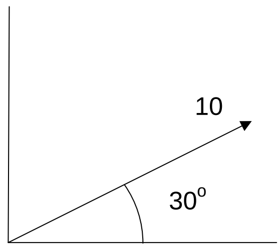
Introduction to Mechanics (0J2)

Example Sheet 4 – solutions

1.



Vertically (upwards): $h = (10\sin 30)t - \frac{1}{2}gt^2$

$\sin 30 = \frac{1}{2}; \quad \cos 30 = \frac{\sqrt{3}}{2}; \quad t = \frac{1}{2}$

$\Rightarrow h = \frac{5}{2} - \frac{9.81}{8} = 1.274$

Horizontally: $x = (10\cos 30)t = \frac{5}{2}\sqrt{3} \;=\; 4.330$

$\therefore h^2 + x^2 = 1.623 + 18.75 \;=\; 20.373$

$\Rightarrow \quad \text{distance} \;=\; \sqrt{20.373} \;=\; 4.514.$

2. $\mathbf{a} = -g\mathbf{j}$ so integrating we get $\mathbf{v} = v_1\mathbf{i} + (v_2 - gt)\mathbf{j}$

where v_1, v_2 are constants of integration.

At $t = 0$, $\quad \mathbf{v} = \mathbf{i} + 2\mathbf{j} \quad \Rightarrow \quad v_1 = 1, v_2 = 2.$

$\therefore \mathbf{v} = \mathbf{i} + (2 - gt)\mathbf{j}$. Now integrate again

$\mathbf{r} = (r_1 + t)\mathbf{i} + (r_2 + 2t - \frac{1}{2}gt^2)\mathbf{j}.$

At $t = 0$, $\quad \mathbf{r} = 0 \quad \Rightarrow r_1 = r_2 = 0.$

Hence $\mathbf{r} = t\mathbf{i} + (2t - \frac{1}{2}gt^2)\mathbf{j}.$

3. Vertically: initial velocity $= U\sin\alpha$ upwards. Using $v^2 = u^2 + 2as$,

at maximum height H we have $\quad 0 = U^2\sin^2\alpha - 2gH.$

$$\therefore H = \frac{U^2\sin^2\alpha}{2g} \quad \text{and} \quad \frac{H}{2} = \frac{U^2\sin^2\alpha}{4g}.$$

Using $s = ut + \frac{1}{2}at^2$ we want to solve

$$\frac{1}{2}H = (U\sin\alpha)t - \frac{1}{2}gt^2$$

$$\therefore \frac{U^2\sin^2\alpha}{4g} = (U\sin\alpha)t - \frac{1}{2}gt^2$$

$$\therefore 2g^2t^2 - 4gtU\sin\alpha + U^2\sin^2\alpha = 0.$$

Solving the quadratic we get

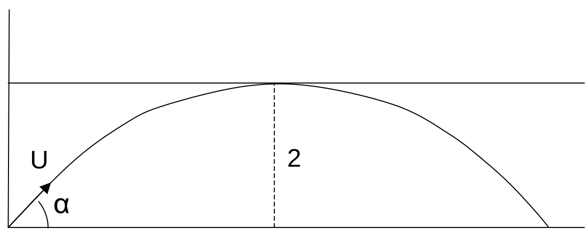
$$t_1, t_2 = \frac{4gU\sin\alpha \pm \sqrt{16g^2U^2\sin^2\alpha - 8g^2U^2\sin^2\alpha}}{4g^2}$$

$$\therefore t_1 = \frac{U\sin\alpha}{g}\left[\frac{4-\sqrt{8}}{4}\right] \text{ and } t_2 = \frac{U\sin\alpha}{g}\left[\frac{4+\sqrt{8}}{4}\right].$$

Thus $t_2 - t_1 = \frac{U\sin\alpha}{g}\sqrt{2}$

Horizontal distance $= (U\cos\alpha)(t_2 - t_1) = \sqrt{2}\frac{U^2}{g}\sin\alpha\cos\alpha.$

4.



Assume it doesn't hit the roof so the maximum distance is when it just grazes the roof.

Maximum height is (see notes or Q3) $\quad \frac{U^2\sin^2\alpha}{2g} \quad = \quad 2.$

$$\therefore \sin^2\alpha = \frac{4g}{U^2}. \quad \therefore \quad \cos\alpha = \frac{U^2 - 4g}{U^2}.$$

Horizontal distance travelled is $(U\cos\alpha)t$.

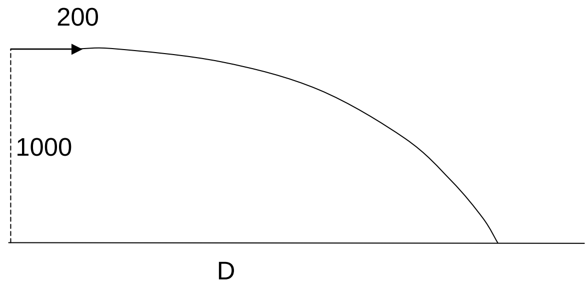
Using $v = u + at$ for the vertical motion the time to maximum height is given by

$$0 = U \sin \alpha - gt \quad \Rightarrow \quad t = \frac{U \sin \alpha}{g}. \qquad \text{Total time is twice this.}$$

Thus distance

$$= \frac{2U^2}{g} \cos \alpha \sin \alpha \quad = \quad \frac{2U^2}{g} \times \frac{2\sqrt{g}}{U} \times \frac{1}{U}\sqrt{U^2 - 4g} \quad = \quad 4\sqrt{\frac{U^2 - 4g}{g}}$$

5.



Vertically, using $s = ut + \frac{1}{2}at^2$, $\quad 1000 = \frac{1}{2}gt^2 = 4.905t^2$.

$$\therefore t^2 \; = \; \frac{1000}{4.905} \; = \; 203.9 \quad \Rightarrow \quad t \; = \; 14.28$$

$$D \quad = 200t \; = \; 2856 \text{ m.}$$

6. $\mathbf{r}_1 = (a \cos \omega t^2 - b, \quad a \sin \omega t^2 - b)$

$\mathbf{v}_1 = (-2\omega t a \sin \omega t^2, \quad 2\omega t a \cos \omega t^2)$

$\mathbf{a}_1 = (-2\omega a \sin \omega t^2 - 4\omega^2 t^2 a \cos \omega t^2, \quad 2\omega a \cos \omega t^2 - 4\omega^2 t^2 a \sin \omega t^2)$

$\therefore |\mathbf{v}_1|^2 = 4\omega^2 t^2 a^2 \sin^2 \omega t^2 + 4\omega^2 t^2 a^2 \cos^2 \omega t^2 \quad = \quad 4\omega^2 t^2 a^2.$

Hence $|\mathbf{v}_1| = 2\omega t a.$

$\mathbf{r}_2 = (At \sin \alpha t, \quad At \cos \alpha t, \quad B \sin \beta t)$

$\mathbf{v}_2 = (A \sin \alpha t + A\alpha t \cos \alpha t, \quad A \cos \alpha t - A\alpha t \sin \alpha t, \quad B\beta \cos \beta t)$

$\mathbf{a}_2 = (2A\alpha \cos \alpha t - A\alpha^2 t \cos \alpha t, \quad -2A\alpha \sin \alpha t - A\alpha^2 t \cos \alpha t, \quad -B\beta^2 \sin \beta t)$

$\therefore |\mathbf{v}_2|^2 = A^2 \sin^2 \alpha t + 2A^2 \alpha t \sin \alpha t \cos \alpha t + A^2 \alpha^2 t^2 \cos^2 \alpha t +$

$A^2 \cos^2 \alpha t - 2A^2 \alpha t \cos \alpha t \sin \alpha t + A^2 \alpha^2 t^2 \sin^2 \alpha t + B^2 \beta^2 \cos^2 \beta t$

$= A^2 + A^2 \alpha^2 t^2 + B^2 \beta^2 \cos^2 \beta t$

Hence $|\mathbf{v}_2| = \sqrt{(1 + \alpha^2 t^2)A^2 + B^2 \beta^2 \cos^2 \beta t}.$

7. (i) $\mathbf{a} = (e^t, \quad \sin t, \quad e^{-t})$. Integrating gives

$$\mathbf{v} \;=\; (v_1 + e^t, \quad v_2 - \cos t, \quad v_3 - e^{-y}).$$

At $t = 0$, $\mathbf{v} = 0$ so constants are $v_1 = -1$, $v_2 = 1$, $v_3 = 1$. Thus

$\mathbf{v} = (e^t - 1, \quad 1 - \cos t, \quad 1 - e^{-t})$. Integrating gives

$$\mathbf{r} = (e^t - t + r_1, \quad t - \sin t + r_2, \quad t + e^{-t} + r_3).$$

At $t = 0$, $\mathbf{r} = 0$ so constants are $r_1 = -1$, $r_2 = 0$, $r_3 = -1$ and so

$$\mathbf{r} = (e^t - t - 1, \quad t - \sin t, \quad t + e^{-t} - 1).$$

(ii) $\frac{1}{2}m\mathbf{v}^2 = |\mathbf{v}|^2 = (e^t - 1)^2 + (1 - \cos t)^2 + (1 - e^{-t})^2$.

$$= e^{2t} - 2e^t + 1 + 1 - 2\cos t + \cos^2 t + 1 - 2e^{-t} + e^{-2t}$$

$$= 2\cosh 2t - 4\cosh t + 3 - 2\cos t + \cos^2 t$$

8. $\mathbf{r} = a\sin pt\,\mathbf{i} + a\cos pt\,\mathbf{j}$

$$\mathbf{v} = pa\cos pt\,\mathbf{i} - pa\sin pt\,\mathbf{j}$$

$$\mathbf{a} = -p^2 a\sin pt\,\mathbf{i} - p^2 a\cos pt\,\mathbf{j}$$

$$\therefore \mathbf{a} = -p^2\mathbf{r}$$

However $\mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2}$ so

$\dfrac{d^2\mathbf{r}}{dt^2} + p^2\mathbf{r} = 0$, which is a constant.

$$\mathbf{r}\cdot\mathbf{v} = pa^2\sin pt\cos pt - a^2 p\sin pt\cos pt = 0$$

so $\mathbf{r}$ is perpendicular to $\mathbf{v}$.

9. $\mathbf{F} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, and $m = 2$ so

$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Integrating: $\mathbf{v} = (2t + v_1)\mathbf{i} + (v_2 - t)\mathbf{j} + (3t + v_3)\mathbf{k}$.

At $t = 0$, $\mathbf{v} = -4\mathbf{i} + \mathbf{j}$ so $v_1 = -4$, $v_2 = 1$, $v_3 = 0$

and $\mathbf{v} = (2t - 4)\mathbf{i} + (1 - t)\mathbf{j} + 3t\mathbf{k}$.

Integrating: $\mathbf{r} = (r_1 + t^2 - 4t)\mathbf{i} + (r_2 + t - \frac{1}{2}t^2)\mathbf{j} + (r_3 + \frac{3}{2}t^2)\mathbf{k}$.

At $t = 0$, $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ so $r_1 = 1$, $r_2 = 2$, $r_3 = -3$

and $\mathbf{r} = (1 - t^2 - 4t)\mathbf{i} + (2 + t - \frac{1}{2}t^2)\mathbf{j} + (\frac{3}{2}t^2 - 3)\mathbf{k}$.

At $t = 4$:

$\mathbf{v} = (8-4)\mathbf{i} + (1-4)\mathbf{j} + 12\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$

$\Rightarrow |\mathbf{v}|^2 = 16 + 9 + 144 = 169 \quad$ so $|\mathbf{v}| = 13$.

$\mathbf{r} = (1 + 16 - 16)\mathbf{i} + (2 + 4 - 8)\mathbf{j} + (3 \times 8 - 3)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 21\mathbf{k}$.

10. Resolving vertically $R = mg$ where R is the normal reaction.

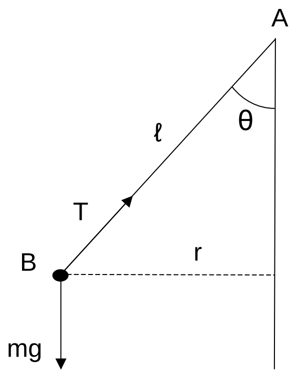
The friction force F is maximum when $F = \mu R$.

Newton's 2nd law: $F = mr\omega^2$ where ω is the angular velocity.

$$\therefore mr\omega^2 = \mu mg$$

$$\therefore \mu = \frac{r\omega^2}{g} = \frac{0.8 \times 9}{9.81} = 0.734.$$

11.



Let the angle be θ and the tension be T.

Resolve vertically: $\quad mg = T\cos\theta$

Horizontal force $= T\sin\theta \quad$ so

Newton's 2nd law: $\quad T\sin\theta = m\omega^2 r$.

Also $\quad r = l\sin\theta$

$\therefore T = m\omega^2 l = 3 \times 64 \times 1.5 = 288$ N.

$$\cos\theta = \frac{mg}{T} = \frac{g}{\omega^2 l} = \frac{9.81}{64 \times 1.5} = 0.1022 \quad \Rightarrow \theta = 84.1°.$$