Introduction to Mechanics (0J2)

Example Sheet 7 – solutions

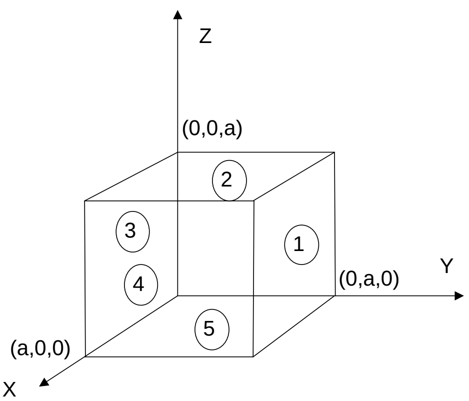
1. 2kg at $(1,4)$, 3kg at $(3,6)$, 4kg at $(2,1)$, so

$$\bar{x} = \frac{2+9+8}{9} = \frac{19}{9}; \qquad \bar{y} = \frac{8+18+4}{9} = \frac{30}{9}.$$

2. 3kg at $(2,-1)$, 2kg at $(3,5)$, 5kg at $(-2,-1)$, 1kg at $(1,-3)$ so

$$\bar{x} = \frac{6+6-10+1}{11} = \frac{3}{11}; \qquad \bar{y} = \frac{-3+10-5-3}{11} = -\frac{1}{11}.$$

3.



Five sides, each with C of M at centre of the respective square and mass m. So C of M of each is:

⑤: (base): $(\frac{a}{2}, \frac{a}{2}, 0)$

①: $(0, a, 0) + (\frac{a}{2}, 0, \frac{a}{2}) = (\frac{a}{2}, a, \frac{a}{2})$

②: $(0, \frac{a}{2}, \frac{a}{2})$

③: $(\frac{a}{2}, 0, \frac{a}{2})$

④: $(a, 0, 0) + (0, \frac{a}{2}, \frac{a}{2}) = (a, \frac{a}{2}, \frac{a}{2})$.

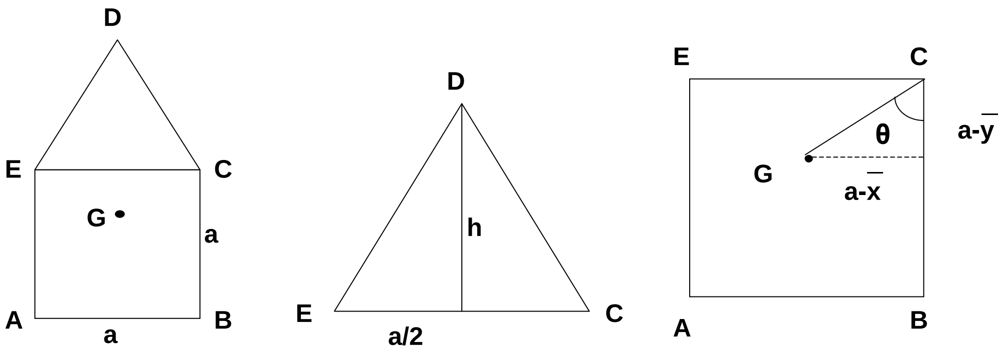
Hence

$\bar{x} = \frac{m}{5m}\{\frac{1}{2}a + \frac{1}{2}a + 0 + \frac{1}{2}a + a\} = \frac{1}{2}a$

$\bar{y} = \frac{m}{5m}\{\frac{1}{2}a + a + \frac{1}{2}a + 0 + \frac{1}{2}a\} = \frac{1}{2}a$

$\bar{z} = \frac{m}{5m}\{0 + \frac{1}{2}a + \frac{1}{2}a + \frac{1}{2}a + \frac{1}{2}a\} = \frac{2}{5}a.$

4.



Let A be at $(0, 0)$, B at $(a, 0)$, C at (a, a), E at $(0, a)$ and D at $(\frac{1}{2}a, a+h)$. Since CDE is an equilateral triangle $h = \frac{\sqrt{3}}{2}a$.

Centre of mass of square ABCE is at $(\frac{1}{2}a, \frac{1}{2}a)$.

Area is a^2 so mass is $a^2\rho$ where ρ is the mass per unit area.

Centre of mass of triangle is at $(\frac{1}{2}a, a + \frac{1}{3}h)$ since it is one third of the way up the median.

Area is $\frac{1}{2}ah = \frac{\sqrt{3}}{4}a^2$ so mass is $\frac{\sqrt{3}}{4}a^2\rho$.

Total mass $M = (1 + \sqrt{3}/4)a^2\rho$.

$$\text{Hence } \bar{y} = \frac{a^2\rho}{M}\{1 \times \frac{1}{2}a + \sqrt{3}/4 \times (a + \frac{1}{3}h)\}$$

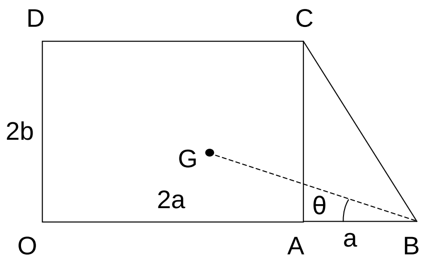
$$= \frac{a^2\rho}{M}\left[\frac{1}{2} + \frac{\sqrt{3}}{4} + \frac{3}{3 \times 8}\right] a \quad = \frac{a^2\rho}{M}\left[\frac{5}{8} + \frac{\sqrt{3}}{4}\right] a.$$

Thus $\bar{y} = \frac{a}{2}\frac{[5+2\sqrt{3}]}{[4+\sqrt{3}]}$. By symmetry $\bar{x} = \frac{1}{2}a$.

Let G be the centre of mass at $(\bar{x}, \bar{y})$. When suspended at C the line CG will be vertically down. Angle θ between CG and CB is given by

$$\tan\theta = \frac{a - \bar{x}}{a - \bar{y}} = \frac{\frac{1}{2}a}{a - \frac{a}{2}(\frac{5+2\sqrt{3}}{4+\sqrt{3}})} = \frac{4 + \sqrt{3}}{3}$$

5.



Take density as ρ (uniform).

Let C of M of whole lamina be at G with coordinates $(\bar{x}, \bar{y})$.

For OADC mass is $4ab\rho$, with C of M at (a, b).

For ABC area is half base $\times$ height so mass $= \frac{1}{2}a(2b)\rho = ab\rho$. Centre of mass is $\frac{1}{3}$ of way from mid-point of AB to C (point of intersection of medians – covered in lecture). Mid-point of AB is $(\frac{5}{2}a, 0)$, C is $(2a, 2b)$ so C of M is at $(\frac{7}{3}a, \frac{2}{3}b)$.

Hence total mass $M = 5ab\rho$ and coordinates of G are

$$\bar{x} = \frac{4a^2b\rho + \frac{7}{3}a^2b\rho}{M} = \frac{19}{15}a$$

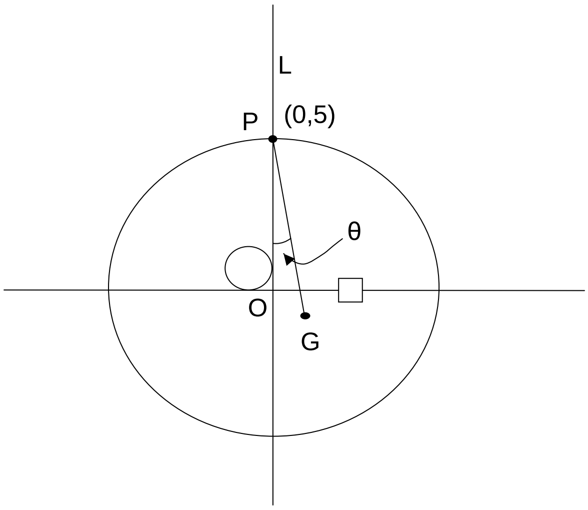
$$\bar{y} = \frac{4ab^2\rho + \frac{2}{3}ab^2\rho}{M} = \frac{14}{15}b$$

Now suspend from B. BG will be vertical. Angle between BO and BG is θ where

$$\tan\theta = \frac{\bar{y}}{3a - \bar{x}} = \frac{14b}{45a - 19a} = \frac{7b}{13a}$$

If $a = 7$ and $b = 13$ then $\tan\theta = 1$ so $\theta = 45°$.

6.



Original disc without holes has mass $25\pi\rho$ and C of M at $(0,0)$.

Square hole has area 1, mass $-\rho$, C of M at $(2,0)$.

Circular hole has area π mass $-\pi\rho$, C of M at $(-1,1)$.

Total mass is $M = (24\pi - 1)\rho$. Coordinates of centre of mass are

$$\bar{x} = \frac{\rho}{M}[25\pi \times 0 - 1 \times 2 - \pi \times (-1)] = \frac{\pi - 2}{24\pi - 1}$$

$$\bar{y} = \frac{\rho}{M}[25\pi \times 0 - 1 \times 0 - \pi \times 1] = \frac{-\pi}{24\pi - 1}$$

If suspended at $P(0,5)$, then PG is vertical. Angle θ is angle original diameter through O makes with vertical where

$$\tan\theta = \frac{\bar{x}}{5 - \bar{y}} = \frac{\pi - 2}{121\pi - 5}$$