Spectra of Jacobi Operators via Connection Coefficients

Marcus Webb

University of Cambridge

Joint work with Sheehan Olver (University of Sydney)

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Funding: EPSRC, PhD Supervisor: Arieh Iserles

Jacobi Operators

A Jacobi operator has matrix form

$$J = \begin{bmatrix} \alpha_0 & \beta_0 & & \\ \beta_0 & \alpha_1 & \beta_1 & \\ & \beta_1 & \alpha_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \quad \alpha_k, \beta_k \in \mathbb{R}, \quad (\beta_k > 0)$$

Jacobi Operators = Orthogonal Polynomials = Probability Densities

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Orthogonal Polynomials

$$\begin{aligned} P_0(x) &= 1 \\ xP_0(x) &= \alpha_0 P_0(x) + \beta_0 P_1(x) \\ xP_k(x) &= \beta_{k-1} P_{k-1}(x) \\ &\quad + \alpha_k P_k(x) + \beta_k P_{k+1}(x) \end{aligned}$$

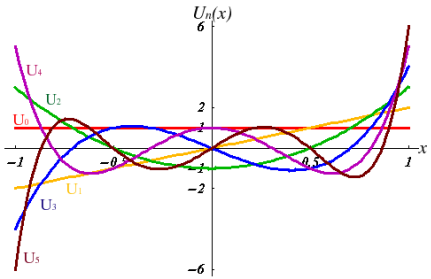
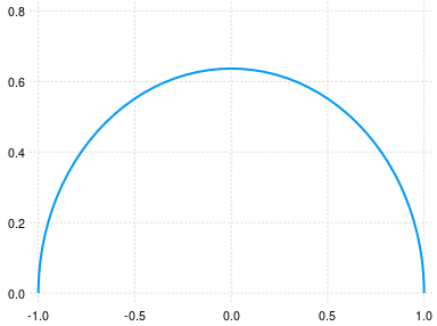
Probability Densities

Spectral measure

$$\mu \in \mathrm{Prob}(\sigma(J))$$

$$\int P_i(x) P_j(x) \, \mathrm{d}\mu(x) = \delta_{ij}$$

Jacobi Operators = Orthogonal Polynomials = Probability Densities

Jacobi Operators	Orthogonal Polynomials	Probability Densities
$\begin{bmatrix} \alpha_0 & \beta_0 & & \\ \beta_0 & \alpha_1 & \beta_1 & \\ & \beta_1 & \alpha_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$	$P_0(x) = 1$ $xP_0(x) = \alpha_0 P_0(x) + \beta_0 P_1(x)$ $xP_k(x) = \beta_{k-1}P_{k-1}(x)$ $+\alpha_k P_k(x) + \beta_k P_{k+1}(x)$	Spectral measure $\mu \in \mathrm{Prob}(\sigma(J))$ $\int P_i(x)P_j(x)\,\mathrm{d}\mu(x) = \delta_{ij}$
Δ	$U_k(x)$ (Chebyshev Polynomials)	$\mu(x) = \frac{2}{\pi}\sqrt{1-x^2}$
$\begin{bmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$		

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  where $e_0 := (1, 0, 0, \ldots)^\top$.
- $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Spectral Measure and Resolvent

The Perron-Stieltjes Theorem:

$$\mu(x) = \frac{1}{2\pi} \lim_{\epsilon \to +0} G(x + i\epsilon) - G(x - i\epsilon)$$

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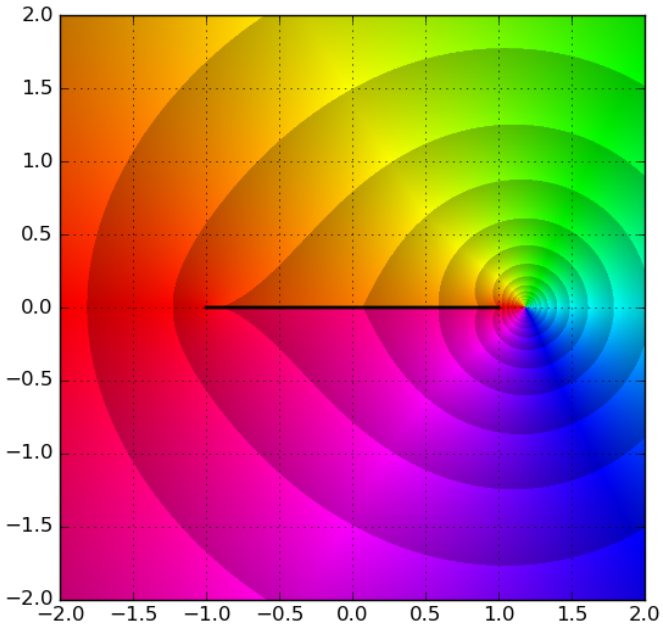

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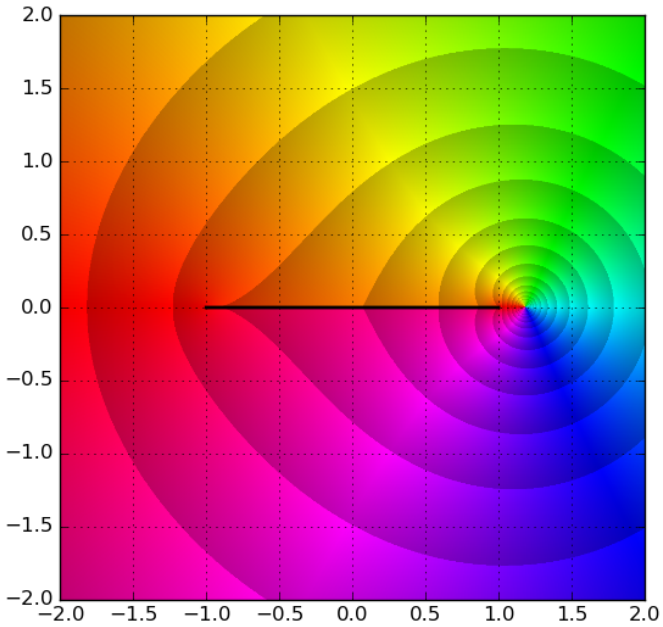

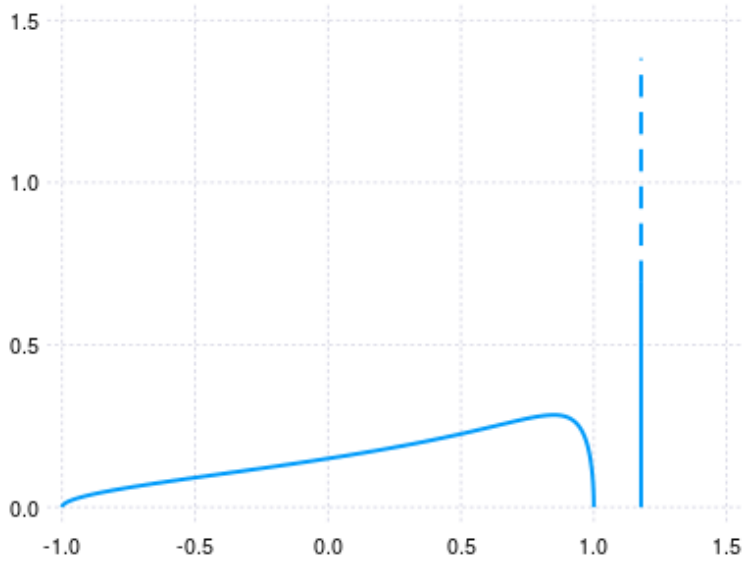

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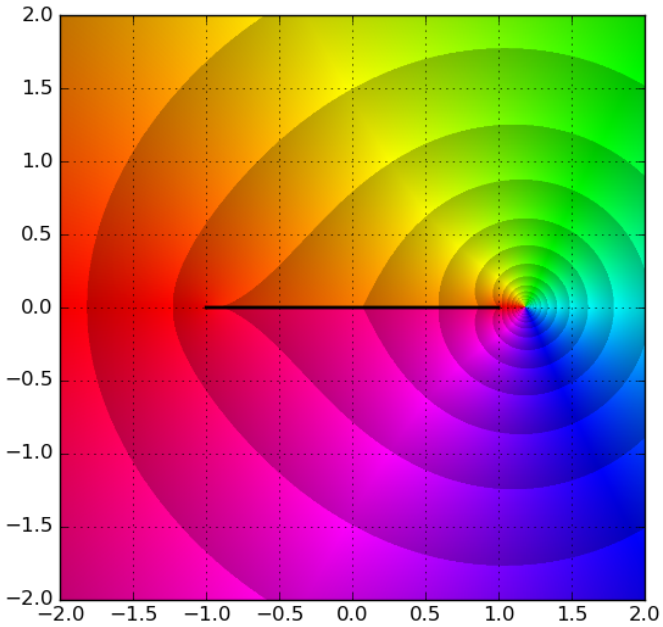

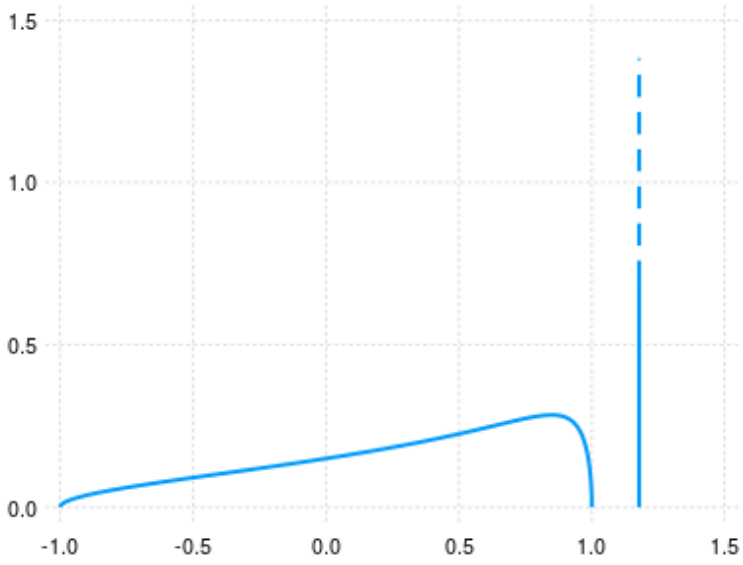

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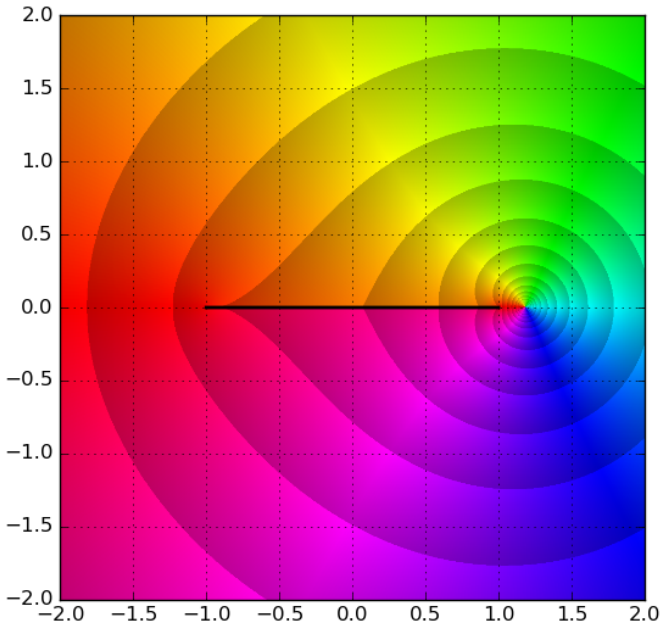

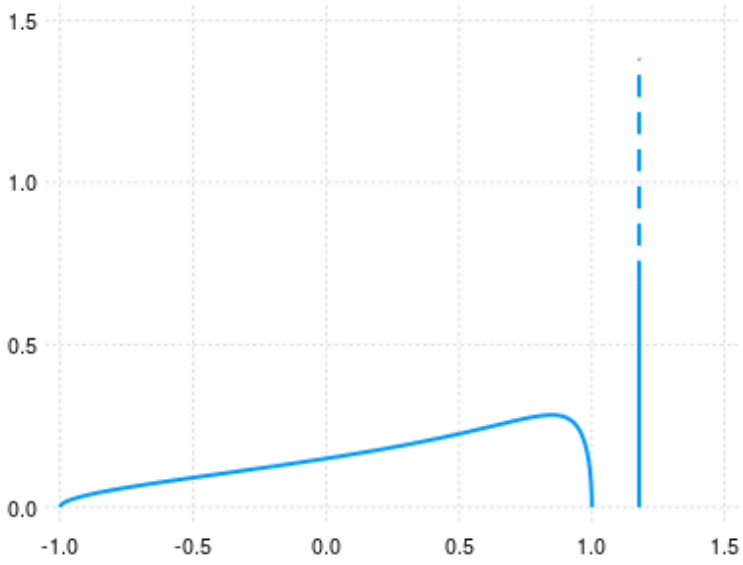

Branch cut in $G \mapsto$ continuous part of μ.
Pole in $G \mapsto$ Dirac mass in μ.

The Jacobi Operators in This Talk

- **For this talk** we restrict to the case where there exists n such that $\alpha_k = 0$, $\beta_{k-1} = \frac{1}{2}$ for all $k > n$. We call this **Pert-Toeplitz**.

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where $m_{\alpha,\beta}$ is a normalisation constant. Jacobi operator has:

$$\begin{aligned}
\alpha_k &= \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)} \to 0 \\
\beta_{k-1} &= 2\sqrt{\frac{k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)^2(2k+\alpha+\beta+1)}} \to \frac{1}{2}
\end{aligned}$$

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- Other relevant authors studying these are Geronimus, Nevai, Chihara, Van Assche.
- Our results extend to trace class too.

Explicit example

The example I will use throughout the talk to help explain is

$$J_{\mathrm{ex}} = \begin{bmatrix} \frac{3}{4} & 1 & & & \\ 1 & -\frac{1}{4} & \frac{3}{4} & & \\ & \frac{3}{4} & \frac{1}{2} & \frac{1}{2} & \\ & & \frac{1}{2} & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} = \Delta + \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & & & \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & & \\ & \frac{1}{4} & \frac{1}{2} & 0 & \\ & & 0 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

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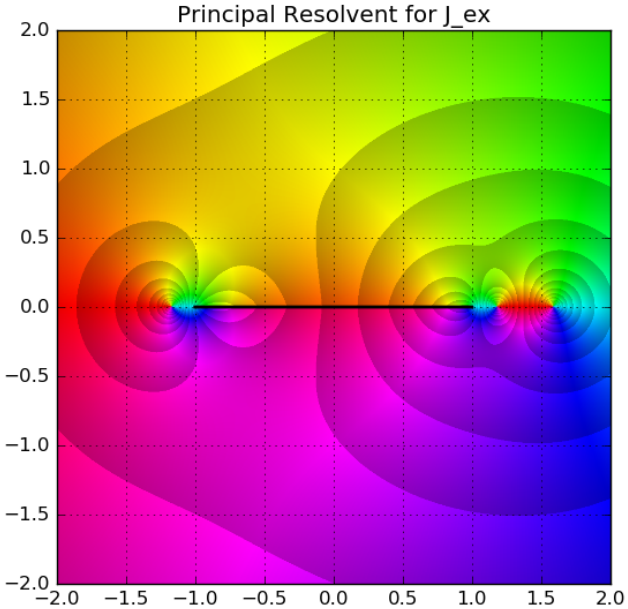

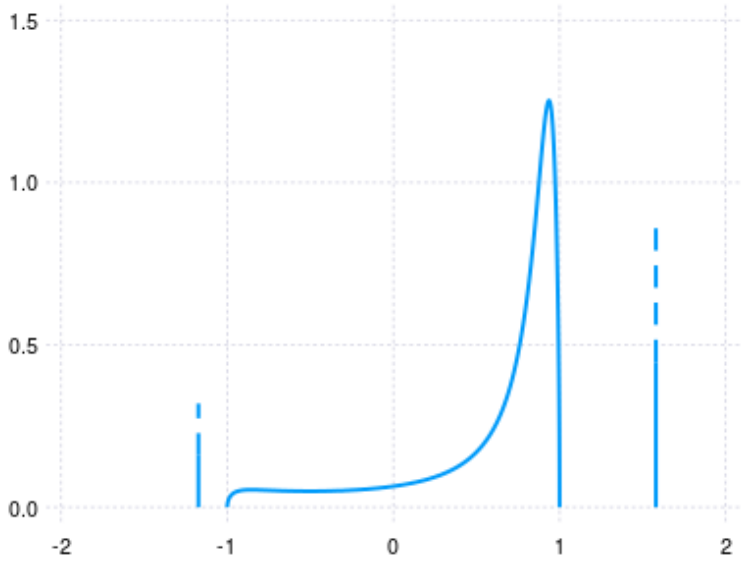

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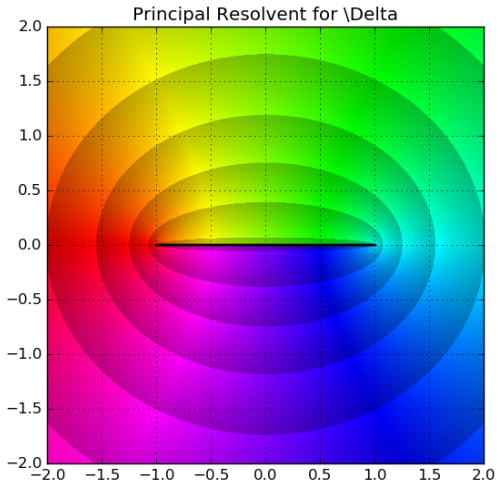
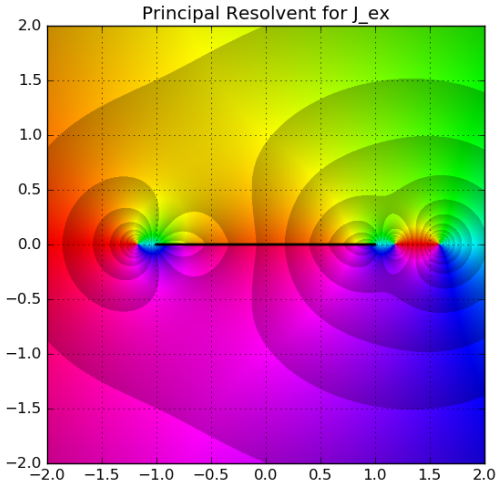
- $G_\Delta(\lambda) = 2\sqrt{\lambda+1}\sqrt{\lambda-1} - 2\lambda \to G(\lambda) = ?$





Connection Coefficients

- Key idea: construct the **connection ceofficients**. Let P_k be the orthogonal polynomials for J and suppose

$$f(x) = \sum_{k=0}^{\infty} a_k^J P_k(x) = \sum_{k=0}^{\infty} a_k^{\Delta} U_k(x).$$

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- $C^\top$ changes basis from U_k to P_k:

$$\begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} c_{00} & 0 & 0 & 0 & \cdots \\ c_{01} & c_{11} & 0 & 0 & \cdots \\ c_{02} & c_{12} & c_{22} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} U_0(x) \\ U_1(x) \\ U_2(x) \\ \vdots \end{bmatrix}$$

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- Entries of C are **computable** by finite difference methods. First column is initial data. Information propagates to the right like a wave.

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$$C_{\mathrm{ex}} = \begin{bmatrix} 1 & -0.75 & -1.25 & 2.04 & -0.08 & -0.33 & 0 & \cdots \\ 0 & 0.5 & -0.33 & -1.33 & 1.71 & -0.08 & -0.33 & \ddots \\ 0 & 0 & 0.33 & -0.66 & -1.33 & 1.71 & -0.08 & \ddots \\ 0 & 0 & 0 & 0.33 & -0.66 & -1.33 & 1.71 & \ddots \\ 0 & 0 & 0 & 0 & 0.33 & -0.66 & -1.33 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0.33 & -0.66 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.33 & \ddots \\ & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

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&= \int (p(x) - p(\lambda))(x - \lambda)^{-1} \, \mathrm{d}\mu(x) + p(\lambda) \int (x - \lambda)^{-1} \, \mathrm{d}\mu(x) \\
&= \qquad p^\mu(\lambda) \qquad\qquad + p(\lambda) \quad G(\lambda)
\end{aligned}
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(μ-derivative of p) (we want G)

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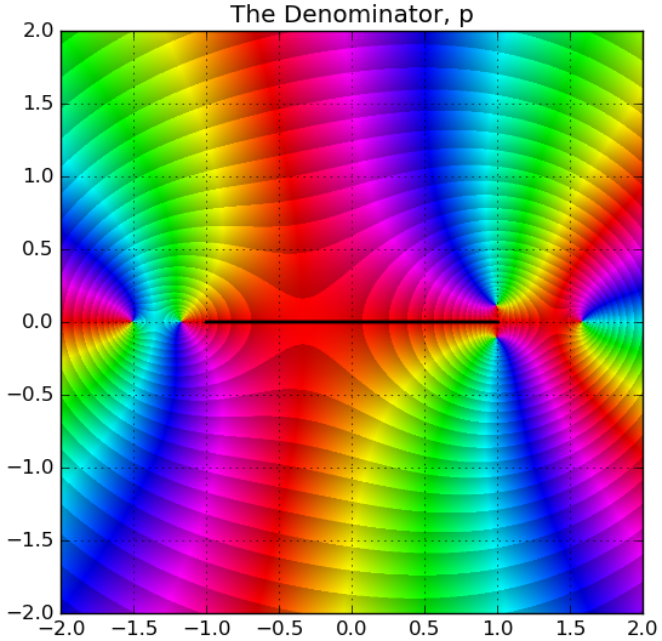

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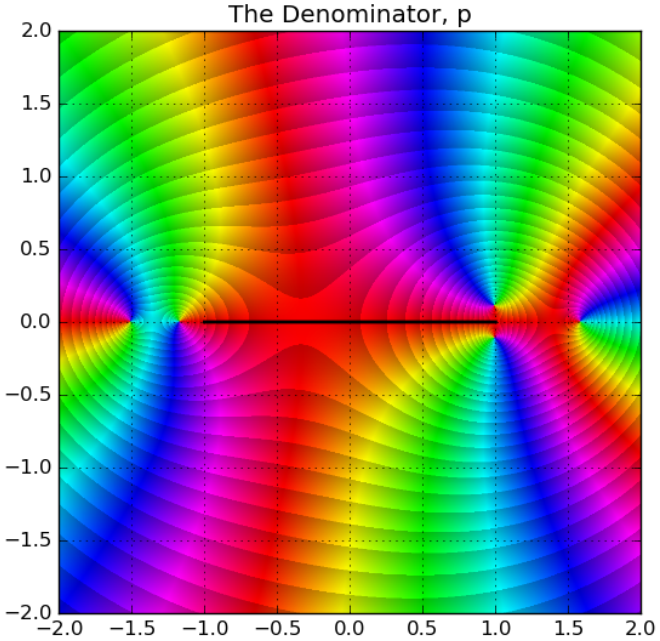

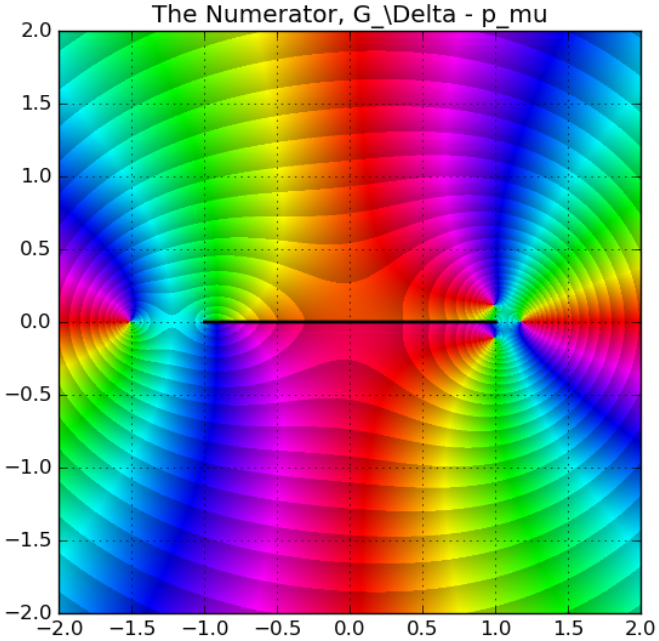

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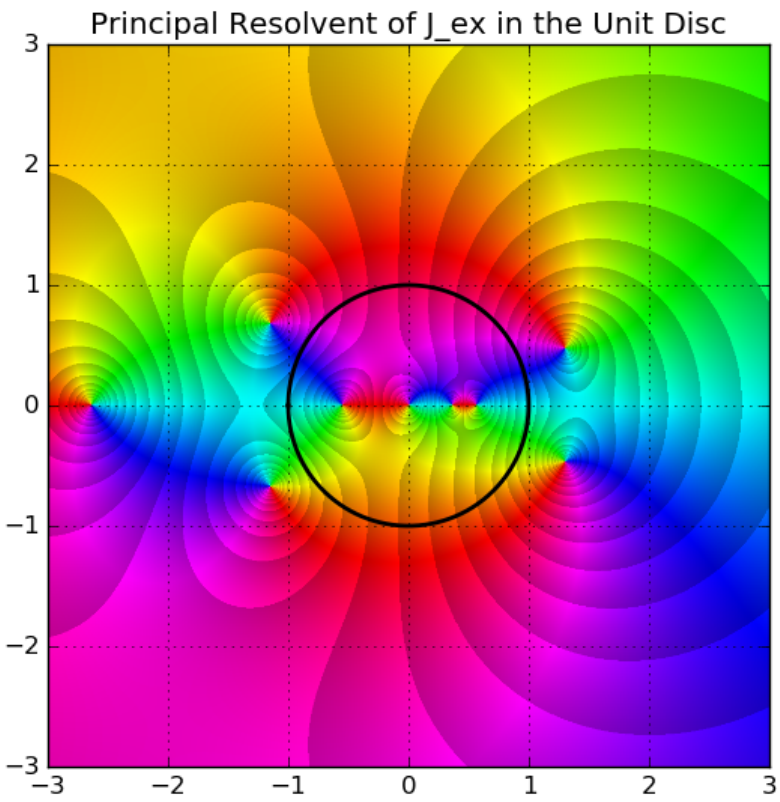

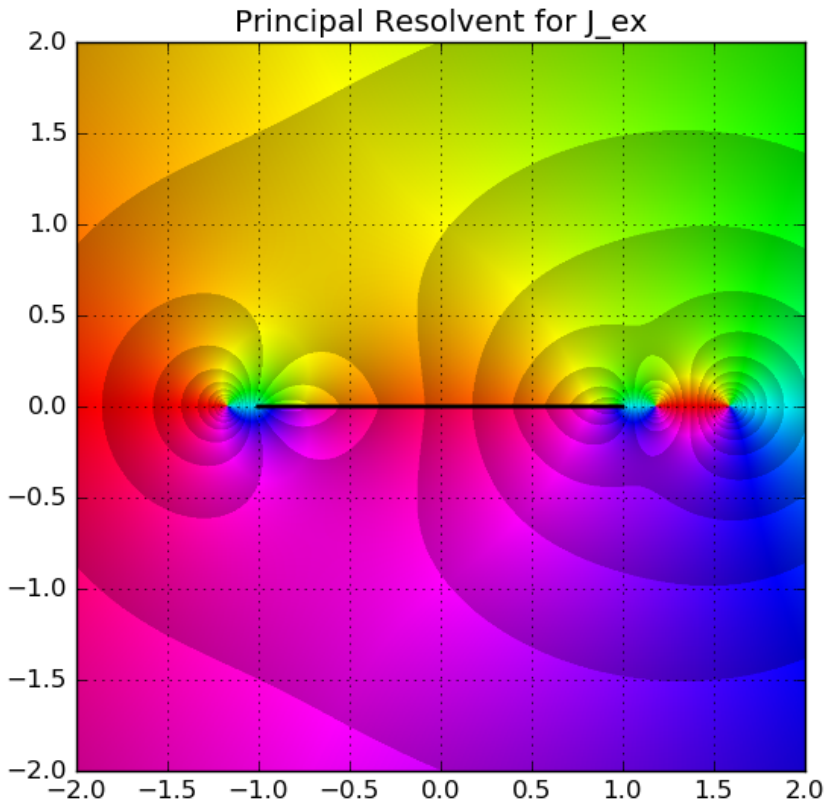

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Connection coefficients matrix C gives us a **formula for the spectral measure** of J (in the case where J is pert-Toeplitz):

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