi_1^2 + \cdots + \xi_n^2)^s \neq \xi_1^{2s} + \cdots + \xi_n^{2s}$ and the Fourier transform is an injective operator. We see further that if we define a fractional Nabla operator by $\nabla^s = (\partial_1^s, \dots, \partial_n^s)$, then we have $(-\Delta)^s \neq -\nabla^r \cdot \nabla^t$ for any r and t; the relationship between the fractional Laplacian and fractional derivatives of these types are not as simple as in the classical case, in which we have $-\Delta = -\nabla \cdot \nabla$.

The classical gradient operator $\nabla = (\partial_1, \dots, \partial_n)$ also has the important property that it contains all the information required to compute directional derivatives. If $\boldsymbol{m}$ is a unit vector in $\mathbb{R}^n$, the directional derivative operator in the direction $\boldsymbol{m}$ is simply $\boldsymbol{m}\cdot\nabla$. Checking the difference between $\boldsymbol{m}\cdot\nabla^s$ and $\mathcal{F}^{-1}\left[(i\boldsymbol{m}\cdot\boldsymbol{\xi})^s \mathcal{F}u\right]$, we see that this is not the case for our naïvely defined fractional Nabla gradient ∇^s. We need more than just the n partial derivatives of order s to adequately describe fractional analogues of the gradient and the divergence. It is for this reason that we do not call fractional-order differential equations in n dimensions PDEs — in general they involve directional derivatives in all directions. From now on we will use the abbreviation FDE, for Fractional Differential Equation.

The lack of these nice properties stem from the fact that fractional-order differential operators are *nonlocal*. One finds that, if $\mathcal{L}$ is a fractional-order operator, for $\boldsymbol{x} \in \mathbb{R}^n$ $\mathcal{L}[u](\boldsymbol{x})$ depends not just on the values of u in any infinitesimally small ball around $\boldsymbol{x}$ (as is the case for classical differential operators), but also on values of u at other points $\boldsymbol{y}$ in the domain, not local to $\boldsymbol{x}$. The fractional Laplacian operator defined by (1.4) turns out to depend on the values of u at *all* points in $\mathbb{R}^n$.

Fractional-order differential operators are not uniquely defined; we do not require much more than that for integrer s the operator is precisely the same as the sth order classical differential operator. The most well-known are the Riemann-Liouville fractional derivative which we study in Chapter 3, and the Riesz symmetric fractional derivative, but there are other fractional calculi, named after authors such as Caputo [22], Hadamard [20], Grünwald, and Letnikov [25, p. 13].

Remark 1.1.1. The term “fractional” can be misleading. It could suggest that the values of s we are considering must be fractions, rational numbers. The name refers to the fractional *part* of a positive real number r, $\{r\} = r - \lfloor r \rfloor$.

1.2 Fractional-order models in science and nature

Fractional-order differential equations occur in a surprising number of real-world models. At the heart of a lot of applications is the phenomenon of *anomalous diffusion.* The *isotropic*[1] *normal diffusion equation* is (with time scaled to remove physical constants):

$$u_t - \Delta u = 0, \tag{1.7}$$

and can be derived in a number of ways: a random walk model, Fick's law of diffusion and the Langevin equation are discussed in [38]. The author has not studied this in depth as part of the project, however according to Vlahos et al., the assumptions for these models are fair for diffusion in homogeneous media, but not for a medium which is highly heterogeneous, a particular case they discuss is when the diffusive system is far from equilibrium.

A more general, relaxed model is the continuous time random walk (CTRW) [16, sect. 4],[38, sect. 5]. The heterogeneity of the medium can be expressed with this model, parametrised by a variable $\alpha \in [1, 2]$ where $\alpha = 2$ corresponds to the diffusion equation above, but other values of α give an FDE of order α:

$$u_t + (-\Delta)^{\alpha/2} u = 0, \tag{1.8}$$

where $(-\Delta)^{\alpha/2}$ is precisely the fractional Laplacian described above in (1.4) with $s = \alpha/2$. We call this equation the *(isotropic) anomalous diffusion equation.* For a short, informal introduction to the underlying stochastic processes see [21].

Benson, Wheatcraft and Meershaert have performed experiments and theoretical studies into contaminant transport in aquifers governed by a fractional advection-dispersion equation, with experimental evidence giving models of orders $\alpha = 1.55$, 1.65 and 1.8 for various aquifer locations [3]. Anomalous diffusion has also been measured experimentally in biological cell processes [39], [33]. The transport of charged particles in a magnetic field can under some circumstances be anomalous [29]. These are just a few examples, but the list of recorded phenomena in the literature is impressive, and growing (see [27] and the introduction to [6]).

Not only physical phenomena can be the result of a diffusive process. Stochastic processes in mathematical finance are often modelled by a Wiener process, or Brownian motion, and these lead to a diffusion-type PDE, but if the stochastic process is so-called *heavy tailed* as opposed to Gaussian, then the governing equations are FDEs [32].

Sometimes the diffusion is not isotropic. For example, diffusion in cardiac tissue is *anisotropic*, because diffusion along tissue fibers happens at a different rate to diffusion across fibers. A directional dependence can be modelled with the anisotropic diffusion equation:

$$u_t - \nabla \cdot (\boldsymbol{A}(\boldsymbol{x})\nabla u) = 0. \tag{1.9}$$

[1] Radially symmetric. From the Greek *iso* (equal) and *tropos* (direction)

Here $\boldsymbol{A}$ is a matrix-valued function defined for $\boldsymbol{x}$ in $\mathbb{R}^n$. A fractional analogue of this can be derived from physical models too; Meerschaert et al. formulate a similar equation to the following using a fractional Fick's law [26]:

$$u_t - \nabla \cdot (J_M^{1-\beta} \nabla u) = 0, \tag{1.10}$$

where for a vector-valued function $\boldsymbol{v}$,

$$J_M^{1-\beta}[\boldsymbol{v}] = \mathcal{F}^{-1} \left[\int_{|\boldsymbol{m}|=1} \boldsymbol{m}(i\boldsymbol{m} \cdot \boldsymbol{\xi})^{\beta-1} \boldsymbol{m} \cdot \hat{\boldsymbol{v}}(\boldsymbol{\xi})\, M(d\boldsymbol{m}) \right]. \tag{1.11}$$

Here $\beta \in (0,1)$ and M is a probability measure on the unit sphere in $\mathbb{R}^n$ that describes the anisotropy of the diffusion. This strange operator is a fractional integral operator of differential order $\beta - 1 < 0$, which we can see from the presence the $(i\boldsymbol{m} \cdot \boldsymbol{\xi})^{\beta-1}$ term; it counteracts the two first-order terms in (1.10), making the FDE of order $1 + \beta$.

Note that the anisotropy doesn't depend on $\boldsymbol{x}$. This is an issue, since most models only obey a constant anisotropy like this on a local scale. We will address this in the next section.

1.3 Fractional-order elliptic problems

Often, a process one wishes to model will involve more than just diffusion. The general PDE modelling this has some extra terms:

$$u_t - \nabla \cdot (\boldsymbol{A}(\boldsymbol{x}) \nabla u) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u + c(\boldsymbol{x}) u = f. \tag{1.12}$$

The extra terms have physical relevance and we paraphrase and modify [17, p. 313] to describe them: $u(\boldsymbol{x}, t)$ can be considered to be the density or concentration of some quantity at position $\boldsymbol{x}$ and time t, such as heat in a metal bar or a chemical in solution. $-\nabla \cdot (\boldsymbol{A} \nabla u)$ represents the diffusion of u, with any anisotropy encoded in the matrix $\boldsymbol{A}$. The first-order term $\boldsymbol{b} \cdot \nabla$ corresponds to the *advection* or transport of the substance due to overall motion in a particular direction described by $\boldsymbol{b}$. The zeroth-order term cu is called the *reaction* term, and describes any general increase or decrease in the concentration and $\boldsymbol{x}$ and time t. Hence these equations are sometimes called *advection-diffusion(-reaction) equations.*

This motivates the study of general *second-order elliptic problems*: For a bounded open subset Ω of $\mathbb{R}^n$, and functions $\boldsymbol{A} \in C^1(\bar{\Omega})^{n \times n}, \boldsymbol{b} \in C^1(\bar{\Omega})^n, c \in C(\bar{\Omega}), f \in L^2(\Omega), g \in C^1(\partial\Omega)$,

$$\begin{aligned} -\nabla \cdot (\boldsymbol{A}(\boldsymbol{x}) \nabla u) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u + c(\boldsymbol{x}) u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1.13}$$

The problem can be thought of as an advection-diffusion problem (1.12) in equilibrium, or simply one where $u_t \equiv 0$. In general an equation in which u_t is assumed to be zero is called the *time-independent* version of the equation. The non-trivial analysis of the time-independent equation can then be extended for time-dependency later.

Finally, we come to the specific aim of the project. We wish to study fractional-order problems of the form:

$$\begin{aligned} \mathcal{D}(\boldsymbol{\Theta}(\boldsymbol{x},\boldsymbol{y})\mathcal{D}^*u) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u + c(\boldsymbol{x})u &= f \quad \text{in } \Omega, \\ u &= h \quad \text{in } \Omega_c \subseteq \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{1.14}$$

developing analyses similar to those in the *C5.1a* and *C12.2b* courses [34], [35], such as notions of a weak solution, well-posedness of the weak problem and finite element approximations.

This FDE is similar to (1.13). The diffusion term has changed, with the intention that it models anisotropic *anomalous* diffusion, and now involves an operator we denote $\mathcal{D}$. This operator is the nonlocal divergence of the nonlocal calculus we describe in Chapter 4, with nonlocal adjoint $\mathcal{D}^*$, both operators being of order $s \in (0, 1)$.

The anisotropy is described by $\boldsymbol{\Theta}$, a matrix-valued function of two variables in $\mathbb{R}^n$, $\boldsymbol{x}$ *and* $\boldsymbol{y}$. This may seem confusing at first, so let us be clear: u is a scalar-valued function of one variable $\boldsymbol{x}$, but the nonlocal operator $\mathcal{D}^*$, when applied to u, gives a function of two variables, $\boldsymbol{x}$ and $\boldsymbol{y}$. The second variable is used to describe nonlocal properties and is "integrated out" by the nonlocal divergence operator $\mathcal{D}$ (which operates on vector-valued functions of two variables to produce a scalar-valued function of one). The $\boldsymbol{\Theta}$ operator has differential order zero, since it is merely a matrix-valued function of $\boldsymbol{x}$ and $\boldsymbol{y}$. That makes the operator $\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^*\cdot)$ of order $2s$.

We also have that the constraints are not imposed over the surface $\partial\Omega$, but over the volume $\Omega_c \subseteq \mathbb{R}^n \setminus \Omega$. This is because in the fractional order Sobolev space $H^s(\Omega)$, a trace operator mapping functions defined on Ω to those defined on $\partial\Omega$ is only well-defined for $s > \frac{1}{2}$. To generalise to problems with $s \leq \frac{1}{2}$, we must consider volume-constrained problems.

We hope the reader is intrigued rather than deterred by our vagueness, as now we carve a rigorous path to a clearer picture. In the next chapter we define a natural setting for functions that may be solutions to fractional-order problems like (1.14), and proceed to discuss fractional-order operators including the Riemann-Liouville derivative and the fractional Laplacian. We then turn to a modern development in the theory of these operators, the nonlocal calculus developed by Du, Gunzburger, Lehoucq and Zhou (2011) for nonlocal volume constrained problems.

Chapter 6 is concerned with a Lax–Milgram-type approach to a proof of well-posedness of fractional-order elliptic equations (under some restrictions consistent with physically relevant models). Chapter 7 is devoted to Galerkin approximation schemes, with a priori error estimates discussed.

Chapter 2

Fractional-Order Function Spaces

In this chapter we create a function space setting that allows us to have a domain for our fractional order operators and a solution space for the FDEs we study later.

2.1 Smoothness and Regularity

Consider the space of continuous functions, $C(\bar{\Omega})$. For any positive integer k, we can take the subspace of k-times continuously differentiable functions to have a function space of differential "order" k. Which subspaces of $C(\bar{\Omega})$ can be considered to be of order s, a positive real number, and what constraints on the elements of $C(\bar{\Omega})$ derive them?

The subspace generated by such a constraint should be $C^k(\bar{\Omega})$ if $s = k$, an integer. Ideally, it would also be endowed with a suitable norm so that it is a Banach space; such spaces satisfy useful functional analysitic properties that incomplete spaces do not. Hölder's solution is as follows [17, p. 255]:

Definition 2.1.1. For a bounded open subset Ω of $\mathbb{R}^n$, the Hölder space of order k (a non-negative integer) with exponent $\sigma \in [0, 1]$ is defined to be:

$$C^{k,\sigma}(\bar{\Omega}) = \left\{ u \in C^k(\bar{\Omega}) : \sum_{|\alpha|=k} |D^\alpha u|_{C^{0,\sigma}(\bar{\Omega})} < \infty \right\}, \tag{2.1}$$

where

$$|u|_{C^{0,\sigma}(\bar{\Omega})} = \sup_{\boldsymbol{x},\boldsymbol{y} \in \bar{\Omega}, \boldsymbol{x} \neq \boldsymbol{y}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\sigma}. \tag{2.2}$$

Functions in $C^{k,\sigma}(\bar{\Omega})$ are k times continously differentiable with the highest order derivatives satisfying condition (2.2). It is easy to see that $C^{0,1}(\bar{\Omega})$ is the space of

Lipschitz continuous functions on $\bar{\Omega}$, $C^{0,0}(\bar{\Omega})$ is the space of continuous functions, and $C^{0,\sigma}(\bar{\Omega})$ for $\sigma \in (0,1)$ is something in between.

Remark 2.1.2. If a function u is Hölder continuous with exponent $\sigma > 1$, then for all $\boldsymbol{x}, \boldsymbol{z} \in \Omega$:

$$\left| \lim_{h \to 0} \frac{u(\boldsymbol{x} + h\boldsymbol{z}) - u(\boldsymbol{x})}{h} \right| \leq \lim_{h \to 0} h^{\sigma - 1} \cdot |\boldsymbol{z}| = 0,$$

so u is constant.

Theorem 2.1.3. *Let $C^{k,\sigma}(\bar{\Omega})$ be endowed with the norm:*

$$\|u\|_{C^{k,\sigma}(\bar{\Omega})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_\infty + \sum_{|\alpha| = k} |D^\alpha u|_{C^{0,\sigma}(\bar{\Omega})}, \tag{2.3}$$

where $\|v\|_\infty = \sup_{\boldsymbol{x} \in \Omega} |v(\boldsymbol{x})|$. Then $C^{k,\sigma}(\bar{\Omega})$ is a Banach space.

Proof. The proof is a straightforward exercise consequence of the fact that $C^k(\bar{\Omega})$ is complete with the norm $\|u\| = \sum_{|\alpha| \leq k} \|D^\alpha u\|_\infty$. The case $k = 0$ was a set problem in the C5.1a course. □

Functions in $C^{k,\sigma}(\bar{\Omega})$ are said to have kth order smoothness with Hölder regularity σ. We can consider $C^{k,\sigma}(\bar{\Omega})$ to have order $s = k + \sigma$, but note $C^{k,1}(\Omega) \neq C^{k+1,0}(\Omega)$; these spaces are a bit more complicated than what we required, because of the subtle difference between smoothness and regularity. In the next section we remove the smoothness component entirely, for a fractional-order space parametrised by a single real number s.

2.2 Fractional Sobolev spaces

As we saw in the C5.1a and C12.2b courses, [34],[35], for the study of differential equations, which is classically performed in $C^k(\bar{\Omega})$, it can be much more fruitful to study weaker forms of equations in *Sobolev spaces* $W^{k,p}(\Omega)$, where notions of smoothness are relaxed to require only the existence of weak derivatives. For the fractional version, we constrain functions in a Lebesgue space using a quotient similar to that used for Hölder spaces.

According to Di Nezza et al. [9], our main reference for this chapter, the following approach is due to Aronsajn, Gagliardo and Slobodeckij.

Definition 2.2.1. For $s \in (0,1)$, $p \in [1, +\infty)$ and $\Omega \subseteq \mathbb{R}^n$ open, the Sobolev space of order s with Lebesgue exponent p is defined by:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \left((\boldsymbol{x}, \boldsymbol{y}) \mapsto \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^{\frac{n}{p} + s}} \right) \in L^p(\Omega \times \Omega) \right\}, \tag{2.4}$$

endowed with norm:

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p \, d\boldsymbol{x} + \int_\Omega \int_\Omega \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+ps}} \, d\boldsymbol{y} \, d\boldsymbol{x} \right)^{\frac{1}{p}}, \tag{2.5}$$

and semi-norm (sometimes called the *Gagliardo* semi-norm):

$$|u|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+ps}} \, d\boldsymbol{y} \, d\boldsymbol{x} \right)^{\frac{1}{p}}. \tag{2.6}$$

Theorem 2.2.2. *$W^{s,p}(\Omega)$ is a Banach space intermediate between $L^p(\Omega)$ and $W^{1,p}(\Omega)$*

Proof. See *Adams, Sobolev Spaces* for a proof [1, 7.36]. It is quite technical. □

The following fact is troublesome: Definition 2.2.1 cannot be used for $s \geq 1$ [9]. This is due to the fact that for such s and any measurable function $u : \Omega \to \mathbb{R}^n$, if

$$\int_\Omega \int_\Omega \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+ps}} \, d\boldsymbol{y} \, d\boldsymbol{x} < \infty,$$

then u is constant on each connected subset of Ω [5, Proposition 2]. Nevertheless, there is a natural way to define fractional Sobolev spaces for all $s > 0$. If k is the unique integer such that $k - 1 < s \leq k$ and $k - s = \sigma$,

$$W^{s,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \sum_{|\alpha| = k} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)} < \infty \right\}. \tag{2.7}$$

We endow this linear space with the natural norm and seminorm.

We are most interested in the case when $p = 2$, as weak formulations for differential equations can be expressed using the L^2 inner product. In this case the fractional Sobolev space is also a Hilbert space and we can define an inner product

$$\langle u, v \rangle = \langle u, v \rangle_{L^2(\Omega)} + \int_\Omega \int_\Omega \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{\frac{n}{2}+s}} \cdot \frac{v(\boldsymbol{x}) - v(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{\frac{n}{2}+s}} \, d\boldsymbol{y} \, d\boldsymbol{x}, \tag{2.8}$$

which the reader can easily check satisfies all of the properties of an inner product. As is customary for integer-order Sobolev spaces, we donote this Hilbert space by $H^s(\Omega)$.

In the next section, we exploit properties of the Fourier transform in $L^2(\mathbb{R}^n)$ to give simple characterisation of $H^s(\mathbb{R}^n)$, and prove some useful results about these spaces.

2.3 Fourier transform approach

Theorem 2.3.1 (Fourier transform characterisation of $H^1(\mathbb{R}^n)$)**.** *Define the space:*

$$\hat{H}^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (\boldsymbol{\xi} \mapsto |\boldsymbol{\xi}|\hat{u}(\boldsymbol{\xi})) \in L^2(\mathbb{R}^n)\}, \tag{2.9}$$

endowed with the norm:

$$\|u\|_{\hat{H}^1(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)|\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \right)^{\frac{1}{2}}, \tag{2.10}$$

and semi-norm:

$$|u|_{\hat{H}^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\boldsymbol{\xi}|^2 |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \right)^{\frac{1}{2}}. \tag{2.11}$$

Then $\hat{H}^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ with equal norms and semi-norms.

Proof. By Plancherel's Theorem A.2 and Proposition A.4, for any $u \in H^1(\mathbb{R}^n)$:

$$\|\partial_j u\|^2_{L^2(\mathbb{R}^n)} = \left\|\widehat{\partial_j u}\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi_j|^2 |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}. \tag{2.12}$$

Therefore,

$$\begin{aligned} |u|^2_{H^1(\mathbb{R}^n)} &= \sum_{j=1}^n \|\partial_j u\|^2_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \left(\sum_{j=1}^n |\xi_j|^2\right) |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^2 |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ &= |u|^2_{\hat{H}^1(\mathbb{R}^n)}. \end{aligned} \tag{2.13}$$

It follows by another application of Plancherel's Theorem that the norms are also equal. So u is also a member of $\hat{H}^1(\mathbb{R}^n)$.

Conversely, suppose that $u \in \hat{H}^1(\mathbb{R}^n)$. Then,

$$\|(i\xi_j)\hat{u}(\boldsymbol{\xi})\|_{L^2(\mathbb{R}^n)} \leq \||\boldsymbol{\xi}|\hat{u}(\boldsymbol{\xi})\|_{L^2(\mathbb{R}^n)} < \infty, \tag{2.14}$$

and we can define $u^{(j)} := \mathcal{F}^{-1}[(i\xi_j)\hat{u}(\boldsymbol{\xi})] \in L^2(\mathbb{R}^n)$. We find, using Parseval's Theorem A.3, that $u^{(j)}$ is the weak derivative of u in the jth component:

$$\begin{aligned} \int_{\mathbb{R}^n} u^{(j)} v \, d\boldsymbol{x} &= \int_{\mathbb{R}^n} (i\xi_j)\hat{u}\bar{\hat{v}} \, d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} \hat{u}\overline{(-i\xi_j)\hat{v}} \, d\boldsymbol{\xi} \\ &= -\int_{\mathbb{R}^n} \hat{u}\overline{\widehat{\partial_j v}} \, d\boldsymbol{\xi} \\ &= -\int_{\mathbb{R}^n} u\partial_j v \, d\boldsymbol{x} \qquad \forall v \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{2.15}$$

Therefore $u \in H^1(\mathbb{R}^n)$ and this completes the proof. □

Remark 2.3.2. This proof is easily extended to show that $H^k(\mathbb{R}^n)$ can be characterised by:

$$\hat{H}^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : (\boldsymbol{\xi} \to |\boldsymbol{\xi}|^k \hat{u}(\boldsymbol{\xi})) \in L^2(\mathbb{R}^n)\right\}, \tag{2.16}$$

with the norm:

$$\|u\|_{\hat{H}^k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\boldsymbol{\xi}|^2)^k |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}\right)^{\frac{1}{2}}, \tag{2.17}$$

and semi-norm:

$$|u|_{\hat{H}^k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{2k} |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}\right)^{\frac{1}{2}}. \tag{2.18}$$

Definition 2.3.3. For any real $s \geq 0$ we define the Fourier fractional Sobolev space:

$$\hat{H}^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (\boldsymbol{\xi} \mapsto |\boldsymbol{\xi}|^s \hat{u}(\boldsymbol{\xi})) \in L^2(\mathbb{R}^n)\}, \tag{2.19}$$

endowed with the norm:

$$\|u\|^2_{\hat{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)^s |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}, \tag{2.20}$$

and semi-norm:

$$|u|^2_{\hat{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{2s} |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}. \tag{2.21}$$

Remark 2.3.4. Note that this definition is valid for all non-negative real s, not just $s \in (0,1)$.

Theorem 2.3.5. *Let $0 \leq s' \leq s$. Then $\hat{H}^s(\mathbb{R}^n) \subseteq \hat{H}^{s'}(\mathbb{R}^n)$ and for any $u \in \hat{H}^s(\mathbb{R}^n)$:*

$$\|u\|_{\hat{H}^{s'}(\mathbb{R}^n)} \leq \|u\|_{\hat{H}^s(\mathbb{R}^n)}. \tag{2.22}$$

Proof. This is obvious because $(1 + |\boldsymbol{\xi}|^2)^{s'} \leq (1 + |\boldsymbol{\xi}|^2)^s$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$. $\square$

This theorem shows that the continuous embedding $H^k(\mathbb{R}^n) \subset H^{k'}(\mathbb{R}^n)$ for non-negative integers $k' \leq k$ is true when we generalise to non-negative real numbers. Now we show that for $s \in (0,1)$, the Fourier fractional Sobolev space coincides with the Gagliardo approach for $\Omega = \mathbb{R}^n$ and $p = 2$.

Theorem 2.3.6 (Equivalence of semi-norms)**.** *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$. There exists a constant $\hat{C}$ depending only on n and s such that:*

$$|u|^2_{\hat{H}^s(\mathbb{R}^n)} = \frac{1}{2}\hat{C}|u|^2_{H^s(\mathbb{R}^n)}. \tag{2.23}$$

This constant is precisely:

$$\hat{C}(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\boldsymbol{\zeta}|^{n+2s}} \, d\boldsymbol{\zeta} \right)^{-1}. \tag{2.24}$$

Proof. We follow the proof in [9, Proposition 3.4]. For every fixed $\boldsymbol{y} \in \mathbb{R}^n$ we can change variables to $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ to get (using Fubini's theorem and Plancherel's theorem):

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}} \, d\boldsymbol{y} \, d\boldsymbol{x} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(\boldsymbol{z} + \boldsymbol{y}) - u(\boldsymbol{y})|^2}{|\boldsymbol{z}|^{n+2s}} \, d\boldsymbol{z} \, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} |\boldsymbol{z}|^{-(n+2s)} \left(\int_{\mathbb{R}^n} |u(\boldsymbol{z} + \boldsymbol{y}) - u(\boldsymbol{y})|^2 \, d\boldsymbol{y} \right) d\boldsymbol{z} \\
&= \int_{\mathbb{R}^n} |\boldsymbol{z}|^{-(n+2s)} \left(\int_{\mathbb{R}^n} |\mathcal{F}[u(\boldsymbol{z} + \cdot) - u(\cdot)](\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \right) d\boldsymbol{z} \\
&= \int_{\mathbb{R}^n} |\boldsymbol{z}|^{-(n+2s)} \left(\int_{\mathbb{R}^n} \left|(e^{i\boldsymbol{z}\cdot\boldsymbol{\xi}} - 1)\hat{u}(\boldsymbol{\xi})\right|^2 \, d\boldsymbol{\xi} \right) d\boldsymbol{z}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|e^{i\boldsymbol{z}\cdot\boldsymbol{\xi}} - 1|^2}{|\boldsymbol{z}|^{n+2s}} \, d\boldsymbol{z} \right) |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(e^{i\boldsymbol{z}\cdot\boldsymbol{\xi}} - 1)(e^{-i\boldsymbol{z}\cdot\boldsymbol{\xi}} - 1)}{|\boldsymbol{z}|^{n+2s}} \, d\boldsymbol{z} \right) |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&= 2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\boldsymbol{z}\cdot\boldsymbol{\xi})}{|\boldsymbol{z}|^{n+2s}} \, d\boldsymbol{z} \right) |\hat{u}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}. \qquad (2.25)
\end{aligned}
$$

Now consider the rotation matrix $\boldsymbol{R}$ for which $\boldsymbol{R}(|\boldsymbol{\xi}|\boldsymbol{e}_1) = \boldsymbol{\xi}$ and perform the substitution $\zeta = |\boldsymbol{\xi}|\boldsymbol{R}^T\boldsymbol{z}$. Then

$$
\boldsymbol{z}\cdot\boldsymbol{\xi} = |\boldsymbol{\xi}|^{-1}\boldsymbol{R}\boldsymbol{\zeta}\cdot\boldsymbol{R}(|\boldsymbol{\xi}|\boldsymbol{e}_1) = \boldsymbol{R}^T\boldsymbol{R}\boldsymbol{\zeta}\cdot\boldsymbol{e}_1 = \zeta_1.
$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{1-\cos(\boldsymbol{z}\cdot\boldsymbol{\xi})}{|\boldsymbol{z}|^{n+2s}} \, d\boldsymbol{z} &= \int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\boldsymbol{\zeta}/|\boldsymbol{\xi}||^{n+2s}} \frac{1}{|\boldsymbol{\xi}|^n} \, d\boldsymbol{\zeta} \\
&= |\boldsymbol{\xi}|^{2s} \int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\boldsymbol{\zeta}|^{n+2s}} \, d\boldsymbol{\zeta} \\
&= |\boldsymbol{\xi}|^{2s}\hat{C}(n,s)^{-1}. \qquad (2.26)
\end{aligned}
$$

Combining (2.25) and (2.26) gives:

$$
|u|^2_{H^s(\mathbb{R}^n)} = 2\hat{C}(n,s)^{-1}|u|^2_{\hat{H}^s(\mathbb{R}^n)}, \qquad (2.27)
$$

which completes the proof. □

Lemma 2.3.7. *Let $a > 0$ and $s \in (0,1)$. Then,*

$$
\frac{1}{2^{1-s}}(1 + a^s) \leq (1+a)^s \leq (1 + a^s) \qquad (2.28)
$$

Proof. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be defined by $f(a) = a^s$. Then $s \in (0,1)$ implies that f is concave. Therefore:

$$
\frac{1}{2}(1 + a^s) \leq \frac{1}{2^s}(1+a)^s. \qquad (2.29)
$$

This gives us the left inequality. For the right inequality, note that $s \in (0,1] \mapsto (1+a^s)^{\frac{1}{s}}$ is monotonic decreasing on $(0,1]$. Hence $1 + a \leq (1+a^s)^{\frac{1}{s}}$ for all $s \in (0,1]$. □

Theorem 2.3.8. *For $s \in (0,1)$, $H^s(\mathbb{R}^n) = \hat{H}^s(\mathbb{R}^n)$ with equivalent norms.*

Proof. Combining Theorem 2.3.6 and Lemma 2.3.7 gives:

$$
\frac{1}{2^{1-s}} \min\left\{1, 2\hat{C}^{-1}\right\} \|u\|^2_{\hat{H}^s(\mathbb{R}^n)} \leq \|u\|^2_{H^s(\mathbb{R}^n)} \leq \max\left\{1, \frac{1}{2}\hat{C}\right\} \|u\|^2_{\hat{H}^s(\mathbb{R}^n)}. \qquad (2.30)
$$

Therefore the two norms are equivalent. □

Corollary 2.3.9. *Let $0 < s' \leq s < 1$. Then $H^s(\mathbb{R}^n) \subseteq H^{s'}(\mathbb{R}^n)$ and for $u \in H^s(\mathbb{R}^n)$:*

$$\|u\|^2_{H^{s'}(\mathbb{R}^n)} \leq \|u\|^2_{H^s(\mathbb{R}^n)}, \tag{2.31}$$

where $C_{\mathrm{emb}}(s', s) = 2^{1-s} \max\left\{1, \frac{1}{2}\hat{C}(n, s')\right\} \max\left\{1, \frac{1}{2}\hat{C}(n, s)\right\}$.

Remark 2.3.10. Continuous embeddings of this form can be proved for general $W^{s,p}(\Omega)$ spaces, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$[9, Props. 2.1, 2.2, 2.3]. However, the constants in the more general inequalities are difficult to find explicitly, which we value highly for the computation of error bounds for approximation schemes.

2.4 Further properties of fractional Sobolev spaces

Here we present more theorems that will be of use to us later.

Theorem 2.4.1. *For any $s \geq 0$, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$*

Proof. See *Sobolev Spaces*, by Adams [1, Thm. 7.38]. □

Theorem 2.4.2. *Let $b \in W^{1,\infty}(\Omega)$. Then b is Lipschitz continuous with Lipschitz constant $\|\nabla b\|_\infty$.*

Proof. Evans remarks that for general open subsets Ω of $\mathbb{R}^n$, $b \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ if and only if b is Lipschitz continuous [17, p. 295]. Since $W^{1,\infty}(\Omega) \subseteq W^{1,\infty}_{\mathrm{loc}}(\Omega)$ we have the theorem. □

Lemma 2.4.3. *Let $u \in H^s(\mathbb{R}^n)$ and $b \in W^{1,\infty}(\mathbb{R}^n)$. Then there exists a constant $C_{b,s}$ depending on s, n and b such that:*

$$\|bu\|_{H^s(\mathbb{R}^n)} \leq C_{b,s} \|u\|_{H^s(\mathbb{R}^n)}. \tag{2.32}$$

Further, the constant can be explicitly stated:

$$C_{b,s} = \|b\|_\infty \cdot \left(1 + \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\left(\frac{4}{s} + \frac{1}{1-s}\left(\frac{\|\nabla b\|_\infty}{\|b\|_\infty}\right)^2\right)\right)^{\frac{1}{2}}. \tag{2.33}$$

Proof. This proof takes ideas from [9, Lemma 5.3]. It is straightforward to prove $\|bu\|_{L^2(\mathbb{R}^n)} \leq \|b\|_\infty \|u\|_{L^2(\mathbb{R}^n)}$. Now for the semi-norm:

$$\begin{aligned} |u|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(\boldsymbol{x})u(\boldsymbol{x}) - b(\boldsymbol{y})u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \\ &\leq 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(\boldsymbol{x})u(\boldsymbol{x}) - b(\boldsymbol{x})u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \\ &\quad +2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(\boldsymbol{x})u(\boldsymbol{y}) - b(\boldsymbol{y})u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \end{aligned}$$

$$\leq \quad 2\left(\|b\|_\infty^2 |u|^2_{H^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(\boldsymbol{x})|^2 \frac{|b(\boldsymbol{x})-b(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x}\right).$$

We used the inequality $(a+b)^2 < 2a^2+2b^2$ for the second line. Now we can do a change of variables $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$, and use the Theorem 2.4.2 for $|\boldsymbol{z}| \leq 1$:

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(\boldsymbol{x})|^2 \frac{|b(\boldsymbol{x})-b(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \quad &= \quad \int_{\mathbb{R}^n} |u(\boldsymbol{x})|^2 \int_{\mathbb{R}^n} \frac{|b(\boldsymbol{x})-b(\boldsymbol{z}+\boldsymbol{x})|^2}{|\boldsymbol{z}|^{n+2s}}\, d\boldsymbol{z}\, d\boldsymbol{x} \\
&\leq \quad \|u\|^2_{L^2(\mathbb{R}^n)} \|\nabla b\|^2_\infty \int_{|\boldsymbol{z}|\leq 1} |\boldsymbol{z}|^{-n+2(1-s)}\, d\boldsymbol{z} \\
&\qquad + \|u\|^2_{L^2(\mathbb{R}^n)}\, 2^2 \|b\|^2_\infty \int_{|\boldsymbol{z}|>1} |\boldsymbol{z}|^{-n-2s}\, d\boldsymbol{z} \\
&= \quad \|u\|^2_{L^2(\mathbb{R}^n)} \|\nabla b\|^2_\infty \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^1 r^{2(1-s)-1}\, dr \\
&\quad +4\, \|u\|^2_{L^2(\mathbb{R}^n)} \|b\|^2_\infty \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_1^\infty r^{-2s-1}\, dr \\
&= \quad \|u\|^2_{L^2(\mathbb{R}^n)} \|\nabla b\|^2_\infty \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{1-s} \\
&\qquad + \|u\|^2_{L^2(\mathbb{R}^n)} \|b\|^2_\infty \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{4}{s}.
\end{aligned}$$

We used the formula for the surface area of the unit hypersphere in $\mathbb{R}^n$ for a change of variables to hyperspherical coordinates. Collecting these inequalities:

$$\begin{aligned}
\|bu\|^2_{H^s(\mathbb{R}^n)} \quad &\leq \quad \|u\|^2_{L^2(\mathbb{R}^n)} \left(\|b\|^2_\infty + 2\|\nabla b\|^2_\infty \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{1-s} + 2\|b\|^2_\infty \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{4}{s}\right) \\
&\qquad +2\|b\|^2_\infty |u|^2_{H^s(\mathbb{R}^n)} \\
&\leq \quad C^2_{b,s}\, \|u\|^2_{H^s(\mathbb{R}^n)}\, .
\end{aligned}$$

For the final step we have used the fact that:

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{4}{s} \geq 1.$$

This completes the proof. □

2.5 Traces and fractional Sobolev spaces

A linear operator T, which maps a function u defined on Ω to $u_{\restriction \partial\Omega}$, its restriction to the boundary, is called a *trace operator*. For continuous functions this is a trivial operation as there is a natural embedding $C(\bar{\Omega}) \hookrightarrow C(\partial\Omega)$.

Now, for a function in $W^{1,p}(\Omega)$, the interpretation of a trace operator isn't so simple. Sobolev spaces are a subspace of Lebesgue spaces, so firstly its elements are not necessarily continuous and secondly they are not necessarily defined everywhere on $\bar{\Omega}$.

Consider the operator $T : C^1(\bar{\Omega}) \to L^p(\partial\Omega)$ defined as above. Recall from the *C5.1a* course [34], that if this operator is continuous, then we can extend T uniquely to the closure of $C^1(\bar{\Omega})$ in $W^{1,p}(\Omega)$, namely the space itself, $W^{1,p}(\Omega)$, giving us the a continuous trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$. We need the following definition.

Definition 2.5.1 ([9]). Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω is of class $C^{k,\sigma}$ for $k \in \mathbb{N}, \sigma \in [0,1]$ if the exists a positive constant M such that for every $\boldsymbol{x} \in \partial\Omega$, there exists $r > 0$ and an isomorphism $S : C \to B(\boldsymbol{x}; r)$ such that:

$$S \in C^{k,\sigma}(\bar{C}), \quad S^{-1} \in C^{k,\sigma}(\overline{B(\boldsymbol{x}; r)}),$$

$$B(\boldsymbol{x}; r) \cap \Omega = S(C_+), \quad B(\boldsymbol{x}; r) \cap \partial\Omega = S(C_0),$$

$$\|S\|_{C^{k,\sigma}(\bar{C})} + \|S^{-1}\|_{C^{k,\sigma}(\overline{B(\boldsymbol{x};r)})} \leq M,$$

where C is the cylinder:

$$C := \left\{\boldsymbol{x} = (\bar{\boldsymbol{x}}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\bar{\boldsymbol{x}}| < 1 \text{ and } |x_n| < 1\right\},$$

and

$$C_+ := \{\boldsymbol{x} \in C : x_n > 0\}, \quad C_0 := \{\boldsymbol{x} \in C : x_n = 0\}.$$

Essentially, if Ω is a domain described above, we can transplant any theorem for the boundary $\partial\Omega$ to a theorem for the flat boundary C_0 of the domain C_+, with the smoothness and regularity assumptions for S controlling how “nice” the transformation is.

We learned in the *C5.1a* course that if Ω is a $C^{1,0}$ domain with bounded boundary, then there is a continuous trace operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Domains with $C^{1,0}$ boundary cannot have any sharp edges, which is restrictive because for approximation schemes we often solve on a polygonal domain.

Thankfully, the result extends to bounded Lipschitz domains (i.e. $C^{0,1}$ domains). These domains are allowed to have sharp edges like a polygon, but not cusps. It also extends to fractional values of s: The following theorem for the special case of $p = 2$ is due to Gagliardo (1957) [18].

Theorem 2.5.2 (The trace theorem)**.** *Let $\frac{1}{2} < s \leq 1$ and let Ω be a bounded Lipschitz domain. Then there exists a continuous, surjective trace operator*

$$T : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega). \tag{2.34}$$

Remark 2.5.3. The theorem is in fact true for $\frac{1}{2} < s < \frac{3}{2}$ (see [10]).

For $s \in (0, \frac{1}{2}]$, we have no such theorem. There is no continuous trace operator on $H^s(\Omega)$ into a Lebesgue space for any $\Omega \subseteq \mathbb{R}^n$. Therefore, boundary conditions such as

$$u = g \quad \text{on } \partial\Omega \tag{2.35}$$

do not make sense for any measurable function g on $\partial\Omega$. Therefore when $s \in (0, \frac{1}{2}]$, we must extend our notion of a boundary condition for the case $s < \frac{1}{2}$. We introduce *volume-constrained problems* in Chapter 5.

2.6 The importance of the case $s \in (0, \frac{1}{2})$

One reason one may wish to study FDEs with weak solutions in the space $H^s(\Omega)$ for $s \in (0, \frac{1}{2})$, despite the difficulties with boundary conditions is that they may have jump discontinuities [11, Sec. 1.1]. Discontinuous solutions occur in many modelling problems (e.g. shocks) and it may be that the discontinuities arise due to the regularity of the modelled function being less than $\frac{1}{2}$. Consider the following simple example:

Proposition 2.6.1. $\chi_{[0,1]} \in H^s(\mathbb{R})$ *if and only if* $s \in (0, \frac{1}{2})$.

Proof. An exercise for the reader. □

This gives us some intuitive reason as to why there are no continuous trace operators on $H^s(\Omega)$ with $s < \frac{1}{2}$. Consider a hypothetical trace operator T from $H^s(0,1)$ to $\{0,1\}$, let $f_n = \chi_{[1/n,1]} \in H^s(0,1)$ for $n = 1, 2\ldots$ and let $f = \chi_{[0,1]} \in H^s(0,1)$. Then, noting that $f = f_n$ on $[\frac{1}{n}, 1]$ and $f_n = 0$ on $[0, \frac{1}{n}]$, we have:

$$\begin{aligned}
\|f - f_n\|^2_{H^s(0,1)} &= \left(\frac{1}{n}\right)^2 + 2\int_0^{\frac{1}{n}} \int_{\frac{1}{n}}^1 \frac{1}{(y-x)^{1+2s}}\, dy\, dx \\
&= \left(\frac{1}{n}\right)^2 + \frac{2}{2s}\int_0^{\frac{1}{n}} \frac{1}{(\frac{1}{n}-x)^{2s}} - \frac{1}{(1-x)^{2s}}\, dx \\
&= \left(\frac{1}{n}\right)^2 \\
&\quad + \frac{2}{2s(1-2s)}\left(\left(\frac{1}{n}\right)^{1-2s} - 1^{1-2s} - 0^{1-2s} + \left(1 - \frac{1}{n}\right)^{1-2s}\right) \\
&\to 0 \text{ as } n \to 0.
\end{aligned}$$

Therefore $f_n \to f$, but $T(f_n)(0) \to 0 \neq 1 = T(f)(0)$. So T is not continuous.

Chapter 3

Fractional-Order Operators

The main aim of this chapter is to note some useful results for operators with fractional order, and to prove the fractional Friedrichs inequality with an explicit constant.

3.1 The Riemann-Liouville fractional derivative

Here we briefly give motivation behind the notion of fractional integration and differentiation due to Riemann and Liouville (independently [31, p. 116]). This section will leave a lot to the reader, should they decide to prove all the propositions. We use phrases like "it is an easy exercise" because we merely want to motivate the technical discussion in the next section, and the proofs are straightforward anyway.

Let $f \in C^\infty([a,b])$ for some real numbers $a < b$ and define the integration operator, $\mathcal{J}_a : C^\infty([a,b]) \to C^\infty([a,b])$:

$$\mathcal{J}_a f = \int_a^x f(t)dt. \tag{3.1}$$

It is an easy exercise to show that $\mathcal{J}_a$ is a continuous linear operator with $\|\mathcal{J}_a\| = b-a$. It is also an easy exercise, using Fubini's theorem, to prove Cauchy's formula for repeated integration:

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{k-1}} f(x_k)\, dx_k \dots dx_2\, dx_1 = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f(t)\, dt. \tag{3.2}$$

In fact, the proof of (3.2) was a set problem in the *B4a Banach Spaces* course in 2010. So we can define an operator $\mathcal{J}_a^{(-k)} : C^\infty([a,b]) \to C^\infty([a,b])$, for each integer $k > 0$, by

$$\mathcal{J}_a^{(-k)} f = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f(t)\, dt. \tag{3.3}$$

We use $-k$ to denote the differential order of $\mathcal{J}_a^{(-k)} = \mathcal{J}_a^k$.

Now we would like to generalise this to obtain a fractional integral operator. Recall the definition of Euler's Gamma function,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \tag{3.4}$$

which extends the factorial function to all positive integers (and in fact, to the complex plane minus the non-positive integers). We have that $\Gamma(1) = 1$ and $s\Gamma(s) = \Gamma(1+s)$ for all $s > 0$, so for $s \in \mathbb{N}$, $\Gamma(s) = (s-1)!$[1].

Now we can define, for $s > 0$,

$$\mathcal{J}_a^{(-s)} f = \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} f(t)\, dt. \tag{3.5}$$

This integral is finite because we are on a bounded domain, and the operator coincides with that in (3.3) for integer values of s. We call (3.5) the *Riemann-Liouville fractional integral* of order s.

What about fractional derivatives? We cannot simply use the same formula for negative values of s; the integral is not finite. However, there is nice trick we can use. If we want a fractional derivative of order $s \in (0,1)$, then we can define the operator as the composition of a fractional integral of order $1-s$ and a first-order derivative:

$$\mathcal{J}_a^{(s)} f = \frac{1}{\Gamma(1-s)} \frac{d}{dx} \int_a^x (x-t)^{-s} f(t)\, dt, \tag{3.6}$$

and similarly for any non-integral $s > 0$, if k is the unique integer such that $k-1 < s < k$, we can define:

$$\mathcal{J}_a^{(s)} f = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-s-1} f(t)\, dt. \tag{3.7}$$

This is called the *Riemann-Liouville fractional derivative* of order s. For integral s we must use the classical definition of a derivative. Notice the nonlocality we discussed in the introduction: $\mathcal{J}_a^{(s)} f(x)$ depends on all the values of f in the range $[a, x]$.

It turns out that we can simplify a lot of the theory that stems from such a definition by setting $a = -\infty$, but still only considering functions on a compact domain. If we want to extract the integral or derivative with arbitrary a, we can use the fact that: $\mathcal{J}_a^{(-s)} f(x) = \mathcal{J}_{-\infty}^{(-s)} f(x) - \mathcal{J}_{-\infty}^{(-s)} f(a)$. This setting also allows us to make the following link between the Gamma function and the fractional derivative. A change of variables in (3.5) gives:

$$\mathcal{J}_{-\infty}^{(-s)} f = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(x-t)\, dt. \tag{3.8}$$

If we let $f(x) = e^x$, then

$$\mathcal{J}_{-\infty}^{(-s)} e^x = \frac{e^x}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t}\, dt = e^x, \tag{3.9}$$

[1]It therefore isn't technically an extension of the factorial function. According to Wikipedia, this quirk is due to Legendre, for reasons that are unclear.

and the same holds for the fractional derivatives. So we can consider the division by $\Gamma(s)$ to be a normalisation of the fractional integral and derivative in order to make e^x invariant under such operations.

3.2 Fractional derivatives

Here we generalise the Riemann-Liouville fractional integral and derivative to partial or directional derivatives on $\mathbb{R}^n$.

Definition 3.2.1. Let $\boldsymbol{m}$ be a unit vector and $u \in C_0^\infty(\mathbb{R}^n)$. The Riemann-Liouville fractional integral $\mathcal{R}_{\boldsymbol{m}}^{-s}[u]$ of order $s > 0$ in the direction $\boldsymbol{m}$ is defined to be:

$$\mathcal{R}_{\boldsymbol{m}}^{-s}[u](\boldsymbol{x}) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} u(\boldsymbol{x} - t\boldsymbol{m})\, dt. \tag{3.10}$$

Note that this integral is finite because u is bounded and has compact support. The special case where $\boldsymbol{m} = \boldsymbol{e}_j$, a canonical basis vector can be written as:

$$\begin{aligned} \mathcal{R}_j^{-s}[u](\boldsymbol{x}) &:= \mathcal{R}_{\boldsymbol{e}_j}^{-s}[u](\boldsymbol{x}) \\ &= \frac{1}{\Gamma(s)} \int_{-\infty}^{x_j} (x_j - t)^{s-1} u(x_1, \ldots, t, \ldots, x_n)\, dt. \end{aligned}$$

If $n = 1$, we recover the Riemann-Liouville fractional integral and derivatives from the previous section. If $\boldsymbol{m} = \boldsymbol{e}_1$ then we have what is known as the left fractional integral (or derivative), and if $\boldsymbol{m} = -\boldsymbol{e}_1$ we have the right fractional integral (or derivative).

For simplicity in the next definition, we define $\mathcal{R}_{\boldsymbol{m}}^{-0}$ to be the identity operator.

Definition 3.2.2. Let $\boldsymbol{m}$ be a unit vector and $u \in C_0^\infty(\mathbb{R}^n)$. The Riemann-Liouville fractional derivative of order $s \geq 0$ in the direction $\boldsymbol{m}$ is defined to be:

$$\mathcal{R}_{\boldsymbol{m}}^{s}[u] := (\boldsymbol{m} \cdot \nabla^k) \mathcal{R}_{\boldsymbol{m}}^{s-k}[u], \tag{3.11}$$

where k is the unique integer such that $k - 1 < s \leq k$.

Remark 3.2.3. We extend Definitions 3.2.1 and 3.2.2 to any measurable function u on $\mathbb{R}^n$, but we make no guarantees about finiteness, continuity or integrability of the resulting functions.

Now we prove an important lemma for finding the Fourier transform of Riemann-Liouville fractional integrals and derivatives. The aim is to show that these fractional derivatives correspond precisely to those tentatively discussed in the introduction using the Fourier transform (1.6).

Lemma 3.2.4. *Let $s \in (0, 1)$. Then*

$$P.V. \int_0^\infty t^{s-1} e^{-it}\, dt = i^{-s}\Gamma(s). \tag{3.12}$$

Note that one cannot simply use the complex substitution $z = it$ [30, p.246].

Proof. Consider the function $f(z) = z^{s-1}e^{-iz}$ for complex z, choosing the branch cut $(-\infty, 0]$ so that arguments lie in the range $(-\pi, \pi)$. Let us consider the contour

$$\Gamma = [\varepsilon, R] + \Gamma_R + [-iR, -i\varepsilon] + \Gamma_\varepsilon,$$

where $0 < \epsilon < R$, which is an annular sector with arguments $[-\frac{\pi}{2}, 0]$ and radius limits $[\epsilon, R]$ [30, Chapter 20]. We orient the contour negatively (clockwise), so that it includes the interval $[\varepsilon, R]$. The function f is holomorphic in and on the contour so by Cauchy's theorem,

$$\int_\Gamma f(z)\, dz = 0.$$

But we also have, using standard arguments from complex analysis:

$$\int_\Gamma f(t)\, dt = \int_\varepsilon^R f(t)\, dt + \int_0^{-\frac{\pi}{2}} f(Re^{i\theta})Rie^{i\theta}\, d\theta + \int_R^\varepsilon f(-it)(-i)\, dt + \int_{-\frac{\pi}{2}}^0 f(\varepsilon e^{i\theta})\varepsilon i e^{i\theta}\, d\theta.$$

The terms on the arcs are bounded by:

$$\begin{aligned}
\left| \int_0^{-\frac{\pi}{2}} (Re^{i\theta})^{s-1} e^{-iRe^{i\theta}} Rie^{i\theta}\, d\theta \right| &\leq R^s \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \\
&\leq R^s \int_0^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\theta}\, d\theta \\
&= R^{s-1}(1 - e^{-R}) \\
&= O(R^{s-1}). \qquad (3.13)
\end{aligned}$$

where we used Jordan's Lemma for the second line. Also note that:

$$\begin{aligned}
\left| \int_0^{-\frac{\pi}{2}} (\varepsilon e^{i\theta})^{s-1} e^{-i\varepsilon e^{i\theta}} \varepsilon i e^{i\theta}\, d\theta \right| &\leq \varepsilon^s \int_0^{\frac{\pi}{2}} e^{-\varepsilon\sin\theta}\, d\theta \\
&\leq \varepsilon^s \cdot \frac{\pi}{2} \\
&= O(\varepsilon^s), \qquad (3.14)
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &= \int_\varepsilon^R t^{s-1}e^{-it}\, dt - (-i)^s \int_\varepsilon^R t^{s-1}e^{-t}\, dt + O(R^{s-1}) + O(\varepsilon^s) \\
&\to \int_0^\infty t^{s-1}e^{-it}\, dt - (-i)^s \int_0^\infty t^{s-1}e^{-t}\, dt \qquad \text{as } \varepsilon \to 0, R \to \infty \\
&= \int_0^\infty t^{s-1}e^{-it}\, dt - i^{-s}\Gamma(s). \qquad (3.15)
\end{aligned}$$

We used the fact that $i^{-1} = -i$ for the last line. □

Theorem 3.2.5. *For $u \in C_0^\infty(\mathbb{R}^n)$, the Fourier transform of the Riemann-Liouville integral of order $s \in (0, 1)$ in the direction $\boldsymbol{m}$ is:*

$$\widehat{\mathcal{R}_{\boldsymbol{m}}^{-s}[u]}(\boldsymbol{\xi}) = (\boldsymbol{m} \cdot i\boldsymbol{\xi})^{-s}\hat{u}(\boldsymbol{\xi}). \qquad (3.16)$$

Proof. We can directly calculate:

$$\begin{aligned}
\widehat{\mathcal{R}_{\boldsymbol{m}}^{-s}[u]}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \frac{1}{\Gamma(s)} \lim_{R\to\infty} \int_0^R t^{s-1} u(\boldsymbol{x}-t\boldsymbol{m})\, dt\, d\boldsymbol{x} \\
&= \frac{1}{\Gamma(s)} \frac{1}{(2\pi)^{n/2}} \lim_{R\to\infty} \int_{\mathbb{R}^n} \int_0^R e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} t^{s-1} u(\boldsymbol{x}-t\boldsymbol{m})\, dt\, d\boldsymbol{x},
\end{aligned}$$

by the Dominated Convergence Theorem because u has compact support. In fact, if u has support in the ball $B(0,T)$ for some $T>0$ then we have that $u(\boldsymbol{x}-t\boldsymbol{m})$ has support in $B(0,T+R)$ for all $t \in [0,R]$, so we have

$$\int_{\mathbb{R}^n} \int_0^R \left| e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} t^{s-1} u(\boldsymbol{x}-t\boldsymbol{m}) \right| \, dt\, d\boldsymbol{x} \le |B(0,T+R)| \cdot \|u\|_\infty \cdot \frac{R^s}{s} < \infty.$$

We can therefore use Fubini's theorem to change the order of integration:

$$\begin{aligned}
\widehat{\mathcal{R}_{\boldsymbol{m}}^{-s}[u]}(\boldsymbol{\xi}) &= \frac{1}{\Gamma(s)} \lim_{R\to\infty} \int_0^R t^{s-1} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} u(\boldsymbol{x}-t\boldsymbol{m})\, d\boldsymbol{x}\, dt \\
&= \frac{1}{\Gamma(s)} \lim_{R\to\infty} \int_0^R t^{s-1} \mathcal{F}[u(\boldsymbol{x}-t\boldsymbol{m})]\, dt \\
&= \left(\frac{1}{\Gamma(s)} \lim_{R\to\infty} \int_0^R t^{s-1} e^{-i(\boldsymbol{m}\cdot\boldsymbol{\xi})t}\, dt \right) \hat{u}(\boldsymbol{\xi}) \\
&= \left(\frac{1}{\Gamma(s)} \lim_{R\to\infty} \int_0^R t^{s-1} e^{-it}\, dt \right) (\boldsymbol{m}\cdot\boldsymbol{\xi})^{-s} \hat{u}(\boldsymbol{\xi}) \\
&= \left(\frac{1}{\Gamma(s)} P.V. \int_0^\infty t^{s-1} e^{-it}\, dt \right) (\boldsymbol{m}\cdot\boldsymbol{\xi})^{-s} \hat{u}(\boldsymbol{\xi}) \\
&= (\boldsymbol{m}\cdot i\boldsymbol{\xi})^{-s} \hat{u}(\boldsymbol{\xi}), \qquad (3.17)
\end{aligned}$$

using Lemma 3.2.4 for the last line. □

Theorem 3.2.6. *For $u \in C_0^\infty(\mathbb{R}^n)$, the Fourier transform of the Riemann-Liouville derivative of order $s \ge 0$ in the direction $\boldsymbol{m}$ is*

$$\widehat{\mathcal{R}_{\boldsymbol{m}}^{s}[u]}(\boldsymbol{\xi}) = (\boldsymbol{m}\cdot i\boldsymbol{\xi})^{s} \hat{u}(\boldsymbol{\xi}). \qquad (3.18)$$

Proof. Let k be the unique integer such that $k-1 < s \le k$. Then

$$\begin{aligned}
\widehat{\mathcal{R}_{\boldsymbol{m}}^{s}[u]}(\boldsymbol{\xi}) &= \mathcal{F}[(\boldsymbol{m}\cdot\nabla^k)\mathcal{R}_{\boldsymbol{m}}^{s-k}[u]](\boldsymbol{\xi}) \\
&= (\boldsymbol{m}\cdot i\boldsymbol{\xi})^k \widehat{\mathcal{R}_{\boldsymbol{m}}^{s-k}[u]}(\boldsymbol{\xi}) \\
&= (\boldsymbol{m}\cdot i\boldsymbol{\xi})^k (\boldsymbol{m}\cdot i\boldsymbol{\xi})^{s-k} \hat{u}(\boldsymbol{\xi}) \\
&= (\boldsymbol{m}\cdot i\boldsymbol{\xi})^{s} \hat{u}(\boldsymbol{\xi}).
\end{aligned}$$

□

Remark 3.2.7. The Fourier transform of the integrals of order $s \geq 1$ are of the same form. It follows easily from the *semi-group* property of the fractional integral operator: $\mathcal{R}_{\boldsymbol{m}}^{-s}\mathcal{R}_{\boldsymbol{m}}^{-s'} = \mathcal{R}_{\boldsymbol{m}}^{-s-s'}$ for all $s, s' > 0$ (see [16, Thm. 2.1]), which we won't prove here. The identity holds when applied to any measurable function u.

Theorem 3.2.8. *The Riemann-Liouville fractional derivative $\mathcal{R}_{\boldsymbol{m}}^{s}$ for $s \geq 0$ can be extended to a continuous linear map from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Proof. First consider $u \in C_0^\infty(\mathbb{R}^n)$. Then, by Theorem 3.2.6 we have:

$$\begin{aligned} \|\mathcal{R}_{\boldsymbol{m}}^{s}[u]\|_{L^2(\mathbb{R}^n)} &= \||\boldsymbol{m}\cdot\boldsymbol{\xi}|^s \hat{u}\|_{L^2(\mathbb{R}^n)} \\ &\leq \||\boldsymbol{\xi}|^s \hat{u}\|_{L^2(\mathbb{R}^n)} \\ &= |u|_{\hat{H}^s(\mathbb{R}^n)}. \end{aligned} \tag{3.19}$$

By Theorem 2.3.8 we have that $\|\mathcal{R}_{\boldsymbol{m}}^{s}[u]\|_{L^2(\mathbb{R}^n)} \leq C\,\|u\|_{H^s(\mathbb{R}^n)}$ for some constant C. By the density Theorem 2.4.1 we can extend $\mathcal{R}_{\boldsymbol{m}}^{s}$ to a continuous linear operator on the whole space $H^s(\mathbb{R}^n)$ with the same norm and the same bound as in (3.19). □

Corollary 3.2.9. *For $u \in H^s(\mathbb{R}^n)$, $s \geq 0$, the Fourier transform of the Riemann-Liouville fractional derivative is the same as in Theorem 3.2.6.*

Proof. Let $u \in H^s(\mathbb{R}^n)$. Then, there is a sequence $\{\phi_j\} \subset C_0^\infty(\mathbb{R}^n)$ such that $\phi_j \to u$ in $H^s(\mathbb{R}^n)$. Then we have

$$\begin{aligned} \left\|(\boldsymbol{m}\cdot i\boldsymbol{\xi})^s\hat{u} - \widehat{\mathcal{R}_{\boldsymbol{m}}^{s}u}\right\|_{L^2(\mathbb{R}^n)} &\leq \left\|(\boldsymbol{m}\cdot i\boldsymbol{\xi})^s(\hat{u}-\hat{\phi}_j)\right\|_{L^2(\mathbb{R}^n)} + \left\|\widehat{\mathcal{R}_{\boldsymbol{m}}^{s}\phi_j} - \widehat{\mathcal{R}_{\boldsymbol{m}}^{s}u}\right\|_{L^2(\mathbb{R}^n)} \\ &\leq 2|u-\phi_j|_{\hat{H}^s(\mathbb{R}^n)} \\ &\to 0 \quad \text{as } j\to\infty. \end{aligned} \tag{3.20}$$

We used the bound from (3.19) to get the second line. □

Theorem 3.2.10 (Inversion property)**.** *Let $s \geq 0$ and $u \in C_0^\infty(\mathbb{R}^n)$. Then,*

$$\mathcal{R}_{\boldsymbol{m}}^{-s}\mathcal{R}_{\boldsymbol{m}}^{s}u = u. \tag{3.21}$$

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$, we can justifiably differentiate under the integral sign in the definition of the fractional derivative to have:

$$\begin{aligned} \mathcal{R}_{\boldsymbol{m}}^{-s}\mathcal{R}_{\boldsymbol{m}}^{s}u &= \mathcal{R}_{\boldsymbol{m}}^{-s}(\boldsymbol{m}\cdot\nabla^k)\mathcal{R}_{\boldsymbol{m}}^{s-k}u \\ &= \mathcal{R}_{\boldsymbol{m}}^{-s}\mathcal{R}_{\boldsymbol{m}}^{s-k}(\boldsymbol{m}\cdot\nabla^k)u. \end{aligned}$$

Then, by the semi-group property mentioned in Remark 3.2.7, we have:

$$\begin{aligned} \mathcal{R}_{\boldsymbol{m}}^{-s}\mathcal{R}_{\boldsymbol{m}}^{s}u &= \mathcal{R}_{\boldsymbol{m}}^{-k}(\boldsymbol{m}\cdot\nabla^k)u \\ &= u, \end{aligned}$$

with the last line being a classical fact. □

3.3 Fractional Friedrichs inequality

Recall from the *C5.1a* course [34, Lemma 6.14] the following Friedrichs inequality:

Theorem 3.3.1 (Friedrichs inequality)**.** *Let $u \in W_0^{k,p}(\Omega)$. Then*

$$\|u\|_{L^p(\Omega)} \leq \frac{d^k}{k!}|u|_{W^{k,p}(\Omega)}, \tag{3.22}$$

where $d = diam(\Omega)$.

We prove an analogue of this for fractional orders and $p = 2$. We denote $(\bar{\boldsymbol{x}}, x_n) := (x_1, x_2, \ldots, x_n)$ and begin with some lemmata.

Theorem 3.3.2. *Let, $f \in L^1(a, b)$ and $g \in L^p(a, b)$ where $p \geq 1$ and $-\infty \leq a < b \leq \infty$. Then $f * g \in L^p(a, b)$ and*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{3.23}$$

Proof. The inequality follows by a straightforward application of Fubini's theorem and Hölder's inequality. □

Lemma 3.3.3. *Let $u \in L^2(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$ be such that* $\operatorname{supp}(u(\bar{\boldsymbol{x}}, \cdot)) \subseteq [0, \infty)$*. Then for any $d > 0$ and any $s \geq 0$,*

$$\|\mathcal{R}_n^{-s}[u](\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)} \leq \frac{d^s}{\Gamma(1+s)}\|u(\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)}. \tag{3.24}$$

Proof. The idea for this proof comes from [15, Lemma 2.6]. We use remark 3.2.3 since u is not necessarily in $C_0^\infty(\mathbb{R}^n)$. Let us define, for $t > 0$:

$$\mathcal{I}^s(t) = \frac{t^{s-1}}{\Gamma(s)}. \tag{3.25}$$

Then we have, using the fact that $u(\bar{\boldsymbol{x}}, t) = 0$ for $t < 0$:

$$\begin{aligned}
\mathcal{R}_n^{-s}[u](\boldsymbol{x}) &= \frac{1}{\Gamma(s)} \int_{-\infty}^{x_n} (x_n - t)^{s-1} u(\bar{\boldsymbol{x}}, t)\, dt \\
&= \int_0^{x_n} \frac{1}{\Gamma(s)} (x_n - t)^{s-1} u(\bar{\boldsymbol{x}}, t)\, dt \\
&= (\mathcal{I}^s * u(\bar{\boldsymbol{x}}, \cdot))(x_n),
\end{aligned} \tag{3.26}$$

where the convolution is over the interval $(0, d)$. Hence, using Theorem 3.3.2,

$$\begin{aligned}
\|\mathcal{R}_n^{-s}[u](\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)} &= \|\mathcal{I}^s * u(\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)} \\
&\leq \|\mathcal{I}^s\|_{L^1(0,d)} \|u(\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)} \\
&= \frac{d^s}{s\Gamma(s)} \|u(\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)} \\
&= \frac{d^s}{\Gamma(1+s)} \|u(\bar{\boldsymbol{x}}, \cdot)\|_{L^2(0,d)}.
\end{aligned} \tag{3.27}$$

□

Lemma 3.3.4. *Let $u \in C_0^\infty(\mathbb{R}^n)$ be such that* $\mathrm{supp}(u(\bar{\boldsymbol{x}},\cdot)) \subseteq [0,\infty)$. *Then for any $s \geq 0$ and $d > 0$ we have:*

$$\|u(\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)} \leq \frac{d^s}{\Gamma(1+s)} \|\mathcal{R}_n^s[u](\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)}. \tag{3.28}$$

Proof. Firstly, by the inversion property (Theorem 3.2.10), we have that:

$$\|u(\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)} = \|\mathcal{R}_n^{-s}\mathcal{R}_n^s[u](\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)}.$$

By Theorem 3.2.8, we have that $\mathcal{R}_n^s[u] \in L^2(\mathbb{R}^n)$. Note also that the property of u that $\mathrm{supp}(u(\bar{\boldsymbol{x}},\cdot)) \subseteq [0,\infty)$ also holds for $\mathcal{R}_n^s[u]$, so by Lemma 3.3.3 we have:

$$\|\mathcal{R}_n^{-s}\mathcal{R}_n^s[u](\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)} \leq \frac{d^s}{\Gamma(1+s)} \|\mathcal{R}_n^s[u](\bar{\boldsymbol{x}},\cdot)\|_{L^2(0,d)}, \tag{3.29}$$

as required. □

Theorem 3.3.5 (Fractional Friedrichs inequality). *Let $s \geq 0$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Then for any $u \in C_0^\infty(\Omega)$ we have the following inequality:*

$$\|u\|_{L^2(\Omega)} \leq \frac{d^s}{\Gamma(1+s)} |u|_{\hat{H}^s(\mathbb{R}^n)}, \tag{3.30}$$

where $d = diam(\Omega)$.

Proof. Without loss of generality, by rotation and translation of u and Ω, we may assume that Ω is contained within the hypercube $Q_n = [0,d]^n$. We use the notation: $Q_{n-1} = [0,d]^{n-1}$. Then,

$$\begin{aligned}
\|u\|_{L^2(\Omega)} &= \|u\|_{L^p(Q_n)} \\
&= \left(\int_{Q_{n-1}} \|u(\bar{\boldsymbol{x}},\cdot)\|^2_{L^2(0,d)} \, d\bar{\boldsymbol{x}} \right)^{\frac{1}{2}} \\
&\leq \frac{d^s}{\Gamma(1+s)} \left(\int_{Q_{n-1}} \|\mathcal{R}_n^s[u](\bar{\boldsymbol{x}},\cdot)\|^2_{L^2(0,d)} \, d\bar{\boldsymbol{x}} \right)^{\frac{1}{2}} \\
&= \frac{d^s}{\Gamma(1+s)} \|\mathcal{R}_n^s[u]\|_{L^2(Q_n)} \\
&\leq \frac{d^s}{\Gamma(1+s)} \|\mathcal{R}_n^s[u]\|_{L^2(\mathbb{R}^n)} \\
&\leq \frac{d^s}{\Gamma(1+s)} |u|_{\hat{H}^s(\mathbb{R}^n)},
\end{aligned} \tag{3.31}$$

where we used (3.19) in the last step. □

Corollary 3.3.6. *For the same u as above,*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq \left(\left(\frac{\hat{C} d^s}{2\Gamma(1+s)} \right)^2 + 1 \right)^{\frac{1}{2}} |u|_{H^s(\mathbb{R}^n)}. \tag{3.32}$$

Remark 3.3.7. This result also holds for the closure of $C_0^\infty(\Omega)$ with respect to the $H^s(\mathbb{R}^n)$ norm. For integer-valued $s = k \geq 1$ this space is $H_0^k(\Omega)$ and we recover the standard Friedrichs inequality. we discuss this space for fractional values of s in Chapter 5.

Remark 3.3.8. Ervin and Roop prove the fractional Friedrichs inequality in one dimension in [15, Cor. 2.15] and two dimensions in [16, Cor. 5.3], but do not specify the constant as we do.

3.4 The fractional Laplacian

Here we discuss the fractional Laplacian operator. For our definition of the fractional Laplacian and derivation of the Fourier transform we follow [9, Sec. 3].

Definition 3.4.1. For $u \in C_0^\infty(\mathbb{R}^n)$ Laplacian of order $s \in (0,1)$ is:

$$(-\Delta)^s u := \hat{C}(n,s) \cdot P.V. \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}, \tag{3.33}$$

where the principle value is taken as the limit of the integral over $\mathbb{R}^n \setminus B_\varepsilon(\boldsymbol{x})$ as $\varepsilon \to 0$.

Remark 3.4.2. The principle value is necessary in the definition if $s \geq \frac{1}{2}$. For the integral to be a Lebesgue integral for a given $\boldsymbol{x}$, we require that in a neighbourhood $B_\varepsilon(\boldsymbol{x})$ of $\boldsymbol{x}$ the following holds for some C, $\delta > 0$:

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \leq C|\boldsymbol{x}-\boldsymbol{y}|^{2s+\delta}, \tag{3.34}$$

for then:

$$\frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}} \leq C \cdot \chi_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) \cdot |\boldsymbol{x}-\boldsymbol{y}|^{\delta-n} + \chi_{\mathbb{R}^n\setminus B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) \cdot 2\|u\|_\infty \cdot |\boldsymbol{x}-\boldsymbol{y}|^{-n-2s},$$

which is integrable over $\boldsymbol{y} \in \mathbb{R}^n$. Conversely, if (3.34) does not hold for any ε, then the integrand is not integrable over any $B_\varepsilon(\boldsymbol{x})$. If $s \in (0, \frac{1}{2})$ then (3.34) is satisfied by the fact that u is Lipschitz. If $s \geq \frac{1}{2}$, then as we noted in Remark 2.1.2, (3.34) implies that u is constant in $B_\varepsilon(\boldsymbol{x})$. Having this for every $\boldsymbol{x} \in \mathbb{R}^n$ implies $u \equiv 0$.

Now, by two changes of variables, $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$ and $\boldsymbol{z'} = \boldsymbol{x} - \boldsymbol{y}$, we can rewrite $(-\Delta)^s u$ as:

$$\begin{aligned}
(-\Delta)^s u &= \frac{1}{2}\hat{C}\left(P.V. \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{z})}{|\boldsymbol{z}|^{n+2s}}\, d\boldsymbol{z} + P.V. \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{x}-\boldsymbol{z'})}{|\boldsymbol{z'}|^{n+2s}}\, d\boldsymbol{z'}\right) \\
&= \frac{1}{2}\hat{C} \cdot P.V. \int_{\mathbb{R}^n} \frac{2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}.
\end{aligned}$$

Using Taylor's theorem to the second order, we have:

$$|2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y})| \leq |\boldsymbol{y}|^2 \cdot \max_{|\alpha|=2} \max_{\boldsymbol{y}\in\mathbb{R}^n} |D^\alpha u(\boldsymbol{y})|. \tag{3.35}$$

Since u has compact support and $|\boldsymbol{y}|^{-n-2s+2}$ is integrable near 0 in $\mathbb{R}^n$, we have the Lebesgue integral:

$$(-\Delta)^s u = \frac{1}{2}\hat{C}\int_{\mathbb{R}^n} \frac{2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{y}. \tag{3.36}$$

We proceed to compute the Fourier transform, but first we need a technical lemma:

Lemma 3.4.3. *The integrand for the fractional Laplacian of $u \in C_0^\infty(\mathbb{R}^n)$ is integrable over $\mathbb{R}^n \times \mathbb{R}^n$:*

$$\left((\boldsymbol{x},\boldsymbol{y}) \mapsto \frac{2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{y}|^{n+2s}}\right) \in L^1(\mathbb{R}^n \times \mathbb{R}^n). \tag{3.37}$$

Proof. In the 19 November 2011 preprint of [9], the authors attempt a proof of this with the following (quoted) argument:

$$\begin{aligned}
& \frac{|u(\boldsymbol{x}+\boldsymbol{y}) + u(\boldsymbol{x}-\boldsymbol{y}) - 2u(\boldsymbol{x})|}{|\boldsymbol{y}|^{n+2s}} \\
\leq \;& 4\left(\chi_{B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{2-n-2s} \cdot \sup_{B_1(\boldsymbol{x})} |D^2 u| + \chi_{\mathbb{R}^n \setminus B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{-n-2s} \sup_{\mathbb{R}^n} |u|\right) \\
\leq \;& C\left(\chi_{B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{2-n-2s} \left(1 + |\boldsymbol{x}|^{n+1}\right)^{-1} + \chi_{\mathbb{R}^n \setminus B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{-n-2s}\right) \\
\in \;& L^1(\mathbb{R}^n \times \mathbb{R}^n).
\end{aligned}$$

This argument is invalid, as the penultimate line is not integrable over the $\boldsymbol{x}$ variable. They do not assume that u has compact support (they are working in what is known as the Schwarz space on $\mathbb{R}^n$, not $C_0^\infty(\mathbb{R}^n)$). We do, so let u have support in the ball $B(0;T) \subset \mathbb{R}^n$ for some $T > 0$. Then

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{|u(\boldsymbol{x}+\boldsymbol{y}) + u(\boldsymbol{x}-\boldsymbol{y}) - 2u(\boldsymbol{x})|}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{x} \\
\leq \;& |B(0;T+1)| \cdot \chi_{B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{2-n-2s} \cdot \sup_{B_1(\boldsymbol{x})} |D^2 u| \\
& \quad + \chi_{\mathbb{R}^n \setminus B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{-n-2s} \cdot 4 \cdot \int_{\mathbb{R}^n} |u(\boldsymbol{x})|\,d\boldsymbol{x} \\
\leq \;& C\left(\chi_{B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{2-n-2s} + \chi_{\mathbb{R}^n \setminus B_1}(\boldsymbol{y}) \cdot |\boldsymbol{y}|^{-n-2s}\right) \\
\in \;& L^1(\mathbb{R}^n).
\end{aligned} \tag{3.38}$$

Then by Fubini's Theorem [23, p. 25], we have the desired result. $\square$

Theorem 3.4.4. *The Fourier transform of the fractional Laplacian of $u \in C_0^\infty(\mathbb{R}^n)$ is:*

$$\mathcal{F}\left[(-\Delta)^s u\right](\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2s}\hat{u}(\boldsymbol{\xi}). \tag{3.39}$$

Proof. We use a similar argument to the proof of Theorem 2.3.6, using the previous lemma to switch the order of integration for the second line:

$$\mathcal{F}\left[(-\Delta)^s u\right] \;=\; \frac{1}{2}\hat{C}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\,\frac{2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{y}\,d\boldsymbol{x}$$

$$\begin{aligned}
&= \frac{1}{2}\hat{C}\int_{\mathbb{R}^n} |\boldsymbol{y}|^{-n-2s}\int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}(2u(\boldsymbol{x}) - u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}-\boldsymbol{y}))\,d\boldsymbol{x}\,d\boldsymbol{y}\\
&= \frac{1}{2}\hat{C}\int_{\mathbb{R}^n} \frac{\mathcal{F}\left[2u(\cdot) - u(\cdot+\boldsymbol{y}) - u(\cdot-\boldsymbol{y})\right]}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{y}\\
&= \frac{1}{2}\hat{C}\int_{\mathbb{R}^n} \frac{(2-\cos(\boldsymbol{y}\cdot\boldsymbol{\xi}))\hat{u}(\boldsymbol{\xi})}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{y}.
\end{aligned}$$

We showed in Theorem 2.3.6 that:

$$\int_{\mathbb{R}^n} \frac{2-\cos(\boldsymbol{y}\cdot\boldsymbol{\xi})}{|\boldsymbol{y}|^{n+2s}}\,d\boldsymbol{y} = 2\hat{C}^{-1}|\boldsymbol{\xi}|^{2s}. \tag{3.40}$$

This gives us the desired result. □

Corollary 3.4.5. *For $s \in (0,1)$, fractional Laplacian operator can be extended to a continuous linear operator on $H^{2s}(\mathbb{R}^n)$ with the same formula for the Fourier transform as above.*

Proof. The proof is based on using the technique in Theorem 3.2.8 and Corollary 3.2.9. □

Remark 3.4.6. This theorem (and corollary) shows us that our "first attempt" at a fractional Laplacian operator in the Introduction does indeed have an explicit form, and it is a continuous operator on the fractional Sobolev space $H^{2s}(\mathbb{R}^n)$.

Remark 3.4.7. The operator can be generalised to functions on any open subset Ω of $\mathbb{R}^n$, by taking the integral to be over Ω rather than $\mathbb{R}^n$. However, the analysis of such an operator is much harder because we cannot use the Fourier transform.

A finite element method for an isotropic anomalous advection-diffusion FDE, involving the fractional Laplacian operator $(-\Delta)^s$, is discussed by Burrage et al. in [6].

Now, as we said in the introduction, we would like to have fractional-order operators that describe anisotropy, and also, operators with which we can create a weak formulation of the advection-diffusion FDE. In the next chapter we describe a framework developed by Du, Gunzburger, Lehouzq and Zhou (2011) [12] which enables us to describe nonlocal, fractional-order operators of this sort.

Chapter 4

A Nonlocal Calculus

In this chapter we describe the nonlocal calculus of Du, Gunzburger, Lehoucq and Zhou [12]. The point of the calculus is to define a general class of operators that behave like the classical divergence, gradient and curl operators. To this end they satisfy nonlocal analogues of the standard calculus theorems such as: integration by parts, Green's identities and Gauss' divergence theorem.

In the classical calculus, these properties are essential for the weak formulation of PDE problems. The main difference will be that the classical theorems involve functions on the boundary of a domain, whereas these nonlocal versions involve functions on a volume complementary to the domain which may or may not include the boundary.

Throughout this chapter we will consider the operators algebraically, and discuss which spaces they act on and their analytical properties such as continuity later. For the sake of argument one could assume all functions are infinitely differentiable with compact support. We will let Ω be an open subset of $\mathbb{R}^n$ throughout, with its regularity and smoothness discussed when necessary.

4.1 Notation

In the nonlocal calculus we consider two types of functions: those we call *point functions*, that take one variable $\boldsymbol{x}$ from $\mathbb{R}^n$, and those we call *two-point functions*, that take two variables $\boldsymbol{x}$ and $\boldsymbol{y}$ from $\mathbb{R}^n$. These functions can map to scalars, vectors or matrices, and so things can become confusing as it is not always clear which domains functions are mapping from or to. A carefully designed notational convention was used in [12], which we follow also. For our purposes their guidelines can be reduced to the following simplified system, which is also summarised in the table:

- Point vectors are denoted $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$.

- Normal-face lower-case denotes scalar-valued functions, bold lower-case denotes vector valued functions and bold upper-case denotes matrix valued functions.

- Point functions are denoted by Roman letters, whereas two-point functions are denoted by Greek letters.

	Scalar-valued	Vector-valued	Matrix-valued	Unspecified
Point function	u, v, f, g	$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{f}, \boldsymbol{g}$	$\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{F}, \boldsymbol{G}$	U, V, F, G
Two-point function	$\varphi, \psi, \theta, \alpha$	$\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}, \boldsymbol{\alpha}$	$\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}, \boldsymbol{A}$	$\mathit{\Phi}, \mathit{\Psi}, \mathit{\Theta}, A$

The notation for other objects can be summarised as follows: domains in $\mathbb{R}^n$ are denoted by upper-case Greek letters (e.g. Ω) and the operators in the calculus are calligraphic Roman letters (e.g. $\mathcal{D}, \mathcal{R}$). If we are in Fourier space then we denote points as $\boldsymbol{\xi}$ or $\boldsymbol{\zeta}$. Notice that we have already been using this convention throughout.

4.2 Nonlocal operators

In this section we will simply define the operators in the nonlocal calculus. The three main operators are denoted $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{C}$, which are nonlocal analogues of the div, grad and curl operators. Here we will only discuss the nonlocal divergence $\mathcal{D}$, because it is the only operator we require for our elliptic FDEs. For information on $\mathcal{G}$ and $\mathcal{C}$, see [12].

Definition 4.2.1 (Nonlocal divergence)**.** Let $\boldsymbol{\alpha}$ be an antisymmetric (i.e. $\boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}) = -\boldsymbol{\alpha}(\boldsymbol{y}, \boldsymbol{x})$), vector-valued two-point function. Then we define the nonlocal divergence operator, which operates on any vector-valued two-point function $\boldsymbol{\psi}$ as follows:

$$\mathcal{D}(\boldsymbol{\psi})(\boldsymbol{x}) = \int_{\mathbb{R}^n} (\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}, \text{ for } \boldsymbol{x} \in \Omega. \tag{4.1}$$

We also define its corresponding *nonlocal interaction operator.*

Definition 4.2.2 (Nonlocal interaction operator)**.** Let $\boldsymbol{\alpha}$ be an antisymmetric, vector-valued two-point function. Then we define the nonlocal interaction operator for the nonlocal divergence, which operates on any vector-valued two-point function $\boldsymbol{\psi}$ as follows:

$$\mathcal{N}(\boldsymbol{\psi})(\boldsymbol{x}) = -\int_{\mathbb{R}^n} (\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}, \text{ for } \boldsymbol{x} \in \mathbb{R}^n \setminus . \tag{4.2}$$

Looking at the relationship between $\mathcal{D}(\boldsymbol{\psi})$ and $\mathcal{N}(\boldsymbol{\psi})$, we see that the only difference in their definition is their sign and domain of definition. In the next section we elucidate this definition.

4.3 Properties of the nonlocal calculus

The Gauss-Green Divergence Theorem for a vector-valued function $\boldsymbol{f} \in H^1(\Omega)$ is as follows [17, p. 711]:

$$\int_\Omega \nabla \cdot \boldsymbol{f}\, d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{m} \cdot \boldsymbol{f}\, dS, \tag{4.3}$$

where $\boldsymbol{m}$ is the outward unit normal vector for the C^1 domain Ω. The following lemma shows how the nonlocal operators are defined specifically to satisfy an analogous theorem.

Lemma 4.3.1 (Fundamental lemma of the nonlocal calculus)**.** *Any two-point function $\Psi \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, which is antisymmetric, has the following property:*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} = 0. \tag{4.4}$$

Proof. First we use antisymmetry, then using Fubini's theorem [23, p. 25] we can the order of integration:

$$\begin{aligned}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} &= -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\boldsymbol{y}, \boldsymbol{x})\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&= -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\boldsymbol{y}, \boldsymbol{x})\, d\boldsymbol{x}\, d\boldsymbol{y} \\
&= -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x}.
\end{aligned} \tag{4.5}$$

The last step is a relabelling of dummy variables. □

Corollary 4.3.2 (Nonlocal divergence theorem)**.** *For $\mathcal{D}$ and $\mathcal{N}$ defined as in section 4.2 and any vector-valued two-point function $\boldsymbol{\psi}$ we have:*

$$\int_\Omega \mathcal{D}(\boldsymbol{\psi})\, d\boldsymbol{x} = \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}(\boldsymbol{\psi})\, d\boldsymbol{x}. \tag{4.6}$$

Proof. Note that $\boldsymbol{\alpha}$ is antisymmetric and $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})$ is symmetric, so $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y})$ is antisymmetric. Hence,

$$\int_\Omega \mathcal{D}(\boldsymbol{\psi})\, d\boldsymbol{x} - \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}(\boldsymbol{\psi})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} = 0.$$

Rearranging gives us the desired equation. □

For the classical calculus, we also have the integration by parts formula. For scalar-valued $u \in H^1(\Omega)$ and vector-valued $\boldsymbol{v} \in H^1(\Omega)$, this is the following:

$$\int_\Omega u(\nabla \cdot \boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega (-\nabla u) \cdot \boldsymbol{v}\, d\boldsymbol{x} = \int_{\partial\Omega} u(\boldsymbol{m} \cdot \boldsymbol{v})\, dS, \tag{4.7}$$

where $\boldsymbol{m}$ is the outward unit normal vector for the C^1 domain Ω. This equation can be interpreted as saying that the negative gradient $-\nabla$ is a formal adjoint to the divergence operator $\nabla \cdot$. Let us define the adjoint of a nonlocal operator in this way, and find it explicitly, to have a nonlocal integration by parts formula.

Definition 4.3.3 (Nonlocal adjoint)**.** For a nonlocal operator $\mathcal{E}$ with associated interaction operator $\mathcal{X}$, its nonlocal adjoint is an operator $\mathcal{E}^*$ such that, for functions $\mathit{\Phi}$ and P:

$$\int_\Omega P\mathcal{E}(\mathit{\Phi})\,d\boldsymbol{x} - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{E}^*(P)\mathit{\Phi}\,d\boldsymbol{y}\,d\boldsymbol{x} = \int_{\mathbb{R}^n\setminus\Omega} P\mathcal{X}(\mathit{\Phi})\,d\boldsymbol{x}. \tag{4.8}$$

Here $\mathit{\Phi}$ and P can be scalar- or vector-valued, depending on $\mathcal{E}$.

Proposition 4.3.4 (Adjoint for nonlocal divergence)**.** *For $\mathcal{D}$ and $\mathcal{N}$ as defined in Section 4.2, the adjoint for the nonlocal divergence is the operator $\mathcal{D}^*$ such that for all scalar-valued functions u:*

$$\mathcal{D}^*(u)(\boldsymbol{x},\boldsymbol{y}) = (u(\boldsymbol{x}) - u(\boldsymbol{y}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n. \tag{4.9}$$

Proof. Let $\boldsymbol{\psi}$ be a vector-valued two-point function. Then, using the fundamental Lemma 4.3.1 to get the penultimate line,

$$\begin{aligned}
&\int_\Omega u\mathcal{D}(\boldsymbol{\psi})\,d\boldsymbol{x} - \int_{\mathbb{R}^n\setminus\Omega} u\mathcal{N}(\boldsymbol{\psi})\,d\boldsymbol{x}\\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(\boldsymbol{x})(\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}))\cdot\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}\\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\boldsymbol{x})\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}) - u(\boldsymbol{y}))\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}))\cdot\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}\\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\boldsymbol{x})\boldsymbol{\psi}(\boldsymbol{y},\boldsymbol{x}) + u(\boldsymbol{y})\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}))\cdot\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}\\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\boldsymbol{x}) - u(\boldsymbol{y}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}\\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^*(u)\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}.
\end{aligned}$$

This completes the proof. □

Corollary 4.3.5. *[Nonlocal integration by parts] Let u be a scalar-valued one-point function and let $\boldsymbol{\psi}$ be a vector-valued two-point function; then:*

$$\int_\Omega u\mathcal{D}(\boldsymbol{\psi})\,d\boldsymbol{x} - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^*(u)\cdot\boldsymbol{\psi}\,d\boldsymbol{y}\,d\boldsymbol{x} = \int_{\mathbb{R}^n\setminus\Omega} u\mathcal{N}(\boldsymbol{\psi})\,d\boldsymbol{x}. \tag{4.10}$$

Proof. This is a restatement of Proposition 4.3.4. □

From the integration by parts formula for classical derivatives, we can derive Green's first identity by setting $\boldsymbol{v} = \boldsymbol{A}\nabla w$ for a matrix-valued function $A \in C^1(\Omega)$ and a scalar-valued function $w \in H^2(\Omega)$:

$$\int_\Omega u(\nabla\cdot\boldsymbol{A}\nabla w)\,d\boldsymbol{x} + \int_\Omega (-\nabla u)\cdot\boldsymbol{A}\nabla w\,d\boldsymbol{x} = \int_{\partial\Omega} u\boldsymbol{m}\cdot\boldsymbol{A}\nabla w\,dS, \tag{4.11}$$

For the nonlocal calculus we can do the same by setting $\boldsymbol{\phi} = \boldsymbol{\Theta}\mathcal{D}^*(v)$ in the integration by parts formula, where v is a scalar-valued two-point function and $\boldsymbol{\Theta}$ is a matrix-valued two-point function (because $\mathcal{D}^*(v)$ is a two-point function).

Theorem 4.3.6 (Nonlocal Green's identity)**.** *Let $\mathcal{D}$ and $\mathcal{N}$ be the nonlocal divergence and interaction operators as defined in Section 4.2, u and v scalar-valued one-point functions and $\boldsymbol{\Theta}$ a matrix-valued two-point functions. Then:*

$$\int_\Omega u\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^*(v))\,d\boldsymbol{x} - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^*(u)\cdot\boldsymbol{\Theta}\mathcal{D}^*(v)\,d\boldsymbol{y}\,d\boldsymbol{x} = \int_{\mathbb{R}^n\setminus\Omega} u\mathcal{N}(\boldsymbol{\Theta}\mathcal{D}^*(v))\,d\boldsymbol{x} \quad (4.12)$$

Proof. Let $\boldsymbol{\psi} = \boldsymbol{\Theta}\mathcal{D}^*(v)$ in Corollary 4.3.5. □

Remark 4.3.7. Note that we use the entire complement of Ω (which is $\mathbb{R}^n \setminus \Omega$) for the domain of the interaction function $\mathcal{N}(\boldsymbol{\psi})$. This means that in general there is no upper limit on the distance of a nonlocal interaction. This need not be the case; for example if $\boldsymbol{\alpha}$ has support in the strip $\{(\boldsymbol{x},\boldsymbol{y}) : |\boldsymbol{x}-\boldsymbol{y}| < \varepsilon\}$ then nonlocal interactions are confined within balls of radius ε. Then the interaction function $\mathcal{N}(\boldsymbol{\psi})$ has support in an ε-thin strip surrounding Ω.

Given the theorems discussed in this section, we see a direct correspondence between the operators of the nonlocal calculus: $\mathcal{D}$, $\mathcal{D}^*$ and $\mathcal{N}$; and those of the classical calculus: div, grad and the normal flux operator. In the next section we discuss how this correspondence can be taken further.

4.4 Inclusion of the classical vector calculus

Du et al. showed that this nonlocal calculus generalises the classical vector calculus in a distributional sense [12]. In other words, there is a choice of $\boldsymbol{\alpha}$ (which is a distribution rather than a function) such that the nonlocal operators are effectively the classical differential operators. By effectively we mean that this correspondence will not be perfect, because nonlocal operators involve two-point functions whereas classical differential operators only involve one-point functions.

We try to be brief here to avoid losing focus; we include this discussion because we take a slightly different approach to the one given in the original paper. Define the antisymmetric $\boldsymbol{\alpha}$ kernel to be the distribution:

$$\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{y}}\delta(\boldsymbol{y}-\boldsymbol{x}). \quad (4.13)$$

This is "physicist's notation" [34, p. 26] for the distribution $T_{\boldsymbol{\alpha}(\boldsymbol{x},\cdot)}$ such that for all $\boldsymbol{w} \in C_0^\infty(\mathbb{R}^n)$,

$$T_{\boldsymbol{\alpha}(\boldsymbol{x},\cdot)}(\boldsymbol{w}) = \nabla\cdot\boldsymbol{w}(\boldsymbol{x}) \quad (4.14)$$

Again in physicist's notation:

$$\begin{aligned} T_{\boldsymbol{\alpha}(\boldsymbol{x},\cdot)}(\boldsymbol{w}) &= \int_{\mathbb{R}^n} \boldsymbol{w}(\boldsymbol{y})\cdot\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \boldsymbol{w}(\boldsymbol{y})\cdot(-\nabla_{\boldsymbol{y}})\delta(\boldsymbol{y}-\boldsymbol{x})\,d\boldsymbol{y} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} (\nabla \cdot \boldsymbol{w})\delta(\boldsymbol{y}-\boldsymbol{x})\, d\boldsymbol{y} \\
&= \nabla \cdot \boldsymbol{w}(\boldsymbol{x}).
\end{aligned}
$$

We also define for each one-point function $\boldsymbol{v}$ a unique two-point function $\boldsymbol{\vartheta_v}$ by the formula:

$$
\boldsymbol{\vartheta_v}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left(\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{y})\right). \tag{4.15}
$$

This next proposition shows that for any $u, \boldsymbol{v} \in C_0^\infty(\mathbb{R}^n)$, we can desribe the divergence, gradient and boundary conditions using the nonlocal opearators with the above distributional definition for $\boldsymbol{\alpha}$.

Proposition 4.4.1. *Let the divergence kernel $\boldsymbol{\alpha}$ be defined by (4.13). Then for each scalar-valued $u \in C_0^\infty(\mathbb{R}^n)$ and each vector-valued $\boldsymbol{v} \in C_0^\infty(\mathbb{R}^n)$, the following hold in a distributional sense:*

$$
\nabla \cdot \boldsymbol{v} = \mathcal{D}(\boldsymbol{\vartheta_v}), \tag{4.16}
$$

$$
-\nabla u = \int_{\mathbb{R}^n} \mathcal{D}^*(u)\, d\boldsymbol{y}, \tag{4.17}
$$

$$
\int_{\partial\Omega} u(\boldsymbol{m}\cdot\boldsymbol{v})\, dS = \int_{\mathbb{R}^n\setminus\Omega} u\mathcal{N}(\boldsymbol{\vartheta_v})\, d\boldsymbol{x}, \tag{4.18}
$$

Proof. These identities can be proven in physicist's notation by formal manipulations, using the theorems discussed in the previous section in a distributional sense, along with classical calculus theorems. □

4.5 Anisotropic fractional Laplacian

In this section we describe the nonlocal operator we call the *anisotropic fractional Laplacian*. Throughout, we let $\boldsymbol{\Theta}$ be a matrix-valued symmetric two-point function i.e. $\boldsymbol{\Theta}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\Theta}(\boldsymbol{y},\boldsymbol{x})$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$.

Consider the following composition of nonlocal operators:

$$
\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)(\boldsymbol{x}) = \int_{\mathbb{R}^n} 2(u(\boldsymbol{x}) - u(\boldsymbol{y}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\Theta}\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y}. \tag{4.19}
$$

The symmetry of $\boldsymbol{\Theta}$ and the antisymmetry of $\boldsymbol{\alpha}$ are essential for this identity. If we let $\boldsymbol{\Theta}$ be the identity matrix, and $\boldsymbol{\alpha}$ be such that:

$$
2|\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})|^2 = \frac{\hat{C}(n,s)}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}},
$$

then the following holds:

$$
\mathcal{D}(\mathcal{D}^* u)(\boldsymbol{x}) = \hat{C}(n,s) \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}. \tag{4.20}
$$

We would like to deduce that the fractional Laplacian operator $(-\Delta)^s$ can be expressed in the nonlocal calculus. However, we have a problem. The fractional Laplacian operator is defined to be the principle value of this integral for $u \in C_0^\infty(\mathbb{R}^n)$ (see Definition 3.4.1) and as we noted in Remark 3.4.2, the fractional Laplacian is only the proper Lebesgue integral for each $\boldsymbol{x} \in \mathbb{R}^n$ if $s \in (0, \frac{1}{2})$, or $u \equiv 0$. This fact has been overlooked by Du et al. in their discussion of the fractional Laplacian in the nonlocal calculus [11, A1], (or perhaps they know of a trivial workaround).

In order to include the fractional Laplacian for $s \geq \frac{1}{2}$ in the nonlocal calculus, we must make a modification: We change the integrals to principal value integrals. So let us redefine the nonlocal divergence $\mathcal{D}$ to be as follows:

$$\mathcal{D}(\boldsymbol{\psi})(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(\boldsymbol{x})} (\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y}, \tag{4.21}$$

and the negative of this for $\mathcal{N}$. This makes the nonlocal divergence and interaction operators well-defined for the case $\boldsymbol{\psi} = \mathcal{D}^* u$, but not much more than this. All of the theorems in Section 4.2 rely on the fundamental lemma, which itself relies on the use of Fubini's theorem, a theorem for Lebesgue integrable functions. In fact, consider the two-point function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:

$$\psi(x, y) = \chi_{[1,\infty)\times[1,\infty)}(x, y) \cdot \frac{x^2 - y^2}{(x^2 + y^2)^2}. \tag{4.22}$$

This two-point function is antisymmetric, but is also a standard integration example (see Part A Integration course) with:

$$0 \neq \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(x, y) \, dy \, dx = -\frac{\pi}{4} \neq \frac{\pi}{4} = \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(x, y) \, dx \, dy. \tag{4.23}$$

Here the integrals are the principal value. We cannot even directly prove the Green's identity as a special case of the fundamental lemma; as we can see from the proof the nonlocal adjoint (Proposition 4.3.4), we rely on the identity:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(\boldsymbol{x})\boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{x}) + u(\boldsymbol{y})\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} \, d\boldsymbol{x} = 0 \text{ for all } u, v, \tag{4.24}$$

which, by setting $\boldsymbol{\psi} = \mathcal{D}^* v$ for Green's identity and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_s$, we have:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(\boldsymbol{x}) + u(\boldsymbol{y}))(v(\boldsymbol{x}) - v(\boldsymbol{y}))}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}} \, d\boldsymbol{y} \, d\boldsymbol{x} = 0 \text{ for all } u, v \tag{4.25}$$

We cannot see any way to prove this, even if we assume $u, v \in C_0^\infty(\Omega)$. This technical issue came to our attention very near to the submission date for this thesis, and we have not been able to resolve it. Everything is well-defined and as it should be for $s \in (0, \frac{1}{2})$, but there certainly is a problem with using the nonlocal calculus for the fractional Laplacian with $s \in [\frac{1}{2}, 1)$. Nonetheless we can still define the anisotropic fractional Laplacian with the principal value version of the nonlocal divergence and have a well-defined operator. Let us define the fractional kernel of order $s \in (0,1)$:

$$\boldsymbol{\alpha}_s(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{n}{2}+s+1}}. \tag{4.26}$$

Definition 4.5.1 (Anisotropic fractional Laplacian)**.** Let $\boldsymbol{\Theta}$ be a matrix-valued two-point function, continuous and symmetric in its two arguments and satisfying the following ellipticity condition for some constants $c_{\boldsymbol{\Theta}}$ and $C_{\boldsymbol{\Theta}}$:

$$\exists c_{\boldsymbol{\Theta}}, C_{\boldsymbol{\Theta}} > 0 \text{ such that } \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n, \quad c_{\boldsymbol{\Theta}} |\boldsymbol{z}|^2 \leq \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\Theta}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} \leq C_{\boldsymbol{\Theta}} |\boldsymbol{z}|^2. \tag{4.27}$$

Let $\mathcal{D}$ be the nonlocal divergence operator with kernel $\boldsymbol{\alpha}_s$, with principal values taken if $s \in [\frac{1}{2},1)$. The anisotropic fractional Laplacian operator for $\boldsymbol{\Theta}$ is the following operator:

$$u \mapsto \mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u). \tag{4.28}$$

Remark 4.5.2. Note that the ellipticity condition on $\boldsymbol{\Theta}$ implies that $\boldsymbol{\Theta}(\boldsymbol{x},\boldsymbol{y})$ is a symmetric, positive definite matrix for every $\boldsymbol{x}$ and $\boldsymbol{y}$. Therefore Θ defines a real inner product matrix for each $\boldsymbol{x}$ and $\boldsymbol{y}$. We can thence use the Cauchy-Schwarz inequality to find:

$$\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{z'} \leq (\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{z})^{\frac{1}{2}}(\boldsymbol{z'}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{z'})^{\frac{1}{2}} \leq C_{\boldsymbol{\Theta}}|\boldsymbol{z}||\boldsymbol{z'}| \quad \forall \boldsymbol{z},\boldsymbol{z'} \in \mathbb{R}^n. \tag{4.29}$$

Explicitly, the definition of the anisotropic fractional Laplacian can be written:

$$\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)(\boldsymbol{x}) = P.V. \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x})-u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{n+2s}} \cdot \frac{(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\Theta}(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\, d\boldsymbol{y}. \tag{4.30}$$

As we can see, this operator generalises the fractional Laplacian by including a positive weighting in the integral. It is difficult to find a space of functions within which 4.30 is well defined and continuous, even for $s \in (0,\frac{1}{2})$; for further work one could try to follow a similar line of argument as in Theorem 3.4.4.

Since we cannot use the nonlocal Green's identity for $s \geq \frac{1}{2}$, we are going to have to define the weak form of the anisotropic fractional Laplacian directly:

Definition 4.5.3. We define the anisotropic fractional Laplacian weakly on an open set Ω for $u \in H^s(\mathbb{R}^n)$ by:

$$\int_\Omega v\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)\, d\boldsymbol{x} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u\, d\boldsymbol{y}\, d\boldsymbol{x} \text{ for all } v \in C_0^\infty(\Omega). \tag{4.31}$$

We can see that this weak operator is bounded on $H^s(\mathbb{R}^n)$ since:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u\, d\boldsymbol{y}\, d\boldsymbol{x} \leq C_{\boldsymbol{\Theta}} \left\|\mathcal{D}^* v\right\|_{L^2(\mathbb{R}^n)} \left\|\mathcal{D}^* u\right\|_{L^2(\mathbb{R}^n)} = C_{\boldsymbol{\Theta}} |v|_{H^s(\mathbb{R}^n)} |u|_{H^s(\mathbb{R}^n)}.$$

We can also see that for $s \in (0,\frac{1}{2})$, the weak operator coincides with the anisotropic fractional Laplacian in Definition 3.4.1 The nonlocal Green's identity implies:

$$\int_\Omega v\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u)\, d\boldsymbol{x} \quad = \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u\, d\boldsymbol{y}\, d\boldsymbol{x} + \int_{\mathbb{R}^n\setminus\Omega} v\mathcal{N}(\boldsymbol{\Theta}u)\, d\boldsymbol{x},$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta} \mathcal{D}^* u \, d\boldsymbol{y} \, d\boldsymbol{x}. \tag{4.32}$$

The following is a useful fact we use in Chapter 6:

Proposition 4.5.4. *For the nonlocal divergence operator $\mathcal{D}$ with kernel α_s, $s \in (0,1)$, we have that $\mathcal{D}^*$ is a continuous linear mapping from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. Explicitly, for a scalar-valued one-point function u,

$$\mathcal{D}^* u(\boldsymbol{x}, \boldsymbol{y}) = (u(\boldsymbol{x}) - u(\boldsymbol{y})) \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^{\frac{n}{2}+s+1}},$$

so:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{D}^* u(\boldsymbol{x}, \boldsymbol{y})|^2 \, d\boldsymbol{y} \, d\boldsymbol{x} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}} \, d\boldsymbol{y} \, d\boldsymbol{x} = |u|^2_{H^s(\mathbb{R}^n)}. \tag{4.33}$$

Therefore, for any $u \in H^s(\mathbb{R}^n)$, $\|\mathcal{D}^* u\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)}$. $\square$

Chapter 5

Volume-Constrained Problems

In this chapter we discuss the general form of some volume-constrained problems, and reduce proving existence and uniqueness of solution to the non-homogeneous Dirichlet volume-constrained problem to that of an homogeneous one.

5.1 Boundary value problems

This section will be slightly vague, as we use it just to motivate our treatment of the volume-constrained problems. Dirichlet boundary-value problems on a bounded domain $\Omega \subset \mathbb{R}^n$ take the form of finding u in a function space V such that:

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

Here f and g are some prescribed functions defined on Ω and $\partial\Omega$ respectively, and $\mathcal{L}$ is a differential operator. One way to start the analysis of such problems is to prove that there exists a function $\tilde{g} \in V$ defined on $\bar{\Omega}$ such that $\tilde{g}_{\restriction \partial\Omega} = g$. This follows from surjectivity of a trace operator from V onto a space containing g (see Theorem 2.5.2). Then we can rewrite the problem as follows:

$$\begin{aligned} \mathcal{L}u_0 &= f_0 \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

where $u_0 = u - \tilde{g}$ and $f_0 = f - \mathcal{L}\tilde{g}$ [17, p. 315]. If we can prove existence and uniqueness of a weak solution for this problem, then we have existence of a weak solution to problem (5.1), which is $u = u_0 + \tilde{g}$. By linearity of $\mathcal{L}$, we also have uniqueness so long as 0 is the unique solution to (5.1) with $f \equiv 0$ and $g \equiv 0$ (which follows from coercivity of the bilinear form for the weak formulation).

For the particular case of $V = H^1(\Omega)$, where Ω is a bounded Lipschitz domain, the weak solutions to the homogeneous problem lie in the space:

$$\tilde{H}_0^1(\Omega) := \left\{ u \in H^1(\Omega) : Tu \equiv 0 \right\}. \tag{5.3}$$

This is the space of zero-trace functions, where T is the trace operator on $H^1(\Omega)$. One can prove that for the case where Ω is a Lipschitz domain, this space is equal to the closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm, which is denoted $H_0^1(\Omega)$ [17, p. 273]. Problems can then be tackled for $C_0^\infty(\Omega)$ functions before generalised to the whole of the solution space by density.

We would like to do something similar for volume constrained problems: reduce the problem to the homogeneous case and show that the homogeneous solution space has dense subspace $C_0^\infty(\Omega)$.

5.2 Volume-constrained problems

Let Ω be a bounded open Lipschitz domain and let $\hat{\Omega}$ be an open set containing the closure of Ω. We denote $\Omega_c = \hat{\Omega} \setminus \Omega \subseteq \mathbb{R}^n \setminus \Omega$. Let $\mathcal{L}$ be a differential operator of order $2s$ where $s \in (0, 1)$, $f \in L^2(\Omega)$ and $h \in H^s(\Omega_c)$. Consider the Dirichlet volume-constrained problem:

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega, \\ u &= h \quad \text{in } \Omega_c. \end{aligned} \tag{5.4}$$

As in the previous section, we would like to reduce this problem to the homogeneous case where $h \equiv 0$. The volume Ω_c shares some of it's boundary with Ω by the way we defined Ω, $\hat{\Omega}$ and Ω_c, which is Lipschitz by assumption. This is sufficient for the existence of an extension of h into the whole of $\hat{\Omega}$ by [9, Thm. 5.4]. Let this extension be $\tilde{h} \in H^s(\hat{\Omega})$. Then we can restate the problem as:

$$\begin{aligned} \mathcal{L}u_0 &= f_0 \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{in } \Omega_c, \end{aligned} \tag{5.5}$$

where $u_0 = u - \tilde{h}$ and $f_0 = f - \mathcal{L}\tilde{h}$. If we can prove existence and uniqueness of weak solutions to this homogeneous Dirichlet problem (5.5) with $u_0 \in H^s(\Omega)$ then we have existence of the solution to the non-homogeneous problem (5.4) which we take to be $u = u_0 + \tilde{h}$. Further, if 0 is the unique solution to the homogeneous problem with $f \equiv 0$ then we have uniqueness of this solution u by linearity of $\mathcal{L}$.

Weak solutions of the homogeneous problem (5.5) lie in the closed subspace:

$$H_\Omega^s(\hat{\Omega}) := \left\{ u \in H^s(\hat{\Omega}) : u = 0 \text{ on } \hat{\Omega} \setminus \Omega \right\}. \tag{5.6}$$

Now, before we go any further, we prove a result which reduces the problem further:

Theorem 5.2.1. *The space $H^s_\Omega(\hat{\Omega})$ is isomorphic to $H^s_\Omega(\mathbb{R}^n)$.*

Proof. Define the linear operator $T_1 : H^s_\Omega(\mathbb{R}^n) \to H^s_\Omega(\hat{\Omega})$ by $T_1 u = u_{\restriction\hat{\Omega}}$. This is a continuous restriction mapping and therefore demonstrates that $H^s_\Omega(\mathbb{R}^n)$ can be continuously embedded into $H^s_\Omega(\hat{\Omega})$.

Conversely consider the linear operator $T_2 : H^s_\Omega(\hat{\Omega}) \to H^s_\Omega(\mathbb{R}^n)$ defined by:

$$T_2 u = \begin{cases} u & \text{in } \hat{\Omega}, \\ 0 & \text{in } \mathbb{R}^n \setminus \hat{\Omega}. \end{cases} \tag{5.7}$$

Then we have $\|T_2 u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\hat{\Omega})}$ and using the fact that $T_2 u = 0$ in $\mathbb{R}^n \setminus \Omega$:

$$\begin{aligned} |T_2 u|^2_{H^s(\mathbb{R}^n)} &= \int_{\hat{\Omega}} \int_{\hat{\Omega}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \hat{\Omega}} \frac{|u(\boldsymbol{x})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \\ &\le |u|^2_{H^s(\hat{\Omega})} + 2 \int_\Omega |u(\boldsymbol{x})|^2 \int_{\mathbb{R}^n \setminus \hat{\Omega}} \frac{1}{\operatorname{dist}(\boldsymbol{y}, \Omega)^{n+2s}}\, d\boldsymbol{y}\, d\boldsymbol{x} \\ &\le |u|^2_{H^s(\hat{\Omega})} + 2\|u\|^2_{L^2(\hat{\Omega})} \int_{\mathbb{R}^n \setminus \hat{\Omega}} \frac{1}{\operatorname{dist}(\boldsymbol{y}, \Omega)^{n+2s}}\, d\boldsymbol{y} \\ &\le C(n, s, \hat{\Omega}) \cdot \|u\|^2_{H^s(\hat{\Omega})}. \end{aligned} \tag{5.8}$$

The last line is follows from the fact that $\hat{\Omega}$ contains the closure of Ω, so since both are open sets, $\operatorname{dist}(\boldsymbol{y}, \Omega) \ge \delta > 0$ for all $\boldsymbol{y} \in \mathbb{R}^n \setminus \hat{\Omega}$. Hence, T_2 is a continuous linear operator demonstrating that $H^s_\Omega(\hat{\Omega})$ can be continuously embedded into $H^s_\Omega(\mathbb{R}^n)$. □

By transitivity, all of the homogeneous Dirichlet volume-constrained spaces for Ω are isomorphic. We see that for pure analysis these homogeneous problems, we can choose whichever volume for $\hat{\Omega}$ we want. In particular, we can choose the space $H^s_\Omega(\mathbb{R}^n)$, which has the useful Fourier transform characterisation.

Now, just as in the case of boundary value problems, we have that infinitely differentiable functions with compact support in Ω is dense in our solution space:

Theorem 5.2.2. *$C_0^\infty(\Omega)$ is dense in $H^s_\Omega(\mathbb{R}^n)$ for $s \in (0, 1)$.*

Proof. This theorem can be found in *Interpolation theory, function spaces and differential operators* by Trievel, [37, p. 317,318]. Trievel uses Besov spaces, a generalisation of fractional Sobolev spaces, so has different notation: $H^s_\Omega(\mathbb{R}^n)$ is denoted by $\tilde{B}^s_{2,2}(\Omega)$. By Section 4.3.2, Theorem 1(b), $C_0^\infty(\Omega)$ is dense in $\tilde{B}^s_{2,2}(\Omega)$, and so we have the result. □

Remark 5.2.3. Trievel proves the theorem assuming the Ω is bounded with C^∞ boundary. This is so that he can consider all s in $\mathbb{R}$ (yes, including negative values!) without having special cases for certain ranges of s. We have not checked fully, but we assume that a bounded Lipschitz domain is sufficient for the theorem in the case $s \in (0, 1)$.

In the next chapter we prove well-posedness of a general class of Dirichlet volume-constrained elliptic problems of order $2s$ for $s \in (0, 1)$.

Chapter 6

Well-Posedness of Elliptic Problems

In this chapter we study a general class of elliptic FDEs of order $2s$ with $s \in (0,1)$. We state the classical form for the homogeneous Dirichlet volume-constrained problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, derive a weak formulation of the problem on the space $H^s_\Omega(\mathbb{R}^n)$ and prove existence and uniqueness of its solution. As explained in the previous chapter, this implies existence and uniqueness for the nonhomogeneous case too, if 0 is the unique solution to the completely homogeneous problem.

6.1 Classical statement of the Dirichlet problem

Let $\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^*\cdot)$ be an anisotropic fractional Laplacian operator defined as in Definition 4.5.3. Let $f \in L^2(\Omega)$, $\boldsymbol{b} = (b_1, \dots, b_n)^{\mathrm{T}} \in C^1(\bar{\Omega})$, and $c \in C(\overline{\Omega})$. We consider two separate cases:

- $s \in [\frac{1}{2}, 1)$, if $\boldsymbol{b} \neq 0$
- $s \in (0,1)$, if $\boldsymbol{b} = 0$

We wish to find a function u defined on $\mathbb{R}^n$ which satisfies the following problem at least in a weak sense:

$$\begin{aligned} \mathcal{L}u := \mathcal{D}(\boldsymbol{\Theta}(\boldsymbol{x},\boldsymbol{y})\mathcal{D}^*u) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u + c(\boldsymbol{x})u = f \quad & \text{on } \Omega, \\ u = 0 \quad & \text{on } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{6.1}$$

The reason for the constraint on s is that the first-order advection term, which is present if and only if $\boldsymbol{b} \neq 0$, forces a weak solution u to require a regularity of at least $\frac{1}{2}$ for our well-posedness proof.

6.2 Weak formulation

In this section we define a bilinear operator on the Dirichlet volume-constrained Sobolev space $H^s_\Omega(\mathbb{R}^n)$ (Definition 5.6) which corresponds to $(u,v) \mapsto \int_\Omega v\mathcal{L}u \, d\boldsymbol{x}$.

Let u be a weak solution to (6.1) and $v \in C_0^\infty(\Omega)$, then:

$$\int_\Omega v\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u) \, d\boldsymbol{x} + \int_\Omega v\boldsymbol{b} \cdot \nabla u \, d\boldsymbol{x} \ + \int_\Omega cvu \, d\boldsymbol{x} = \int_\Omega vf \, d\boldsymbol{x}. \tag{6.2}$$

By Definition 4.5.3,

$$\int_\Omega v\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* u) \, d\boldsymbol{x} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u \, d\boldsymbol{y} \, d\boldsymbol{x}. \tag{6.3}$$

For the advection term, first note that $v = 0$ on $\mathbb{R}^n \setminus \Omega$, so:

$$\begin{aligned} \int_\Omega v\boldsymbol{b} \cdot \nabla u \, d\boldsymbol{x} &= \int_{\mathbb{R}^n} v\boldsymbol{b} \cdot \nabla u \, d\boldsymbol{x} \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} vb_j \tfrac{\partial u}{\partial x_j} \, d\boldsymbol{x}. \end{aligned} \tag{6.4}$$

Now, using Parseval's Theorem (A.3) along with the fact that $v\boldsymbol{b}$ is a real-valued function, and then the Fourier transform of a partial derivative (Theorem A.4) we have:

$$\begin{aligned} \sum_{j=1}^n \int_{\mathbb{R}^n} vb_j \tfrac{\partial u}{\partial x_j} \, d\boldsymbol{x} &= \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{\widehat{vb_j}} \, \widehat{\tfrac{\partial u}{\partial x_j}} \, d\boldsymbol{x} \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{\widehat{vb_j}} i\xi_j \hat{u} \, d\boldsymbol{\xi} \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{(-i\xi_j)^{\frac{1}{2}}\widehat{vb_j}} (i\xi_j)^{\frac{1}{2}} \hat{u} \, d\boldsymbol{\xi}. \end{aligned} \tag{6.5}$$

Here we used the following:

$$\overline{(i\xi_j)^{\frac{1}{2}}} = -i(i\xi_j)^{\frac{1}{2}} = (-i)^{\frac{1}{2}}(-i^2)^{\frac{1}{2}}(\xi_j)^{\frac{1}{2}} = (-i\xi_j)^{\frac{1}{2}}. \tag{6.6}$$

Using Corollary 3.2.9, we can express this in terms of Riemann-Liouville fractional derivatives:

$$\begin{aligned} \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{(-i\xi_j)^{\frac{1}{2}}\widehat{vb_j}} (i\xi_j)^{\frac{1}{2}} \hat{u} \, d\boldsymbol{\xi} &= \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{\widehat{\mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)}} \widehat{\mathcal{R}^{\frac{1}{2}}_{j}(u)} \, d\boldsymbol{\xi} \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \overline{\mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)} \mathcal{R}^{\frac{1}{2}}_{j}(u) \, d\boldsymbol{x} \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \mathcal{R}^{\frac{1}{2}}_{-j}(vb_j) \mathcal{R}^{\frac{1}{2}}_{j}(u) \, d\boldsymbol{x}. \end{aligned} \tag{6.7}$$

$\mathcal{R}^{\frac{1}{2}}_j$ is fractional derivative operator (3.2.2) of order $\frac{1}{2}$ in the direction of the canonical basis vector $\boldsymbol{e}_j$ and that which we denote $\mathcal{R}^{\frac{1}{2}}_{-j}$ is that in the opposite direction, $-\boldsymbol{e}_j$.

Now we are ready to define a bilinear form to express the weak formulation of (6.1).

Definition 6.2.1. Let $\mathcal{D}(\boldsymbol{\Theta}\mathcal{D}^* \cdot)$ be an anisotropic fractional Laplacian operator defined as in Definition 4.5.3. Let $\boldsymbol{b} \in W^{1,\infty}(\Omega)$ and $c \in L^\infty(\Omega)$. We define the bilinear form for $u, v \in H^s_\Omega(\mathbb{R}^n)$ by:

$$a(u,v) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u \, d\boldsymbol{y}\, d\boldsymbol{x} + \sum_{j=1}^{n} \int_{\mathbb{R}^n} \mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)\mathcal{R}^{\frac{1}{2}}_{j}(u)\, d\boldsymbol{x} + \int_\Omega cvu\, d\boldsymbol{x}, \quad (6.8)$$

and for $f \in L^2(\Omega)$, the linear functional l on $H^s_\Omega(\mathbb{R}^n)$:

$$l(v) := \int_\Omega vf\, d\boldsymbol{x}. \quad (6.9)$$

Definition 6.2.2 (Weak solution)**.** Let a and l be as defined in (6.8) and (6.9). We call $u \in H^s_\Omega(\mathbb{R}^n)$ a weak solution to (6.1) if the following holds:

$$a(u,v) = l(v) \quad \forall v \in H^s_\Omega(\mathbb{R}^n). \quad (6.10)$$

6.3 Well-posedness of the weak formulation

Lemma 6.3.1. *The bilinear form a is continuous.*

Proof. By Theorem B.1 we only need to show that there exists a constant c_1 such that $|a(u,v)| \le c_1 \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}$ for all u and v. Using the inequality derived in Remark 4.5.2 and the continuity of $\mathcal{D}^*$ (Proposition 4.5.4), the diffusion term is controlled by Sobolev semi-norms:

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* u|\, d\boldsymbol{y}\, d\boldsymbol{x} &\le C_{\boldsymbol{\Theta}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{D}^* v||\mathcal{D}^* u|\, d\boldsymbol{y}\, d\boldsymbol{x} \\ &\le C_{\boldsymbol{\Theta}} \|\mathcal{D}^* v\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \|\mathcal{D}^* u\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \\ &= C_{\boldsymbol{\Theta}} |v|_{H^s(\mathbb{R}^n)} |u|_{H^s(\mathbb{R}^n)}. \end{aligned} \quad (6.11)$$

If $\boldsymbol{b} \neq 0$, then by continuity of the Riemann-Liouville derivatives (Theorem 3.2.8) the advection term is controlled by Sobolev norms:

$$\begin{aligned}\sum_{j=1}^{n} \int_{\mathbb{R}^n} |\mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)\mathcal{R}^{\frac{1}{2}}_{j}(u)|\, d\boldsymbol{x} &\le \sum_{j=1}^{n} \left(\int_{\mathbb{R}^n} |\mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)|^2\, d\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\mathcal{R}^{\frac{1}{2}}_{-j}(u)|^2\, d\boldsymbol{x}\right)^{\frac{1}{2}} \\ &\le \sum_{j=1}^{n} |b_j v|_{\hat{H}^{\frac{1}{2}}(\mathbb{R}^n)} |u|_{\hat{H}^{\frac{1}{2}}(\mathbb{R}^n)} \\ &= \frac{1}{2}\hat{C}\left(n, \frac{1}{2}\right) \sum_{j=1}^{n} |b_j v|_{H^{\frac{1}{2}}(\mathbb{R}^n)} |u|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \\ &\le \frac{1}{2}\hat{C}\left(n, \frac{1}{2}\right) \sum_{j=1}^{n} \|b_j v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\hat{C}\left(n, \frac{1}{2}\right)\sum_{j=1}^{n} C_{b_j,\frac{1}{2}} \|v\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \|u\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \\
&\leq C(\boldsymbol{b}, n, s) \|v\|_{H^s(\mathbb{R}^n)} \|u\|_{H^s(\mathbb{R}^n)}, \qquad (6.12)
\end{aligned}
$$

where

$$
C(\boldsymbol{b}, n, s) = C_{\text{emb}}\left(s, \frac{1}{2}\right)\hat{C}\left(n, \frac{1}{2}\right)\sum_{j=1}^{n} C_{b_j,\frac{1}{2}}. \qquad (6.13)
$$

The constants and inequalities here come from Theorem 2.3.6, Theorem 2.4.3 and the embedding Corollary 2.3.9. Finally, the reaction term is controlled by L_2 norms:

$$
\begin{aligned}
\int_\Omega |cvu|\, d\boldsymbol{x} &\leq \|c\|_\infty \int_\Omega |vu|\, d\boldsymbol{x} \\
&\leq \|c\|_\infty \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\
&= \|c\|_\infty \|v\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}. \qquad (6.14)
\end{aligned}
$$

Combining these bounds, we have:

$$
\begin{aligned}
a(u,v) &\leq C_{\boldsymbol{\Theta}}|v|_{H^s(\mathbb{R}^n)}|u|_{H^s(\mathbb{R}^n)} + C(\boldsymbol{b}, n, s)\|v\|_{H^s(\mathbb{R}^n)}\|u\|_{H^s(\mathbb{R}^n)} + \|c\|_\infty\|v\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)} \\
&\leq \left(\left(C_{\boldsymbol{\Theta}}|v|_{H^s(\mathbb{R}^n)}\right)^2 + \left(C(\boldsymbol{b}, n, s)\|v\|_{H^s(\mathbb{R}^n)}\right)^2 + \left(\|c\|_\infty\|v\|_{L^2(\mathbb{R}^n)}\right)^2\right)^{\frac{1}{2}} \\
&\quad \cdot \left(|u|^2_{H^s(\mathbb{R}^n)} + \|u\|^2_{H^s(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}} \\
&\leq c_1 \|v\|_{H^s(\mathbb{R}^n)}\|u\|_{H^s(\mathbb{R}^n)}, \qquad (6.15)
\end{aligned}
$$

where

$$
c_1 = 2^{\frac{1}{2}} \cdot \left(\max\left\{C^2_{\boldsymbol{\Theta}}, \|c\|^2_\infty\right\} + C(\boldsymbol{b}, n, s)^2\right)^{\frac{1}{2}}. \qquad (6.16)
$$

Note that we have derived an explicit form for this constant, which depends on $\boldsymbol{\Theta}$, $\boldsymbol{b}$, c, n, s, but not Ω. $\square$

Lemma 6.3.2. *If $c - \frac{1}{2}\nabla \cdot \boldsymbol{b} \geq 0$ then the bilinear functional a is coercive in $H^s_\Omega(\mathbb{R}^n)$, i.e. $\exists c_2 > 0$ such that $a(v,v) \geq c_2 \|v\|^2_{H^s(\mathbb{R}^n)}$ for all $v \in H^s_\Omega(\mathbb{R}^n)$.*

Proof. Assume for now that $v \in C_0^\infty(\Omega)$. We deal with the advection term with a standard argument using integration by parts:

$$
\begin{aligned}
\sum_{j=1}^{n}\int_{\mathbb{R}^n} \mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)\mathcal{R}^{\frac{1}{2}}_j(v)\, d\boldsymbol{x} &= \int_\Omega v\boldsymbol{b}\cdot\nabla v\, d\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{b}\cdot\frac{1}{2}\nabla v^2\, d\boldsymbol{x} \\
&= \int_\Omega \left(-\frac{1}{2}\nabla\cdot\boldsymbol{b}\right) v^2\, d\boldsymbol{x}. \qquad (6.17)
\end{aligned}
$$

Then we directly show coercivity:

$$
a(v,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* v\, d\boldsymbol{y}\, d\boldsymbol{x} + \sum_{j=1}^{n}\int_{\mathbb{R}^n} \mathcal{R}^{\frac{1}{2}}_{-j}(vb_j)\mathcal{R}^{\frac{1}{2}}_j(v)\, d\boldsymbol{x} + \int_\Omega cv^2\, d\boldsymbol{x}.
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{D}^* v \cdot \boldsymbol{\Theta}\mathcal{D}^* v \, d\boldsymbol{y}\, d\boldsymbol{x} + \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\boldsymbol{b}\right) v^2 \, d\boldsymbol{x} \\
&\geq c_{\boldsymbol{\Theta}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{D}^* v|^2 \, d\boldsymbol{y}\, d\boldsymbol{x} \\
&= c_{\boldsymbol{\Theta}} |v|^2_{H^s(\mathbb{R}^n)} \\
&\geq c_2 \|v\|^2_{H^s(\mathbb{R}^n)} . \qquad (6.18)
\end{aligned}
$$

The last step is by the corollary of the fractional Friedrichs inequality for $C_0^\infty(\Omega) \subset H^s_\Omega(\mathbb{R}^n)$ (Corollary 3.3.6). Note that we have derived an explicit form for the constant c_2, which depends on $\boldsymbol{\Theta}$, $\boldsymbol{b}$, c, s and d, the diameter of Ω, but not n.

Now consider $v \in H^s_\Omega(\mathbb{R}^n)$. $C_0^\infty(\Omega)$ is dense in $H^s_\Omega(\mathbb{R}^n)$ (Theorem 5.2.2), so we have a sequence $\{v_m\}_{m=0}^\infty \subset C_0^\infty(\Omega)$ such that $v_m \to v$ in $H^s(\mathbb{R}^n)$. By Lemma 6.3.1, a is continuous, so:

$$
\begin{aligned}
a(v,v) &= a(\lim_{m\to\infty}(v_m, v_m)) \\
&= \lim_{m\to\infty} a(v_m, v_m) \\
&\geq c_2 \lim_{m\to\infty} \|v_m\|^2_{H^s(\mathbb{R}^n)} \\
&= c_2 \|v\|^2_{H^s(\mathbb{R}^n)} . \qquad (6.19)
\end{aligned}
$$

Therefore we have coercivity. □

Lemma 6.3.3. *The linear functional l is continuous.*

Proof. Let $v \in H^s_\Omega(\mathbb{R}^n)$. Then:

$$
\begin{aligned}
|l(v)| &= |\int_\Omega v f \, d\boldsymbol{x}| \\
&\leq \int_\Omega |vf| \, d\boldsymbol{x} \\
&\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} \|v\|_{H^s(\mathbb{R}^n)} . \qquad (6.20)
\end{aligned}
$$

□

Theorem 6.3.4 (Existence and uniqueness)**.** *If $c - \frac{1}{2}\nabla\cdot\boldsymbol{b} \geq 0$, then there exists a unique weak solution to the homogeneous Dirichlet volume-constrained problem* (6.1)*.*

Proof. Collecting Lemmata 6.3.1, 6.3.2 and 6.3.3 gives us satisfaction of the conditions for the Lax-Milgram Theorem (B.2). Therefore problem (6.10) has a unique solution. □

Theorem 6.3.5 (Well-posedness)**.** *The weak formulation (6.10) is well-posed problem in the sense of Hadamard [19], i.e. there exists a unique solution u in $H^s_\Omega(\mathbb{R}^n)$ that depends continuously on the data, f.*

Proof. We showed in Theorem 6.3.4 above that there exists a unique solution to this problem. For continuity, let u_j be the solution to (6.10) with data f_j for $j = 1, 2$. Then by linearity $u_1 - u_2$ is the solution to (6.10) with $l(v) = \int_\Omega v(f_1 - f_2)\, d\boldsymbol{x}$ and so:

$$\|u_1 - u_2\|_{H^s(\mathbb{R}^n)} \leq \frac{|a(u_1 - u_2, u_1 - u_2)|}{c_2\|u_1 - u_2\|_{H^s(\mathbb{R}^n)}} = \frac{|l(u_1 - u_2)|}{c_2\|u_1 - u_2\|_{H^s(\mathbb{R}^n)}} \leq c_2^{-1}\|f_1 - f_2\|_{L^2(\Omega)}. \tag{6.21}$$

□

Remark 6.3.6. Ervin and Roop [16, Sec. 6] study elliptic FDEs in two dimensions and have a similar proof for existence and uniqueness of solution. The differences are: the problems they study are boundary value problems, they only consider the case $s \in (\frac{1}{2}, 1)$ and they use a different operator for the order $2s$ term, which is anisotropic, but the anistropy does not depend on the location in the domain.

Chapter 7

Galerkin Approximation

7.1 Approximation of the problem

We wish to approximate the the weak formulation of the problem (6.10) by restricting our consideration for our solution u and test functions v to a k-dimensional subspace V_k of $H^s_\Omega(\mathbb{R}^n)$ (for $s \in (0,1)$), calling the approximate solution u_k. We would then like to prove that as k tends to infinity, these approximate solutions converge to the solution of the actual problem in some suitable norm. This is a general approach called *Galerkin's method* [8, p. 36].

For now, we will assume nothing about V_k except its dimension and that it is a subspace of $H^s_\Omega(\mathbb{R}^n)$. Firstly, recall the weak formulation of our problem: Find $u \in H^s_\Omega(\mathbb{R}^n)$ such that

$$a(u,v) = l(v) \text{ for all } v \in H^s_\Omega(\mathbb{R}^n). \tag{7.1}$$

Definition 7.1.1. We say that $u_k \in V_k$ is the Galerkin approximation to the solution to (7.1) if:

$$a(u_k, v_k) = l(v_k) \text{ for all } v_k \in V_k. \tag{7.2}$$

Theorem 7.1.2. *There exists a unique Galerkin approximation to u for each k.*

Proof. We can use the Lax-Milgram Theorem (B.2), because the fact that a and l satisfy the conditions for the theorem on $H^s_\Omega(\mathbb{R}^n)$ imply they satisfy them on the subspace V_k. □

Remark 7.1.3. Recall from the *C12.2b* course that since V_k is finite dimensional, it has a basis, and therefore the Galerkin approximation u_k can be found by solving a system of linear equations.

Definition 7.1.4. We say that the Galerkin approximation u_k *converges* to the solution u of (7.1) in the norm $\|\cdot\|$ on $H^s_\Omega(\mathbb{R}^n)$ if:

$$\|u - u_k\|_{H^s_\Omega(\mathbb{R}^n)} \to 0 \text{ as } k \to \infty. \tag{7.3}$$

7.2 Convergence in the H^s norm

The following lemma will give us a sufficient condition for convergence of the finite element approximation in the $H^s(\mathbb{R}^n)$ norm.

Lemma 7.2.1 (Céa's lemma)**.** *Let u_k be the finite element approximation of (7.1), and let c_1 and c_2 be the constants found in Lemmas 6.3.1 and 6.3.2 respectively for the bilinear form a. Then:*

$$\|u - u_k\|_{H^s(\mathbb{R}^n)} \leq \frac{c_1}{c_2} \inf_{v_k \in V_k} \|u - v_k\|_{H^s(\mathbb{R}^n)}. \tag{7.4}$$

Suppose futher that a is symmetric (i.e. $\boldsymbol{b} = 0$). Then:

$$\|u - u_k\|_{H^s(\mathbb{R}^n)} \leq \sqrt{\frac{c_1}{c_2}} \inf_{v_k \in V_k} \|u - v_k\|_{H^s(\mathbb{R}^n)}. \tag{7.5}$$

Proof. See *Finite Element Methods for Elliptic Problems*, by Ciarley [8, Thm. 2.4.1, Rmk. 2.4.1]. □

Theorem 7.2.2. *A sufficient condition for convergence of the finite element approximation u_k to the solution of (7.1) in the $H^s(\mathbb{R}^n)$ norm is that there exists a sequence of operators $\mathcal{P}_k : H^s_\Omega(\mathbb{R}^n) \to V_k$ such that:*

$$\|v - \mathcal{P}_k(v)\|_{H^s(\mathbb{R}^n)} \to 0 \text{ as } k \to \infty, \text{ for all } v \in H^s_\Omega(\mathbb{R}^n). \tag{7.6}$$

Now we discuss examples of subspaces V_k and operators $\mathcal{P}_k$ such that this sufficient condition holds.

7.3 Finite elements

As seen in the *C12.2b* course, we can consider the finite element spaces. We assume that Ω is a polygonal domain, and triangulate it as usual with $h > 0$ being the length of the longest side of any triangle in the mesh. The finite element space $V_h \subset H^s_\Omega(\mathbb{R}^n)$ is the space of all continuous, piecewise polynomials of degree m on the triangulation, where m is a positive integer.

Theorem 7.3.1. *Let $s \in (0, 1)$, $u \in H^s_\Omega(\mathbb{R}^n))$. Then for real σ such that $0 \leq s \leq \sigma \leq m$ there exists a constant $C_\mathcal{I}$ depending only on Ω such that:*

$$\|u - \mathcal{I}_h u\|_{H^s(\mathbb{R}^n)} \leq C_\mathcal{I} h^{\rho - s} \|u\|_{H^\sigma(\Omega)} \tag{7.7}$$

Proof. See [4, Sec. 14.3] for a proof of:

$$\|u - \mathcal{I}_h u\|_{H^s(\Omega)} \leq C h^{\rho - s} \|u\|_{H^\sigma(\Omega)}, \tag{7.8}$$

for some constant $C > 0$. Then consider the theorem in [24, Thm. 11.4]. This theorem implies that there exists a constant C_Ω such that for any $u \in H^s_\Omega(\mathbb{R}^n)$

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C_\Omega \|u\|_{H^s(\Omega)}. \tag{7.9}$$

This gives the desired result with $C_\mathcal{I} = C \cdot C_\Omega$.

The authors of [16] seem to have overlooked this issue. They quote the result (7.8), but then use it as if the $H^s(\Omega)$ norm is equivalent to their $H^s_0(\Omega)$ norm without justification, which they define with the Fourier transform on $\mathbb{R}^2$. □

7.4 Legendre polynomials

Let us consider the special case where $\Omega = [-1, 1]$, $\boldsymbol{b} = 0$ and $s \in (0, \frac{1}{2})$. The case $s < \frac{1}{2}$ is interesting because the galerkin approximation need not be zero on the boundary (see Section 2.6). It also simplifies matters because we don't have to force our approximations to have roots at -1 and 1. We can define the space P_k of polynomials of degree at most $k \in \mathbb{N}$ and the operator $\mathcal{P}_k$ is the polynomial interpolant in $k+1$ Legendre points in Ω.

Theorem 7.4.1. *Let $s \in (0, 1)$, $u \in H^s_\Omega(\mathbb{R}^n)$. Then for any real σ such that $0 \leq s \leq \sigma$, there exists a constant $C_\mathcal{P}$ such that*

$$\|u - \mathcal{P}_k u\|_{H^s(\mathbb{R})} \leq C_\mathcal{P} k^{3s/2 - \sigma} \|u\|_{H^\sigma(\Omega)}. \tag{7.10}$$

Proof. See [7, Thm. 2.4] for the bound:

$$\|u - \mathcal{P}_k u\|_{H^s(\Omega)} \leq C k^{3s/2 - \sigma} \|u\|_{H^\sigma(\Omega)}, \tag{7.11}$$

for some $C > 0$. Then note, as in Theorem 7.3.1, the continuous embedding of $H^s(\Omega)$ into $H^s_\Omega(\mathbb{R}^n)$ from [24, Thm. 11.4]. This gives the required inequality. □

Corollary 7.4.2. *Suppose that u is the solution to (7.1) for $\Omega = [-1, 1]$, $\boldsymbol{b} = 0$ and $s \in (0, \frac{1}{2})$. Suppose further that $u \in H^\sigma(\Omega)$ where $\sigma > 3s/2$. Then the Galerkin approximation u_k to the problem (7.1) in the space P_k converges, and satisfies:*

$$\|u - u_k\|_{H^s(\mathbb{R}^n)} \leq \sqrt{\frac{c_1}{c_2}} C_\mathcal{P} k^{3s/2 - \sigma} \|u\|_{H^\sigma(\Omega)} \tag{7.12}$$

Proof. Combine Céa's Lemma 7.2.1 with the interpolation error estimate in Theorem 7.4.1. □

Chapter 8

Conclusion

8.1 Aims of the project

When we proposed this project, we had an aim to develop a-posteriori error estimates for a finite element method for an anomalous diffusion-type equation using the nonlocal calculus. With these computable error estimates we wanted to implement an efficient adaptive mesh algorithm.

However, the analysis of FDEs turned out to be much tougher than expected, primarily because of the nonlocality of the fractional-order norms and operators, but it certainly is interesting and it gave us a lot to study and write about. As a result, the majority of this dissertation is devoted to the analytical side of FDEs, with just one chapter devoted to approximation considerations.

8.2 Further work

We present ideas for further work in the form of a list:

- The issue with the nonlocal calculus for expressing the fractional Laplacian for $s \in [\frac{1}{2}, 1)$ needs to be resolved. It is very frustrating that the authors have not clarified this issue in their paper.
- The extension of the theoretical work to the time dependent case.
- Consider nonlocal Neumann-type volume constraints, as in [11].
- Practical implementation issues for Galerkin approximation of FDEs.
- Elliptic regularity estimates for fractional-order elliptic problems.
- Aubin-Nietsche argument for error bounds in the L^2 norm.

Appendix A

The Fourier Transform

In this appendix we make clear which definition of the Fourier transform we use and state some useful properties. We follow *Partal Differential Equations*, by Evans [17, p. 187-190]. We consider all functions to be complex-valued.

Definition A.1. For $u \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\mathcal{F}[u] = \hat{u}$ and its inverse Fourier transform $\mathcal{F}^{-1}[u] = \check{u}$ by:

$$\hat{u}(\boldsymbol{\xi}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} u(\boldsymbol{x})\, d\boldsymbol{x}, \tag{A.1}$$

$$\check{u}(\boldsymbol{x}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} u(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{A.2}$$

For $u \in L^2(\mathbb{R}^n)$, the Fourier transform and inverse Fourier transform is defined using the density of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ (see [17, p. 189]).

Theorem A.2 (Plancherel)**.** *Assume $u \in L^2(\mathbb{R}^n)$. Then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}. \tag{A.3}$$

Theorem A.3 (Parseval's identity)**.** *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} u\bar{v}\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}\, d\boldsymbol{\xi}. \tag{A.4}$$

Proposition A.4 (Fourier transform of directional derivatives)**.** *Assume $u \in H^k(\mathbb{R}^n)$ for some integer $k \geq 1$ and $\boldsymbol{m}$ a unit vector in $\mathbb{R}^n$. Let $(\boldsymbol{m} \cdot \nabla^k)$ denote the kth-order directional derivative in the direction m. Then*

$$\mathcal{F}\left[(\boldsymbol{m} \cdot \nabla^k) u\right] = (\boldsymbol{m} \cdot i\boldsymbol{\xi})^k \hat{u} \tag{A.5}$$

Appendix B

Bilinear Forms on Hilbert Spaces

Lemma B.1 (Criterion for continuity)**.** *Let H be a real Hilbert space and let $a : H \times H \to \mathbb{R}$ be a bilinear functional. Then a is continuous with respect to the product norms on $H \times H$ if and only if there exists a constant $C > 0$ such that:*

$$a(u, v) \leq C\|u\|_H \|v\|_H \text{ for all } u, v \in H. \tag{B.1}$$

Proof. Taught in the *B4b Hilbert Spaces* course, Hilary Term 2011. □

Theorem B.2 (Lax-Milgram)**.** *Let H be a real Hilbert space, let $l : H \to \mathbb{R}$ be a continuous linear functional, and let $a : H \times H \to \mathbb{R}$ be a continuous bilinear functional, that is also coercive i.e. there exists a constant $c > 0$ such that:*

$$a(v, v) \geq c\|v\|_H^2 \text{ for all } v \in H. \tag{B.2}$$

Then there exists a unique solution to the problem of finding $u \in H$ such that:

$$a(u, v) = l(v) \text{ for all } v \in H. \tag{B.3}$$

Proof. See *Finite Element Methods for Elliptic Problems*, by Ciarlet [8, p. 8] or *Partial Differential Equations*, by Evans [17, p. 316]. □

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