Integral Quadratic Constraint Theorem: A topological separation approach

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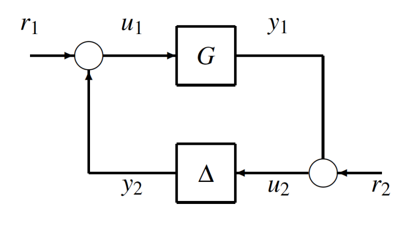
Lurye system(1943)

Lurye Problem

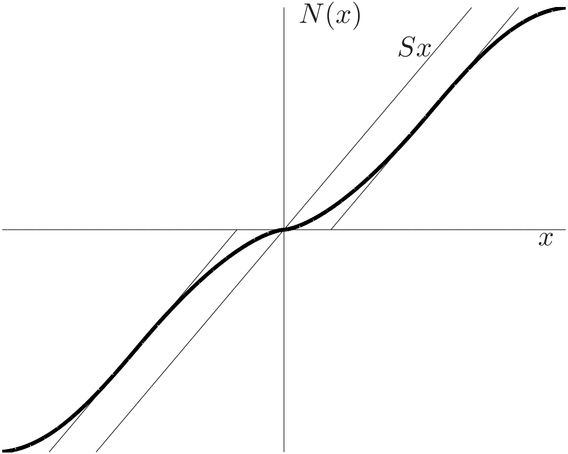
Find condition on G such that this feedback interconnection is stable for all Δ within some class of nonlinearities.



Stability vs absolute stability

Are we worried about stability of two system or stability of a class systems?

Slope-restricted nonlinearities



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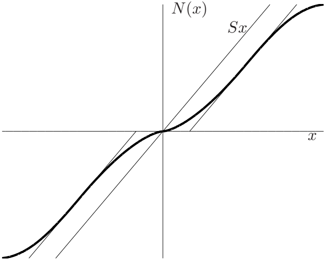
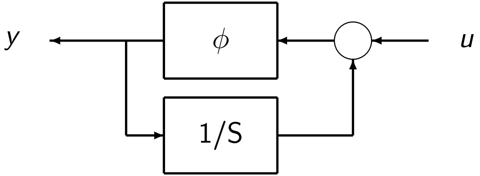


- The memoryless operator ϕ is represented by $N : \mathbb{R} \to \mathbb{R}$
- Slope condition: $0 \leq \frac{dN(x)}{dx} \leq S$
- Let us assume that it is odd, i.e. $N(x) = -N(-x)$

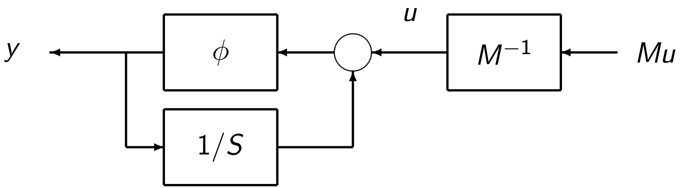
Classical motivation



- $y = N_c(u)$
- Correlation relationship (Falb-Zames, 1967)

$$-\int_{-\infty}^{\infty} u(t)y(t)dt \leq \int_{-\infty}^{\infty} u(t+\tau)y(t)dt \leq \int_{-\infty}^{\infty} u(t)y(t)dt$$

Noncausal multipliers (O'Shea 67)

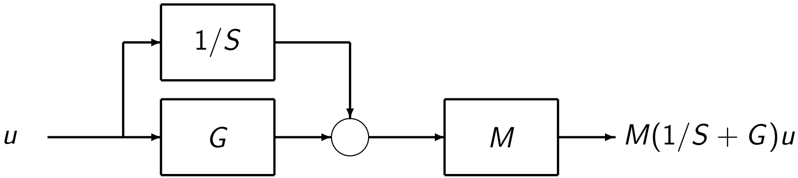


- Take M as a convolution operator with impulse response $m(t) = \delta(t) - h(t)$

$$\int_{-\infty}^{\infty} (Mu)(t)y(t)dt \geq \underbrace{\left(1 - \int_{-\infty}^{\infty} \|h(\tau)\| d\tau\right)}_{>0} \int_{-\infty}^{\infty} u(t)y(t)dt$$

- If we only analyse the nonlinearity, there is no reason to assume that M is causal, i.e. $h(t) = 0$ if $t < 0$.
- Zames and Falb (1968) fix some issues and show the technicalities to use noncausal multipliers: $M = M_{ac}M_c$, where M_{ac}^*, M_{ac}^{-*}, M_c, and M_c^{-1} are stable.

Noncausal multipliers (Zames-Falb, 68)



- Use the factorization to convert the single multiplier M into causal and stable systems.
- The system will be stable if

$$M(j\omega)(1/S + G(j\omega)) > 0$$

for all ω, for some convolution operator M with impulse response $m(t) = \delta(t) - h(t)$ such that

$$\left(1 - \int_{-\infty}^{\infty} \|h(\tau)\| d\tau\right) > 0$$

- For more details: Carrasco, Turner, and Heath (EJC, 2016)

Megretski and Rantzer TAC'97

IQC definition

A bounded system Δ satisfies the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \Delta\hat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \Delta\hat{u}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $u \in \mathcal{L}_2[0, \infty)$

IQC Theorem TAC'97

Let $\Pi \in \mathbf{RL}^{(m+l)\times(m+l)}$, with $\Pi(j\omega) = \Pi^*(j\omega)$. Let $G \in \mathbf{RH}^{m\times l}$ and $\Delta : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^l$ be a bounded and causal system. Assume that

1. The feedback interconnection of G and $\tau\Delta$ is well-posed for all $\tau \in [0,1]$.
2. For all $\tau \in [0,1]$, the system $\tau\Delta$ satisfies that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \widehat{\tau\Delta u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \widehat{\tau\Delta u}(j\omega) \end{bmatrix} d\omega \geq 0$$

   for all $u \in \mathcal{L}_2[0,\infty)$.
3. The system G satisfies that there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < -\epsilon I$$

   for all $\omega \in \mathbb{R}$.

Then the feedback interconnection is $\mathcal{L}_2$-stable.

Unifying framework

Small Gain Theorem

$$\Pi(j\omega) = \begin{bmatrix} 1 & 0 \\ 0 & -1/\|\Delta\|^2 \end{bmatrix}$$

Passivity Theorem

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Sector-restricted nonlinearities (Jösson, 1995)

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 - \nu s \\ 1 + \nu s & 2/K \end{bmatrix}$$

Slope-restricted nonlinearities

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & (M(j\omega) + M^*(j\omega))/S \end{bmatrix}$$

Some claims and main trick

Claim 1

"...the applicability of many of the results has been limited by computational problems and by restrictive causality conditions used in the multiplier theory."

Claim 2

"The purpose of this paper is to address the second obstacle to efficient analysis by proving that multipliers can be introduced in a less restrictive manner, without causality constraints."

Trick

Don't think of Π as an operator, just an algebraic object $\Pi(j\omega)$. Remove this object before using truncations, then use the small gain theorem.

Dancing around the time-domain

Time-domain version

It is commented that a factorization $\Pi(j\omega) = \Phi^*(j\omega)M\Phi(j\omega)$, where Φ is stable, leads to a time-domain version

$$\int_0^T \begin{bmatrix} u(t) \\ \Delta u(t) \end{bmatrix}^\top \Phi^\sim M \Phi \begin{bmatrix} u(t) \\ \Delta u(t) \end{bmatrix} dt \geq 0$$

with the following classification:

- Hard if it holds for any $T > 0$.
- Soft if it just holds for $T = \infty$

Input-Output approach: Frequency/time division

We like time since...

...stability is a time domain concept. What is going to happen if I don't stop the experiment? Time runs in one direction!

We like frequency since...

...it is very powerful representation. Without LMI solver, it helps you to optimise. NB. We have noncausal transfer function but we can't have noncausal state-space representation.

Theories

- Passivity, Dissipativity, or topological separation: Pure time domain.
- IQCs v1.0: mixed approach.
- Multiplier theory: mixed approach.
- IQC v2.0: Pure frequency domain.

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Time interval

Either $\mathcal{L}_2(-\infty, \infty)$ or $\mathcal{L}_{2e}(T_0, \infty)$

Stability, instability, invertibility, and causality by Willems SIAM'69. To guarantee stability, we:

1. Either show that bounded inverse is causal.
2. Or show that causal inverse is bounded.

Stable or causal

$$G(s) = \frac{1}{s-1}$$

ROC is missing in control textbooks since feedback loops require causal systems. Example from Georgiou and Smith (TAC'95).

Missing concept in several textbooks

Vinnicombe is one of the few textbooks with a nice discussion on this subject.

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Passivity, dissipativity, and topological separation

Stability theorems (Time)

1. Passivity (Zames(1966))
2. Dissipativity (Willems (1972))
3. Input/output Dissipativity (Hill and Moylan (1977), Vidyasagar (1977))
4. Topological separation (Safonov (1980), Teel (1996), Georgiou and Smith (1997))

Absolute stability versions (Time/Frequency)

1. Original multiplier (Popov(1961))
2. Multiplier interpretation (Brockett and Willems (JL) (1965))
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IQC v1.0 and IQC v2.0

IQC v1.0 by Yakubovich (Time/Frequency)

One condition in time domain (nonlinear system) and another condition in the frequency domain (linear system). Altshuller et al. (2004) and Altshuller (2011) developed IQC v1.1 (Delay-IQC) which seems the natural extension to include convolution results, i.e., (noncausal) multipliers. The results in this framework rely on the S-procedure.

IQC v2.0 by Megretski & Rantzer, TAC'97 (Frequency)

Both conditions are given in the frequency domain and a homotopy argument is used to show stability. Noncausal multipliers are a natural tool. Mixing nonlinearities and multipliers is then trivial.

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Links between the theories

- Goh and Safonov, CDC'95 and CDC'96, propose a triangular factorization, suggesting the equivalence between the IQC v2.0 and Safonov's topological separation.
- Fu et al. (Automatica, 2005) provide some links between IQC and multiplier approach.
- Seiler et al. (CDC'10) propose to develop a dissipativity inequality using the above triangular factorization. However, the existence of a storage function cannot be ensured.
- Carrasco et al. (Automatica, 2012) provide the equivalence between passivity using multipliers and their associated IQC under a mild assumption on the nonlinearity.
- Veenman and Scherer (CDC'13) propose a solution to the issue in Seiler et al. proposing a loop transformation similar to the canonical factorization.
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Where is the problem with previous attempts?

Goh (CDC'96) factorization

$$\Phi(j\omega) = \begin{bmatrix} R(s) & 0 \\ Q(s) & P(s) \end{bmatrix},$$

where Φ is stable and $R(s)^{-1}$ is also stable.

Good thing

$\Pi(j\omega) = \Phi^*(j\omega) M \Phi(j\omega)$ is a hard-factorization since R^{-1} is stable.

Where does it fail?

The linear condition must be also truncated as the nonlinear condition; however, it is not really appreciated in literature. Hard/soft factorization is quite misleading!

What are we proposing here?

- Recovering Goh and Safonov's interpretation.
- Follow Teel (TAC'96) for the technicalities.
- Use the factorization proposed by Seiler (TAC'15), $\Phi, \Phi^{-1} \in \mathbf{RH}_\infty$ (doublely-hard!).
- Consider positive-negative IQC-multipliers, i.e. if

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix},$$

  then we assume that $\Pi_{11} \geq 0$ (which is required in the IQC v2.0) and $\Pi_{22} \leq 0$.
- Then, the IQC v2.0 can be expressed as a time domain result.
- The condition in the IQC theorem can be expressed as a dissipativity condition.

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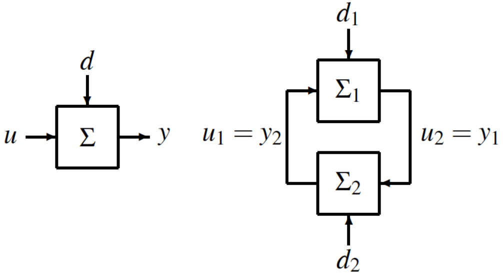
Graphs and distance between Graphs (Teel'96)



$$\mathcal{G}(d) := \{(u, y) \in \mathcal{L}_{2e} \times \mathcal{L}_{2e} : (d, u, y) \in \Sigma\}$$
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$$\delta_T(z) := \begin{cases} \inf\limits_{(d_1, d_2) \in \mathcal{D}(z)} \|(d_1, d_2)_T\| & \text{if } \mathcal{D}(z) \neq \emptyset; \\ \infty & \text{otherwise.} \end{cases}$$

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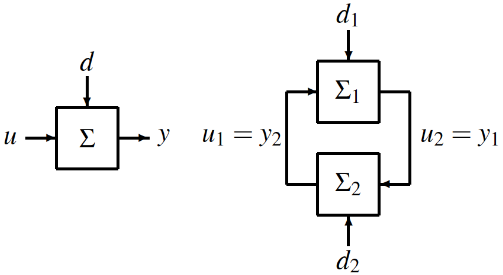

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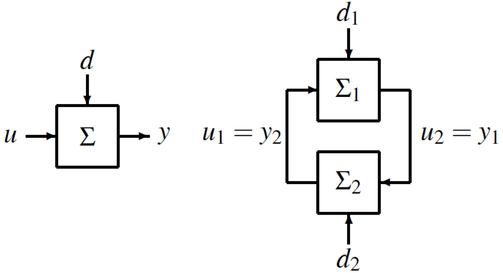

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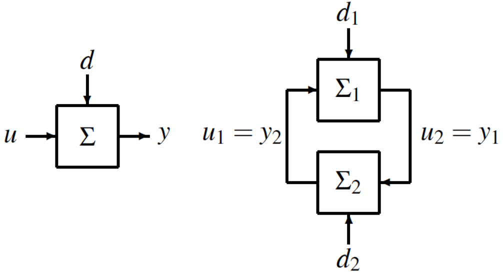

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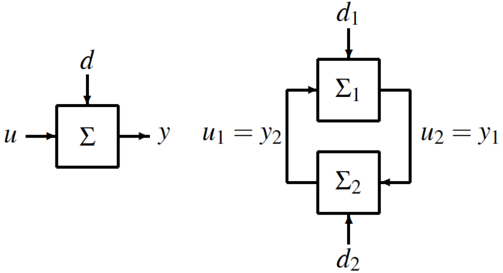

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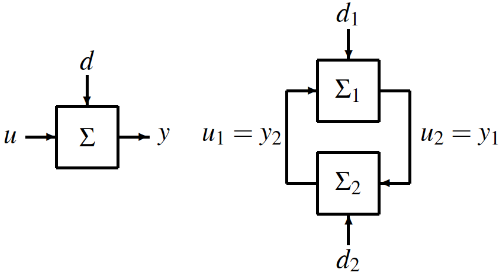

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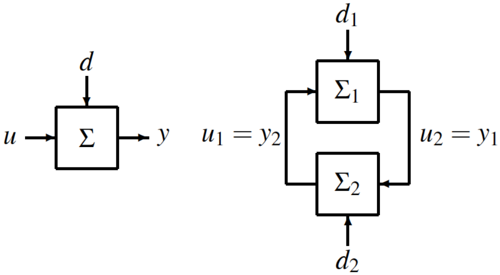

Theorem (Teel'96)

The feedback interconnection is $\mathcal{L}_2$-stable if and only if for each $T > 0$ and output $z \in \mathcal{L}_{2e} \times \mathcal{L}_{2e}$ satisfying $\max\{\|z_T\|, \delta_T(z)\} < \infty$ we have $\|z_T\| \leq C\delta_T(z)$ for some positive constant $C > 0$.

Result involving inner products (Teel'96)

Definition

A graph $\mathcal{G}(0)$ is said to be positive with respect to (Ω_l, Ω_r) if $\langle \Omega_l z_T, \Omega_r z_T \rangle \geq 0$, for all $T > 0$ and $z \in \mathcal{G}(0)$.

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Corollary – "Operator" dissipativity theorem

Suppose that $\mathcal{G}_1^I(0)$ is strictly negative with respect to (Ω_l, Ω_r) and $\mathcal{G}_2(0)$ is positive with respect to (Ω_l, Ω_r). Under these conditions the feedback interconnection in is $\mathcal{L}_2$-stable.

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IQC-like result in the graph framework

Let $\Psi, \Psi^{-1} \in \mathbf{RH}^{(m+l)\times(m+l)}$. Let $G : \mathcal{L}_{2e}^{l} \to \mathcal{L}_{2e}^{m}$ and $\Delta : \mathcal{L}_{2e}^{m} \to \mathcal{L}_{2e}^{l}$ be two causal systems. Assume that

1. The feedback interconnection of G and Δ is well-posed.
2. The system Δ satisfies that

$$\int_0^T \begin{bmatrix} u_T \\ \Delta u_T \end{bmatrix}^\top \Psi^\sim J_{l,m} \Psi \begin{bmatrix} u_T \\ \Delta u_T \end{bmatrix} dt \geq 0$$

   for all $u \in \mathcal{L}_{2e}^{l}$ and $T > 0$ where $J_{n,m} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix}$.

3. The system G satisfies that there exists $\epsilon > 0$ such that

$$\int_0^T \begin{bmatrix} Gu_T \\ u_T \end{bmatrix}^\top \Psi^\sim J_{n,m} \Psi \begin{bmatrix} Gu_T \\ u_T \end{bmatrix} dt \leq -\epsilon \int_0^T \begin{bmatrix} Gu_T \\ u_T \end{bmatrix}^\top \begin{bmatrix} Gu_T \\ u_T \end{bmatrix} dt$$

   for all $u \in \mathcal{L}_{2e}^{m}$ and $T > 0$.

Then the feedback interconnection is $\mathcal{L}_2$-stable.

J-spectral factorization

Seiler, TAC'15

Let $\Pi \in \mathbf{RL}_\infty^{(l+m)\times(l+m)}$ be positive-negative multiplier. Then there exists Ψ such that $\Psi, \Psi^{-1} \in \mathbf{RH}_\infty^{(l+m)\times(l+m)}$ and $\Pi(j\omega) = \Psi^\sim(j\omega) J_{l,m} \Psi(j\omega)$ where $J_{l,m} = \mathrm{diag}(I_l, -I_m)$.

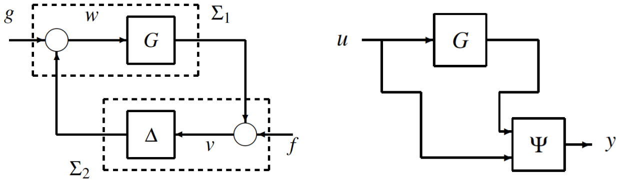
This is a hard factorization

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \hat{\Delta u}(j\omega) \end{bmatrix}^* \Psi^\sim J_{l,m} \Psi \begin{bmatrix} \hat{u}(j\omega) \\ \hat{\Delta u}(j\omega) \end{bmatrix} d\omega \geq 0$$

is equivalent to

$$\int_0^T \begin{bmatrix} u_T \\ \Delta u_T \end{bmatrix}^\top \Psi^\sim J_{l,m} \Psi \begin{bmatrix} u_T \\ \Delta u_T \end{bmatrix} dt \geq 0 \quad \forall T > 0$$

IQC as a dissipativity result



Theorem

Let $G \in \mathbf{RH}^{m \times n}$ and $\Delta : \mathcal{L}_{2e}^{m} \to \mathcal{L}_{2e}^{n}$ a bounded causal operator. Assume that:

1. The interconnection of G and Ψ is well-posed.
2. Δ satisfies the IQC defined by Π.
3. The system $\Psi \begin{bmatrix} G \\ I \end{bmatrix}$ is dissipative with respect to the supply rate $s(u, y) = -y^{\top} J_{l,m} y = y_2 - y_1$.

Then the feedback interconnection is $\mathcal{L}_2$-stable.

Conclusions

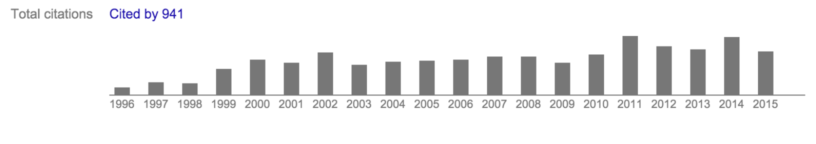
- There are already several connections in the literature between IQC theorem and dissipativity. Here it is a old one (Goh-Safonov) with a new result (Seiler).
- The graph separation is a very general framework, and can cover the IQC theorem if the multiplier is positive-negative.
- It could be claimed that it is quite more general than the IQC framework, as the multiplier can be either linear or nonlinear.
- However, the IQC framework could argue that a nonlinear multiplier makes no sense. The absolute stability framework tries to split the system into linear and nonlinear so the property to check is over the linear part only.

Open questions

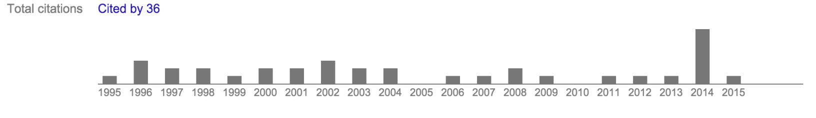
- What if Π is not positive-negative?
- Is hard/soft factorization the right classification?
- Can graph theory be used for synthesis?
- Can we use LTV operators as multipliers?
- Can we analyse open-loop unstable systems?
- Can we analyse local stability as in Georgiou and Smith (TAC'97) with multipliers?
- What about instability?

Citation summary

Megretski and Rantzer (1997): 941



Teel (1996): 36



Many thanks for your attention!

Integral Quadratic Constraint Theorem: A topological separation approach

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14 Jan 2015