State-Space Control

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Contents

Notation

Symbols

Symbol	Meaning
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^n$	Real coordinate space of n dimension
$\mathbb{C}^n$	Complex coordinate space of n dimension
$\mathbb{R}^{n\times m}$	Real matrices with n rows and m columns
$\mathbb{C}^{n\times m}$	Complex matrices with n rows and m columns
$A^\top$	Transpose of A
A^*	Complex conjugate of A
$\dot{x}$	$\frac{dx}{dt}$
$x = (x_1, x_2)$	$x = \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top$
$\mathrm{diag}(d_1, d_2, \ldots, d_n)$	Diagonal matrix with $(d_1, d_2, \ldots, d_n)$ as elements of the diagonal
z^*	Complex conjugate of the complex number z
$\bar{v}$	Vector whose element are the complex conjugate of the elements of v, i.e. $\bar{v}_i = v_i^*$

Acronyms

Acronyms	Meaning
ODE	Ordinary Differential Equation
LTI	Linear Time Invariant

Introduction

This set of notes in State-Space Control summarises and extends undergraduate concepts on this topic. It has been developed following different textbook at different sections. Some of them are:

- "Feedback systems" by Åström and Murray.
- "Feedback Control of Dynamical Systems" by Franklin et al.
- "State Variables for Engineers" by DeRusso et al.
- "A Linear Systems Primer" by Antsaklis and Michel.
- "Linear Systems" by Kailath.

The last but one is recommended for students willing to extend their knowledge in this topic. The last one is the classical reference in this topic but its approach is very mathematical.

I have tried to produce a self-contained set of notes, explaining some of the mathematical concepts that we will use during the course. Nevertheless, the student may need further help in Linear Algebra. Any undergraduate textbook in this topic should be fine, but I would recommend "Introduction to Linear Algebra" by G. Strang, and "Matrix Analysis and Applied Linear Algebra" by C. D. Meyer[1]. For advanced students, the book "Matrix Analysis" by R. A. Horn and C. R. Johnson should provide all required knowledge in this topic, but it is far beyond the scope of this unit.

State-Space Control provides a mathematical framework for studying how to design controllers and observer for dynamical systems. This representation will not be as intuitive as the transfer function representation of the dynamical systems but it will provide other advantages. For example, the state-space representation is not limited to linear systems. We will be required to use several geometrical concepts, which are very powerful but can lead to complex and artificial problems.

Whereas classical control was developed using transfers functions as the representation of the systems; in the 1950s-1960s-1970s, there was a strong development of tools in the state-space. Powered by the development of computer, State-Space Control became very popular. For a while, it seemed that frequency methods were something in the past and Laplace transform suffered a temporally exile from the control realm ("The Laplace transform is dead and

[1] Available at http://www.matrixanalysis.com/DownloadChapters.html

buried", R. E. Kalman, 1959). During the 1970s, frequency Jedi (watch Star Wars!) fought to defend the simplicity and intuition of the Laplace transform. One of these Jedis was Prof. H. H. Rosenbrock, a pioneer in computer aided-control techniques and MIMO frequency methods, who founded the Control Systems Centre at UMIST, nowadays University of Manchester. Currently, sophisticated design methods, such as $\mathbf{H}_\infty$, use advantages of the state-space representation to solve the frequency design problem. In fact, other fields such as operator theory have adopted this representation since it provides valuable advantages over other representations.

This part of the unit has the following structure:

Chapter 1 The basic notions of state-space representations are given. Since the system is represented by a linear application, the representation of a system is associated with the basis used to represent this linear application. There are some forms of the state-space representation that will be very important to design controllers and observers, and they are referred to as canonical forms. We will show how to derive these canonical forms of a system represented by an ODE or a transfer function.

Chapter 2 The solution of the state-space representation is shown as well as an alternative form to express the system: the modal form. It introduces the concept of modes of the systems, which are equivalent to the poles of the system in the frequency framework. Stability then will be analysed using the dynamics of the modes, which are given by the eigenvalues of a matrix. The Laplace transform of the state-space representation provides the transfer function associated with the state-space representation.

Chapter 3 When the system is nonlinear, state-space representation is essential. In this unit, we will study how to analyse nonlinear systems by linearising the system. Two linearisations will be given: around equilibrium points and around operating points.

Chapter 4 Four concepts that are essential for the design techniques in State-Space control are introduced in this chapter: controllability, stabilizability, observability, and detectability. These will be introduced in a mathematical framework, but their meaning will be fully understood in Chapter 5.

Chapter 5 The problem of designing controllers and observers is solved. Concepts introduced in Chapter 4 ensure either a solution for any arbitrary design specification or suitable solutions with some restriction on the design specification.

Chapter 6 The last chapter introduces MIMO systems and their state-space representation. If it is trivial to obtain a state-space representation of a MIMO system, a new concept turns up: minimality. Concepts presented in Chapter 4 are exploited to define the minimality of the system. Gilbert's realization is required to obtain representation without redundant states.

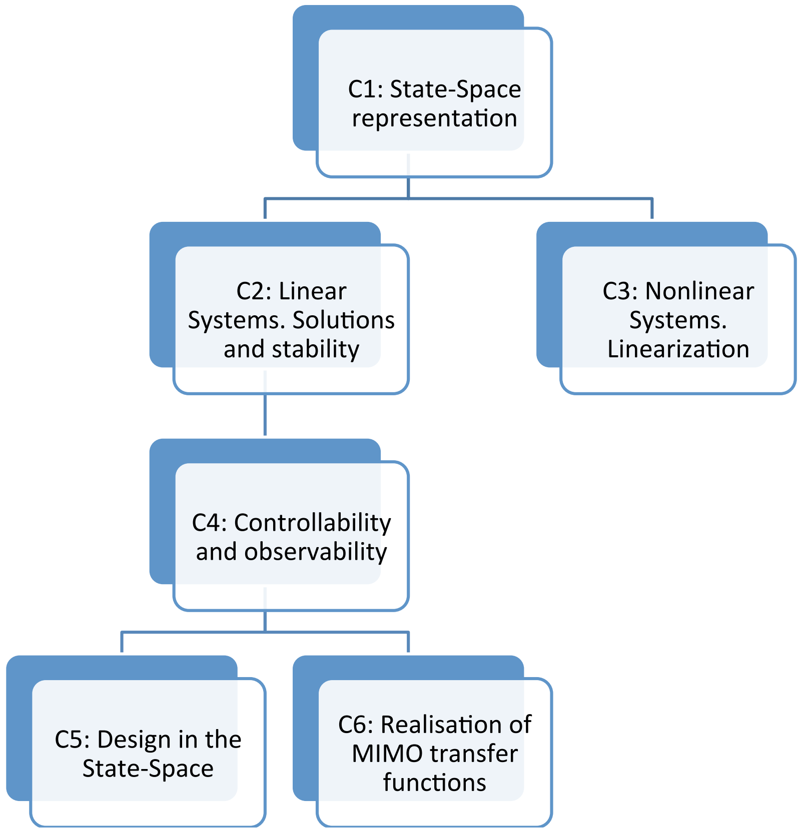


Figure 1: Structure of this set of notes.

Finally, three appendices are given: Appendix A contains tutorials for each chapter; the software lab is given in Appendix B; and the notes conclude with previous exam papers.

Chapter 1

Introduction to state-space representation

1.1 Dynamical systems

1.1.1 Definition of dynamical systems

A dynamical system is a system that changes as time evolves. A system can be a part of the universe, a set of memory bits, etc. A dynamical system consists of two elements:

1. A non-empty space $\mathcal{D}$, e.g. $\mathbb{R}^2$.

2. A map from this space and the time into the same space: $f : \mathcal{D} \times \mathbb{R} \mapsto \mathcal{D}$.

Then, the dynamical system would be described by the differential equation

$$\dot{x}(t) = f(x(t), t). \tag{1.1}$$

Mathematically, it is a geometrical concept. It is not surprising that some of the concepts presented in this unit are also referred to as Geometrical Control by some authors. Loosely speaking, for every point of the space $x \in \mathcal{D}$, the function $f(x, t)$ provides the information about the evolution of the system at the instant t. Given an initial condition, the trajectory of the state follows the *field of velocities* $f = f(x, t)$. When the function f does not depend on the time, i.e. $f = f(x)$, then the system is said to be a *time-invariant system*. Henceforth, we will focus our attention on time-invariant systems.

A useful concept of a dynamical system is the *state* of the system. It can be defined as the minimal information that determines the future of the system. For the dynamical system (1.1), the state of the system is given by $x(t) \in \mathcal{D}$. Hence the state of the system is a point of the space $\mathcal{D}$.

Example 1.1.1. Consider the free fall problem:

$$m\frac{d^2y}{dt^2}(t) = -mg. \tag{1.2}$$

We will see that it can be represented as (1.1) where $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$. You should have studied at some point of you life that the future of the system is determined by its position and speed at some instant. Hence the state of the system $x \in \mathbb{R}^2$.

This unit will be focused on dynamical systems with this property; however, it is not general.

Example 1.1.2. Let us consider a dynamical systems with input delay, then the differential equation is

$$\dot{x}(t) = f(x, t-\tau). \tag{1.3}$$

The state of the system is a function $g : [-\tau, 0) \mapsto \mathcal{D} \times [-\tau, 0]$, i.e. we need the past of the system from the time $[-\tau, 0)$ in order to determine the evolution of the system at the instant 0 .

As the system (1.1) is isolated from the rest of the universe, its evolution only depends on itself, and we say that the system (1.1) is an *autonomous system*.

1.1.2 Autonomous linear system and transformations

During most of the part of this unit, we will focus on linear time invariant dynamical systems. In this case, f is a linear map of the coordinates x, hence $f(x(t)) = Ax(t)$ where A is a square matrix, i.e.

$$\dot{x}(t) = Ax(t). \tag{1.4}$$

In the jargon, such systems are referred to as Linear Time-Invariant (LTI). Loosely speaking, the function f is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$ [1]. A natural question arises: can we express this system using another set of coordinates?

Let us assume that we have two different bases on $\mathbb{R}^n$, x and z [2]. Then there exists a nonsingular square matrix T such that $z = Tx$. The dynamical system (1.4) can also be represented in the basis z as follows

$$\dot{z} = T\dot{x} = TAx = TAT^{-1}z. \tag{1.5}$$

[1]Any linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ is represented by a m-by-n matrix.

[2]We are highly abusive of the notation of the bases.

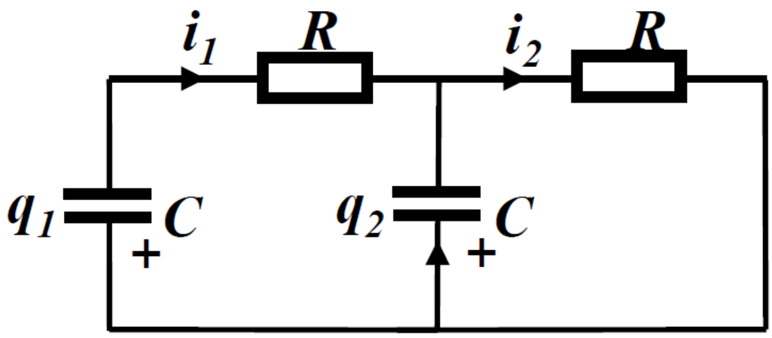


Figure 1.1

As a result, the dynamical system can be represented in any basis of $\mathbb{R}^n$. Considering two bases x and z where $z = Tx$, two different matrices A_x and A_z will describe the same dynamical system, and they are related as $A_z = TA_xT^{-1}$.

1.1.3 Worked example

Let us consider the electrical circuit in Fig. 1.1. The dynamics of the circuit can be described using infinite set of coordinates, but two sets seem straightforward: the charges at the capacitors $q = (q_1, q_2)$ and the mesh current $i = (i_1, i_2)$. In this example, we are going to model the same circuit using both sets of coordinates and check the theoretical result that we have obtained in the above section.

Using q Applying Kirchoff's Voltage Law (see Wikipedia for more details) on the left mesh

$$\sum_i V_i^{\text{left}} = \frac{1}{C}q_1 + i_1R - \frac{1}{C}q_2 = 0, \qquad i_1 = -\frac{1}{CR}q_1 + \frac{1}{CR}q_2 \tag{1.6}$$

and using KVL on the right mesh

$$\sum_i V_i^{\text{right}} = i_2R + \frac{1}{C}q_2 = 0, \qquad i_2 = -\frac{1}{CR}q_2 \tag{1.7}$$

Moreover, both charges and currents are related as follows

$$\dot{q}_1 = i_1 = -\frac{1}{CR}q_1 + \frac{1}{CR}q_2, \tag{1.8}$$

$$\dot{q}_2 = i_2 - i_1 = -\frac{1}{CR}q_2 + \frac{1}{CR}q_1 - \frac{1}{CR}q_2. \tag{1.9}$$

or equivalently

$$\dot{q} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{2}{RC} \end{bmatrix} q \tag{1.10}$$

Using i The time-derivatives of (1.7) and(1.6) are given by

$$\dot{i}_2 R + \frac{1}{C}\dot{q}_2 = 0, \qquad \dot{q}_2 = -RC\dot{i}_2; \tag{1.11}$$

and

$$\frac{1}{C}\dot{q}_1 + \dot{i}_1 R - \frac{1}{C}\dot{q}_2 = 0, \qquad \dot{q}_1 = -RC\dot{i}_1 - RC\dot{i}_2. \tag{1.12}$$

The dynamical equations in the capacitors can be written as

$$\begin{aligned} i_1 &= \dot{q}_1 = -RC\dot{i}_1 - RC\dot{i}_2, && (1.13)\\ i_2 - i_1 &= \dot{q}_2 = -\frac{1}{CR}q_2 + \frac{1}{CR}q_1 - \frac{1}{CR}q_2. && (1.14) \end{aligned}$$

Reordering the above equation, the desired result is reached

$$\dot{i} = \begin{bmatrix} -\frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} i \tag{1.15}$$

Transformation From the KVL, we can deduce the transformation between charges and currents

$$\begin{aligned} i_1 &= -\frac{1}{CR}q_1 + \frac{1}{CR}q_2 && (1.16)\\ i_2 &= -\frac{1}{CR}q_2 && (1.17) \end{aligned}$$

or

$$i = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ 0 & -\frac{1}{RC} \end{bmatrix} q \tag{1.18}$$

Therefore, applying the transformation result to the system (1.10), we should recover (1.15):

$$\dot{i} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{2}{RC} \end{bmatrix} \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ 0 & -\frac{1}{RC} \end{bmatrix}^{-1} i \tag{1.19}$$

Trick. The inverse of a 2-by-2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1.20}$$

■

$$\begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{2}{RC} \end{bmatrix} \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ 0 & -\frac{1}{RC} \end{bmatrix}^{-1} =$$

$$\frac{1}{RC} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} =$$

$$\frac{1}{RC} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \frac{1}{(-1)(-1) - 0} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} =$$

$$\frac{1}{RC} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}. \quad (1.21)$$

Hence, performing the basis transformation we obtain the same result as in (1.15). This example has demonstrated the relationship between two representations of the same system.

1.2 System modelling

The key point in control engineering and system theory is interaction. We are interested in studying the dynamical evolution of interconnected systems. In particular, feedback systems will attract much of our attention. Therefore, we would like to model our system as a dynamical system including explicitly input u and output y:

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^{n_x},\ u \in \mathbb{R}^{n_u}; \quad (1.22)$$

$$y = h(x, u) \qquad y \in \mathbb{R}^{n_y} \quad (1.23)$$

where n_x is the number of state coordinates, n_u is the number of inputs, and n_y is the number of outputs. This representation of a system is very general and most real systems can be modelled by (1.22) and (1.23). These equations are referred to as *the system equation* and *the output equation*, respectively.

In contrast with the transfer function representation of a system, the state-space representation is not limited to linear systems and it can also cover time-varying systems if $f = f(x, u, t)$ and $h = h(x, u, t)$.

Nevertheless, this first encounter with state-space representation will be focused on either linear systems or how to approximate a nonlinear system with a linear system, i.e. linearisation.

1.2.1 State-space representation of a linear system

The general definition of a dynamical system can be used to describe the behaviour of a linear system as follows:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^{n_x},\ u \in \mathbb{R}^{n_u}; \tag{1.24}$$

$$y = Cx + Du \qquad y \in \mathbb{R}^{n_y} \tag{1.25}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, and $D \in \mathbb{R}^{n_y \times n_u}$. Equations (1.24) and (1.25) are said to be the *state-space representation* of a linear system. In short, we will say that the four matrices (A,B,C,D) represent a time-invariant linear (LTI) system.

For systems with single input and output, i.e. $n_u = n_y = 1$, B is column vector, C is a row vector and D is a number. These systems are referred to as Single-Input Single-Output (SISO). Systems with a single input but several outputs, i.e. $n_u = 1$ $n_y > 1$, are referred to as Single-Input Multiple-Output (SIMO). Systems with several inputs but a single output, i.e. $n_y = 1$ $n_u > 1$, are referred to as Multiple-Input Single-Output (MISO). Finally, systems with several inputs and several outputs, i.e. $n_u > 1$ $n_y > 1$, are referred to as Multiple-Input Multiple-Output (MIMO).

Restricting our attention to SISO systems, we have the following result:

Result 1.2.1. *Any ordinary differential equation in the form*

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u \tag{1.26}$$

with $m \leq n$ has an equivalent state-space representation. ■

Worked example: Ideal Mass-spring-damper system

Let us consider an ideal mass-spring-damper system where a external force F is applied on the mass (See Fig. 1.2). The output of the system is the position of the mass y. Applying Newton's second law, the dynamics of the system is given by

$$\sum_i F_i = ma = m\ddot{y},$$

There are three forces in the direction of y: the spring force $(-ky)$, the damper force $(-\beta y)$, and the external force F. It follows

$$F + (-ky) + (-\beta\dot{y}) = m\ddot{y}, \tag{1.27}$$

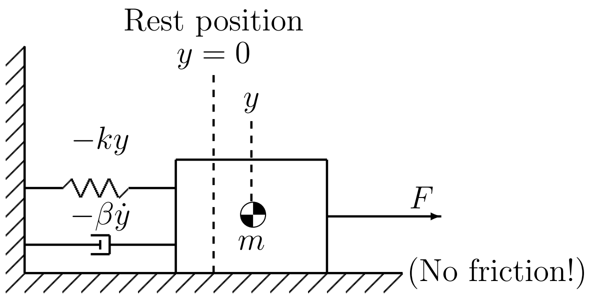


Figure 1.2: Ideal Mass-Spring-Damper system

equivalently

$$\ddot{y} + \frac{\beta}{m}\dot{y} + \frac{k}{m}y = \frac{F}{m}. \tag{1.28}$$

Let us define the set of states as

$$x_1 = y, \tag{1.29}$$

$$x_2 = \dot{y}; \tag{1.30}$$

then we can find a state-space representation of this system as follows. From the definition of both coordinates, it is trivial that $\dot{x}_1 = x_2$; then (1.28) can be rewritten in terms of x_1, x_2, and $\dot{x}_2$

$$\dot{x}_2 + \frac{\beta}{m}x_2 + \frac{k}{m}x_1 = \frac{F}{m}. \tag{1.31}$$

As a result, the system is described by two first order simultaneous differential equation

$$\dot{x}_1 = 0x_1 + x_2 + 0F, \tag{1.32}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{\beta}{m}x_2 + \frac{1}{m}F. \tag{1.33}$$

Now we rewrite these two equation using matrices and the state $x = (x_1, x_2)$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F. \tag{1.34}$$

Using (1.29)-(1.30), the output equation is given by

$$y = x_1 + 0x_2 + 0F = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0F \tag{1.35}$$

In summary, the state-space representation of an ideal mass-spring-damper system is given by

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0. \tag{1.36}$$

It is worth highlighting that the state-space is a mathematical description of a system that could be different to the real space where the system performs its trajectory. In this example, the system moves along the y-axis, which is one-dimensional space. However, the state of the system, as a mathematical concept, evolves on a two-dimensional space.

1.2.2 Transformation of state-space representation

In the same manner as we can use different bases to express autonomous systems, we can also use basis transformations with input-output systems.

Let us consider the dynamical system given by

$$\dot{x} = A_x x + B_x u \tag{1.37}$$

$$y = C_x x + D_x u \tag{1.38}$$

and the new set of coordinates $z = Tx$, where $T \in \mathbb{R}^{n_x \times n_x}$ is a nonsingular matrix. Then, we can express the above dynamical system in the basis z as follows

$$\dot{z} = T\dot{x} = TA_x x + TB_x u = TA_x T^{-1} z + TB_x u \tag{1.39}$$

$$y = C_x x + D_x u = C_x T^{-1} z + D_x u \tag{1.40}$$

As a result, the state-space representation has been transformed as $A_z = TA_xT^{-1}$, $B_z = TB_x$, $C_z = C_xT^{-1}$, and $D_z = D_x$.

1.3 Canonical forms

As we have stated in Result 1.2.1, there is a space-space representation for all ODEs. Usually, we prefer to express a differential equation in the Laplace domain, and we speak about transfer functions. In conclusion, any transfer function can be represented by an infinite number of state-space representations. Among all possible state-space representations of a system, three are very important: controller canonical form, observer canonical form, and modal form (Chapter 2).

Before starting the different representations of a transfer function, we must comment that the term D is independent of the state-space representation, as we have shown in Section 1.2.2. Therefore, we will focus our attention on system where $D = 0$. For instance, any transfer function can be decomposed as

$$G(s) = \bar{G}(s) + G(\infty) \tag{1.41}$$

where $\bar{G}$ is a strictly proper system, i.e. $\bar{G}(\infty) = 0$. For any representation of $G(s)$, the matrix D will be given by $D = G(\infty)$.

1.3.1 Controller canonical form

Definition

Consider the system described by

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}u(t)}{dt^{n-2}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t), \tag{1.42}$$

or equivalently,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}; \tag{1.43}$$

where $b_i \neq 0$ for at least one $1 \leq i < n$.

Then its controller canonical form is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{1.44}$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} x(t). \tag{1.45}$$

Other versions of this form can be found in the literature by renaming the state in opposite order. In the following, we show its development.

Development: Simplest case

First, let us consider the case where the ODE does not contain derivatives of the input

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y = u, \tag{1.46}$$

or equivalently, a transfer function with no bounded zeros

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{1.47}$$

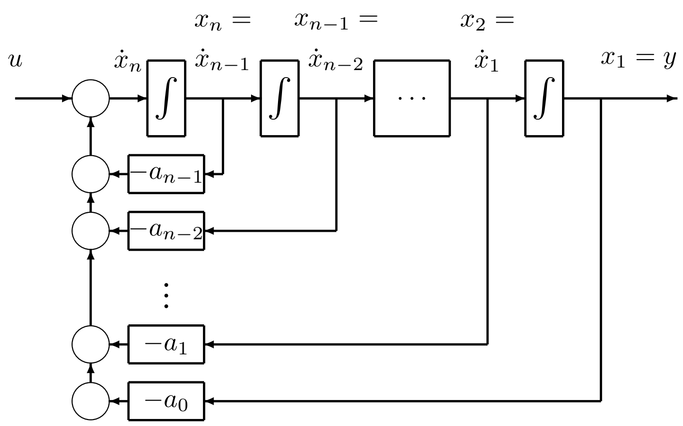


Figure 1.3: Block diagram representation of the control canonical form.

Let us define the set of state coordinates as

$$x_1(t) = y(t), \tag{1.48}$$
$$x_2(t) = \frac{dy(t)}{dt} = \dot{x}_1(t), \tag{1.49}$$
$$\vdots \tag{1.50}$$
$$x_n(t) = \frac{d^{n-1}y(t)}{dt^{n-1}} = \dot{x}_{n-1}(t). \tag{1.51}$$

Then, substituting the above states in (1.46), it follows

$$\dot{x}_n(t) + a_{n-1}x_n(t) + \cdots + a_1x_2(t) + a_0x_1(t) = u(t), \tag{1.52}$$

where the derivative of the last state is

$$\dot{x}_n(t) = -a_{n-1}x_n(t) - \cdots - a_1x_2(t) - a_0x_1(t) + u(t). \tag{1.53}$$

In summary, the controller canonical form of (1.46) or (1.47) is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{1.54}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} x(t) \tag{1.55}$$

Since integration blocks are standard in electrical circuits, so we can think of this procedure as the practical implementation of an ODE or transfer function. Therefore, this procedure is classically referred to as **realisation**.

Remark 1.3.1. Different authors use different structures for the definition of controller canonical form, so the interested student should go to different textbooks and understand that different structures have a common point, only one state depends directly on the input and the rest of the states only depend directly on one state.

Development: General case

Once we have developed this particular case, we have got the tools to deal with the most general case. Let us consider the differential equation given by

$$\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_1\frac{dy(t)}{dt}+a_0y(t) = b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}}+b_{n-2}\frac{d^{n-2}u(t)}{dt^{n-2}}+\cdots+b_1\frac{du(t)}{dt}+b_0u(t), \tag{1.56}$$

or equivalently, a transfer function with no bounded zeros:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}; \tag{1.57}$$

where $b_i \neq 0$ for at least one $1 \leq i < n$.

Let us consider an intermediate variable ξ such that

$$\frac{d^n\xi(t)}{dt^n} + a_{n-1}\frac{d^{n-1}\xi(t)}{dt^{n-1}} + \cdots + a_1\frac{d\xi(t)}{dt} + a_0\xi(t) = u(t). \tag{1.58}$$

By linearity, it can be shown that

$$y(t) = b_{n-1}\frac{d^{n-1}\xi(t)}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}\xi(t)}{dt^{n-2}} + \cdots + b_1\frac{d\xi(t)}{dt} + b_0\xi(t), \tag{1.59}$$

however, this linearity argument may not be straightforward for the student. It can be easy to understand the following procedure is the frequency domain. Let us introduce an intermediate variable Ξ (Laplace transform of ξ) as follows

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}\frac{\Xi(s)}{\Xi(s)}, \tag{1.60}$$

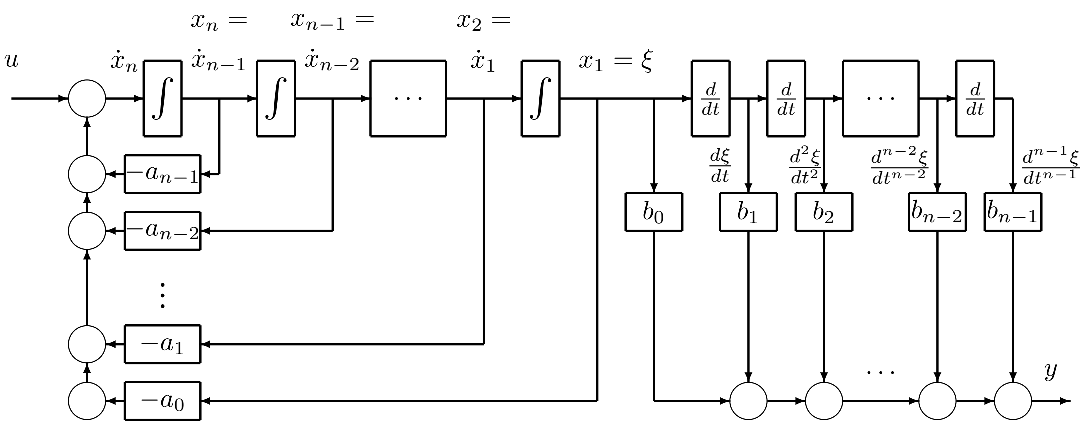
hence we can rewrite the system between y and u as a system between u and ξ followed by a sequence of derivatives

$$\frac{\Xi(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{1.61}$$

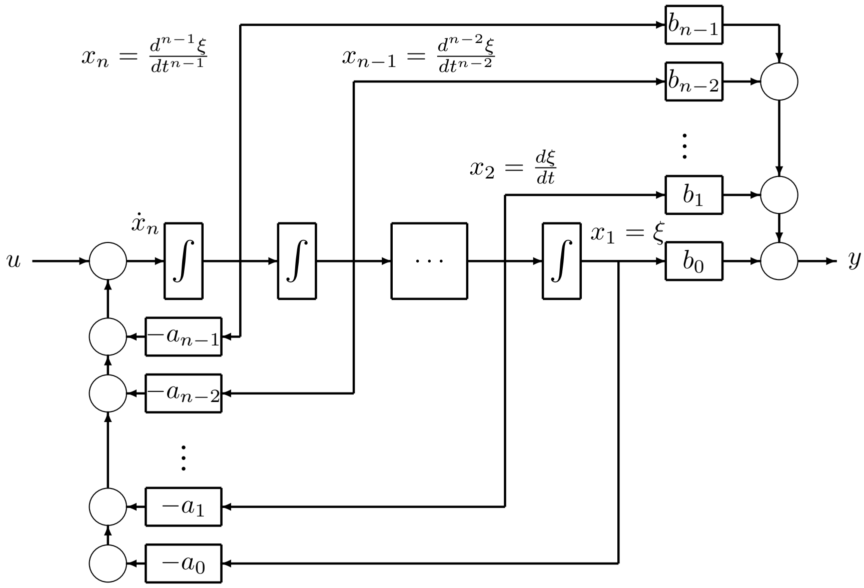
$$Y(s) = (b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0)\Xi(s). \tag{1.62}$$

Now, it is clear that (1.61) and (1.62) are the equivalent of (1.58) and (1.59) in the Laplace domain.

Therefore, the realization of the general case can be reduced to the previous case by using the intermediate variable ξ. One could think that the brute force implementation of (1.62) is



(a) Unrealisable control canonical form.



(b) Realisable control canonical form.

Figure 1.4: Block diagram of control canonical form

indeed infeasible (see Fig. 1.4a), and hence the system (1.56) unrealizable. Nonetheless, $\xi = x_1$ and $x_2 = \dot{x}_1$, then $\dot{\xi} = x_2$, and so on; thus the realization of (1.56) is carried out without the implementation of (1.62) (see Fig. 1.4b).

As a result, the control canonical form of the ODE (1.56) or transfer function (1.57) is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{1.63}$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} x(t). \tag{1.64}$$

Worked example

Let us consider the dynamical system given by the following ODE

$$\frac{d^3y(t)}{dt^3} + 9\frac{d^2y(t)}{dt^2} + 26\frac{dy(t)}{dt} + 24y(t) = 3\frac{d^2u(t)}{dt^2} + 6\frac{du(t)}{dt} + 4u(t), \tag{1.65}$$

or equivalently, by the following transfer function

$$G(s) = \frac{3s^2 + 6s + 4}{s^3 + 9s^2 + 26s + 24}. \tag{1.66}$$

Let us consider the system with input u and output ξ defined by

$$\frac{d^3\xi(t)}{dt^3} + 9\frac{d^2\xi(t)}{dt^2} + 26\frac{d\xi(t)}{dt} + 24\xi(t) = u(t), \tag{1.67}$$

Now, let us defined state coordinates as follows:

$$x_1(t) = \xi(t), \tag{1.68}$$

$$x_2(t) = \frac{d\xi(t)}{dt} = \dot{x}_1(t), \tag{1.69}$$

$$x_3(t) = \frac{d^2\xi(t)}{dt^2} = \dot{x}_2(t); \tag{1.70}$$

and replace these states in (1.67)

$$\dot{x}_3(t)+9x_3(t)+26x_2(t)+24x_1(t) = u(t) \text{ and so } \dot{x}_3(t) = -24x_1(t)-26x_2(t)-9x_3(t)+u(t) \tag{1.71}$$

As a result, the time derivative of the vector state x is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \tag{1.72}$$

By linearity arguments[3], it can be shown that the output y is given as a linear combination of $\xi = x_1$, $\frac{d\xi(t)}{dt} = x_2$, and $\frac{d^2\xi(t)}{dt^2} = x_3$

$$y(t) = 3\frac{d^2\xi(t)}{dt^2} + 6\frac{d\xi(t)}{dt} + 4\xi(t) \text{ and so } y(t) = 4x_1(t) + 6x_2(t) + 3x_3(t) \tag{1.73}$$

or, in vector form,

$$y(t) = \begin{bmatrix} 4 & 6 & 3 \end{bmatrix} x(t) \tag{1.74}$$

[3] If you are still not sure of how this argument works, use the same argument as used in (1.60)-(1.61)-(1.62)

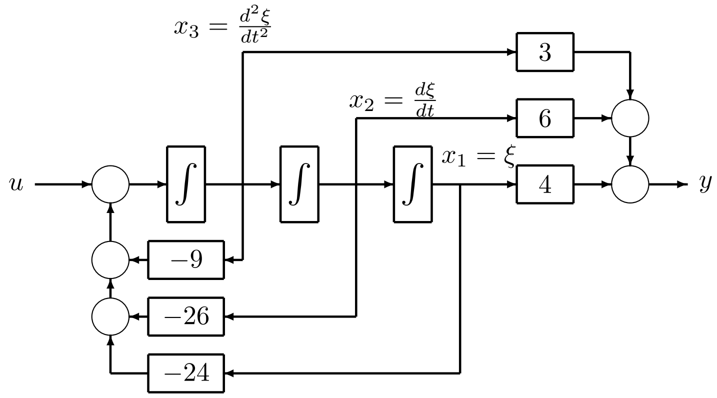


Figure 1.5: Block diagram representation of the control canonical form for the ODE (1.65) or transfer function (1.66).

Exercise 1.3.2. Given the state-space realization defined by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 4 & 6 & 3 \end{bmatrix}, \text{ and } D = 0; \tag{1.75}$$

use the MATLAB commands `ss` and `tf` to find the transfer function of this realisation. Check whether the result corresponds with (1.66). Develop Simulink model using: (a) state-space representation block, (b) integrators as in Fig. 1.5, and (c) interpreted MATLAB function to define the state derivative followed by an integrator.

1.3.2 Observer canonical form

Once again, two different cases could be developed. However, since the general case is simpler than in the previous case, we will develop the Observer canonical form for the general case, without any intermediate case. To simplify the notation, we will omit expression of the dependence on t.

Definition

Consider again the system described by

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}u(t)}{dt^{n-2}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t), \tag{1.76}$$

or equivalently,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}; \tag{1.77}$$

where $b_i \neq 0$ for at least one $1 \leq i < n$.

Then the observer canonical form is given by

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u \tag{1.78}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} x. \tag{1.79}$$

Development

We are going to define a more sophisticated set of state coordinates for the ODE

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_1\frac{du}{dt} + b_0 u. \tag{1.80}$$

Let us reorder the above equation in a fancy way. All the terms with time-derivatives on the left-hand side, and the rest on the right-hand side, so it follows

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} - b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} - b_{n-2}\frac{d^{n-2}u(t)}{dt^{n-2}} - \cdots - b_1\frac{du(t)}{dt} = b_0 u(t) - a_0 y. \tag{1.81}$$

As every term on the left-hand side has a time-derivative, then one time-derivative can be taken as a common factor, so

$$\frac{d}{dt}\left(\frac{d^{n-1}y}{dt^{n-1}} + a_{n-1}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_1 y - b_{n-1}\frac{d^{n-2}u}{dt^{n-2}} - b_{n-2}\frac{d^{n-3}u}{dt^{n-3}} - \cdots - b_1 u\right) = b_0 u(t) - a_0 y. \tag{1.82}$$

The first state has magically turned up! We will consider our first state as everything inside of the brackets on the left-hand side, i.e.

$$x_1 = \frac{d^{n-1}y}{dt^{n-1}} + a_{n-1}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_1 y - b_{n-1}\frac{d^{n-2}u}{dt^{n-2}} - b_{n-2}\frac{d^{n-3}u}{dt^{n-3}} - \cdots - b_1 u, \tag{1.83}$$

and

$$\dot{x}_1 = -a_0 y + b_0 u. \tag{1.84}$$

Let us reorder (1.83) following the same procedure, but now terms with time-derivative will stay on the right-hand side of the equation, whereas terms without derivatives will be moved to the left-hand side, and so

$$x_1 - a_1 y + b_1 u = \frac{d^{n-1}y}{dt^{n-1}} + a_{n-1}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_2\frac{dy}{dt} - b_{n-1}\frac{d^{n-2}u(t)}{dt^{n-2}} - b_{n-2}\frac{d^{n-3}u}{dt^{n-3}} - \cdots - b_2\frac{du}{dt}, \tag{1.85}$$

Once again, our fancy re-aggregation of term allows us to take a common derivative on the right-hand side, so

$$x_1 - a_1 y + b_1 u = \frac{d}{dt}\left(\frac{d^{n-2}y}{dt^{n-2}} + a_{n-1}\frac{d^{n-3}y}{dt^{n-3}} + \cdots + a_2 y - b_{n-1}\frac{d^{n-3}u}{dt^{n-3}} - b_{n-2}\frac{d^{n-4}u}{dt^{n-4}} - \cdots - b_2 u\right), \tag{1.86}$$

and the second state is born as everything inside of the bracket on the left-hand side, i.e.

$$x_2 = \frac{d^{n-2}y}{dt^{n-2}} + a_{n-1}\frac{d^{n-3}y}{dt^{n-3}} + \cdots + a_2 y - b_{n-1}\frac{d^{n-3}u}{dt^{n-3}} - b_{n-2}\frac{d^{n-4}u}{dt^{n-4}} - \cdots - b_2 u, \tag{1.87}$$

and

$$\dot{x}_2 = x_1 - a_1 y + b_1 u. \tag{1.88}$$

So forth and so on, we will obtain the state x_3, x_4,..., x_{n-2} and

$$x_{n-1} = \frac{dy}{dt} + a_{n-1}y - b_{n-1}u, \tag{1.89}$$

which can be reordered in the same manner, i.e

$$x_{n-1} - a_{n-1}y + b_{n-1}u = \frac{dy}{dt}, \tag{1.90}$$

an equation that gives birth to our last state

$$x_n = y, \tag{1.91}$$

and

$$\dot{x}_n = x_{n-1} - a_{n-1}y + b_{n-1}u. \tag{1.92}$$

During this procedure, some students would have been very worried, since one could think that time-derivative of the states can only depend on states and input. However, (1.91) will have calmed these students since it allows us to rewrite the time-derivative of the states as follows

$$\begin{aligned}
\dot{x}_1 &= -a_0 x_n + b_0 u, & (1.93)\\
\dot{x}_2 &= x_1 - a_1 x_n + b_1 u, & (1.94)\\
&\vdots & (1.95)\\
\dot{x}_{n-1} &= x_{n-2} - a_{n-2}x_n + b_{n-2}u. & (1.96)\\
\dot{x}_n &= x_{n-1} - a_{n-1}x_n + b_{n-1}u; & (1.97)
\end{aligned}$$

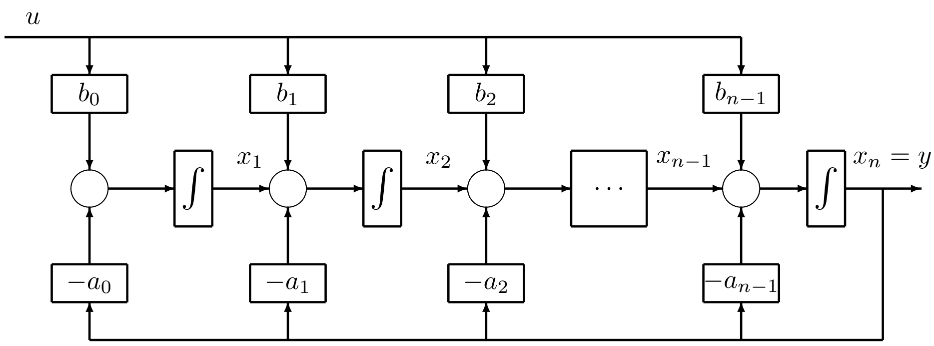


Figure 1.6: Observer canonical form of ODE (1.76) or transfer function (1.77) with the new set of coordinates.

or equivalently

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} \xi \tag{1.98}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} x. \tag{1.99}$$

This form is known as the observer canonical realization of of ODE (1.76) or transfer function (1.77). See Fig. 1.6 for a block diagram.

Worked example

Let us consider the dynamical systems given by the following ODE

$$\frac{d^3y}{dt^3} + 9\frac{d^2y}{dt^2} + 26\frac{dy}{dt} + 24y = 3\frac{d^2u}{dt^2} + 6\frac{du}{dt} + 4u, \tag{1.100}$$

or equivalently, by the following transfer function

$$G(s) = \frac{3s^2 + 6s + 4}{s^3 + 9s^2 + 26s + 24}. \tag{1.101}$$

We are going to use this example to uncover our magic trick. Let us rewrite (1.100) as follows

$$\frac{d}{dt}\left(\ddot{y} + 9\dot{y} + 26y - 3\dot{u} - 6u\right) = -24y + 4u, \tag{1.102}$$

then we can choose

$$x_1 = \ddot{y} + 9\dot{y} + 26y - 3\dot{u} - 6u, \tag{1.103}$$

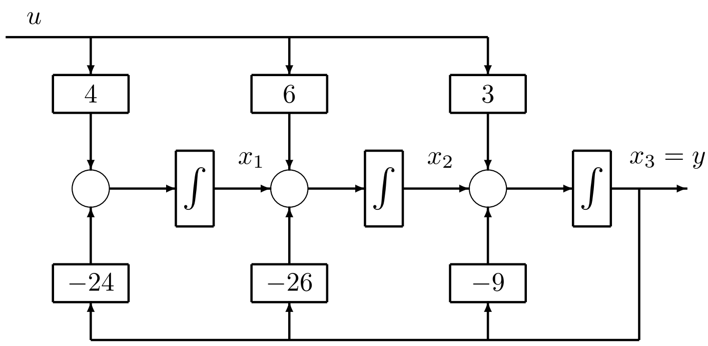


Figure 1.7: Observer canonical form of ODE (1.100) or transfer function (1.101) with the new set of coordinates.

where it is clear that $\dot{x}_1 = -24y + 4u$. Once again, let us rewrite (1.103) as follows

$$\frac{d}{dt}\left(\dot{y} + 9y - 3u\right) = x_1 - 26y + 6u, \tag{1.104}$$

then we can choose

$$x_2(t) = \dot{y} + 9y - 3u, \tag{1.105}$$

then $\dot{x}_2 = -26y + x_1 + 6u$. And, finally,

$$\frac{d}{dt}\left(y\right) = x_2 - 9y + 3u, \tag{1.106}$$

then $x_3 = y$ and $\dot{x}_3 = -9y + x_2 + 3u$.

As a result,the final expression to the obsever canonical form is

$$\dot{x} = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix} x + \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} u \tag{1.107}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \tag{1.108}$$

Exercise 1.3.3. Given the state-space realization defined by the matrices

$$A = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ \text{and } D = 0; \tag{1.109}$$

use the MATLAB commands `ss` and `tf` to find the transfer function of this realisation. Check whether the result corresponds with (1.101). Develop Simulink model using: (a) state-space representation block, (b) integrators as in Fig. 1.7, and (c) interpreted MATLAB function to define the state derivative followed by an integrator.

1.4 Learning Outcomes

The learning outcomes of this chapter can be summarised as follows:

- The state-space representation is a very powerful method to model any system, linear or nonlinear.
- When the system is linear and time invariant (LTI), then the representation of the system is given by four matrices A, B, C and D with

$$\dot{x} = Ax + Bu, \tag{1.110}$$

$$y = Cx + Du. \tag{1.111}$$

- Any transfer function or ODE has a state-space representation.
- The state of the system is the minimal information that allows us to determine the future of the system.
- The evolution of the system is represented by the trajectory of the state in the state-space.
- There are three important forms of representing an LTI system: control canonical form, observer canonical form and modal form (Chapter 2).
- In the control canonical form, the input only directly affects one state and the output is a linear combination of the state coordinates.
- In the observer canonical form, the output is one of the state coordinates and the input may directly affect the dynamic of all states.

Chapter 2

Solutions in the state-space

This chapter is devoted to finding the solution of a dynamical system in the state-space representation. Two methods are presented: firstly, we will solve the ODE in the time-domain and; secondly, we will do so in the frequency-domain.

2.1 Modal form

2.1.1 Definition of the modal form

We have skipped this form in the previous chapter since this form is related with the solution of the system. The modal form of a system is obtained when the matrix A is represented by its diagonal form.

Definition 2.1.1. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be diagonalizable if there exist a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that

$$\Lambda = V^{-1}AV. \tag{2.1}$$

The diagonal elements of Λ, λ_i, are called eigenvalues of the matrix A and they satisfy $\det(A - \lambda_i I) = 0$ for $i = 1, 2, \ldots, n$. The column vectors of V are the eigenvectors of the matrix A.

There are matrices that cannot be diagonalised, but we are not going to consider these details. Interested students can check any Linear Algebra textbook, e.g. Chapter 7 in Meyer's book available online (http://www.matrixanalysis.com/DownloadChapters.html). We will restrict our attention to matrices with different eigenvalues, i.e., $\lambda_i = \lambda_j$ if and only if $i = j$; then we can ensure that there is a diagonal form on the matrix.

Result 2.1.2. *If $A \in \mathbb{R}^{n\times n}$ has n different eigenvalues, then A is diagonalizable.*

Let us consider the system with a state-space representation given by

$$\dot{x} = Ax + Bu, \tag{2.2}$$

$$y = Cx + Du. \tag{2.3}$$

where $A \in \mathbb{R}^{n\times n}$ has n different eigenvalues. Then, A is diagonalizable, so let us find V such that $\Lambda = V^{-1}AV$, with Λ diagonal. Applying the change of variable $x = Vq$, we obtain

$$\dot{q} = \Lambda q + V^{-1}Bu, \tag{2.4}$$

$$y = CVq + Du. \tag{2.5}$$

The coordinates q are referred to as *system modes* and the matrix Λ as the *modal matrix*. Note that this expresion is equivalent to the previous chapter expresion for change of variable if we use $T = V^{-1}$.

Whereas x are assumed to be real, we cannot longer assume that q will be real as the eigenvalues can be complex. Some authors propose a linear combination of the modes q to recover realness. In this case Λ is no longer diagonal.

2.1.2 Worked example: Modal form of a system

Find the modal form of the system with the following state representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \tag{2.6}$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.7}$$

The first step is to find the eigenvalues of the matrix A, i.e., find λ such that $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -2-\lambda & 1 \\ 2 & -3-\lambda \end{vmatrix} = (-2-\lambda)(-3-\lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) \tag{2.8}$$

hence the eigenvalues of A are given by the roots of the polynomial $(\lambda+4)(\lambda+1)$, i.e. -1 and -4. The second step is to find the eigenvectors. The eigenvector associated with the eigenvalue $\lambda_1 = -1$ is given by any of the infinite solutions of the simultaneous equation $Ax = \lambda_1 x$ or $(A - \lambda_1 I)x = 0$, i.e.

$$(A - (-1)I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.9}$$

hence

$$\begin{aligned} -x + y &= 0, && (2.10) \\ 2x - 2y &= 0. && (2.11) \end{aligned}$$

It is clear that both equations provide the same information; thus the system of simultaneous equations will have infinite solutions. In this case, any vector where $x = y$ will be an eigenvalue of λ_1, for instance

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.12}$$

Following the same procedure for $\lambda = -4$

$$A - (-4)I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.13}$$

Students with basic knowledge in Linear Algebra will not be surprised by the fact that there are infinite solutions again. The definition of eigenvector ensures that there exist infinite eigenvectors. In this case

$$\begin{aligned} 2x + y &= 0, && (2.14) \\ 2x + y &= 0; && (2.15) \end{aligned}$$

resulting in that any vector such that $y = -2x$ is a eigenvector of λ_2. For instance, let us take

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \tag{2.16}$$

As a result, we have obtained that the transformation matrix V is given by

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \implies V^{-1} = \frac{1}{-2-1} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.17}$$

hence

$$\Lambda = V^{-1}AV = \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 2 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \tag{2.18}$$

So we have found a matrix V such that $\Lambda = V^{-1}AV$ is diagonal. In the jargon, A it is said to be similar to Λ. As V is not unique, the modal form is not unique, but Λ, i.e. the A matrix of the state-space representation, will be unique up to ordering of its diagonal elements.

In summary, the modal form is given by applying the transformation $q = V^{-1}x$

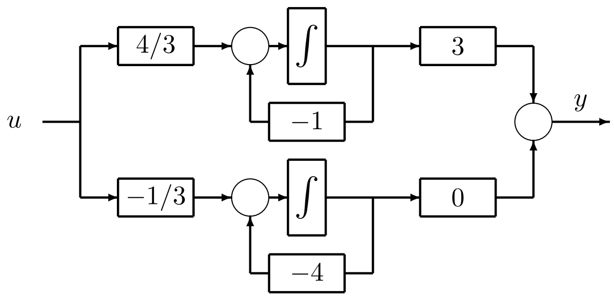


Figure 2.1: Block diagram of the modal form.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{4}{3} \\ \frac{-1}{3} \end{bmatrix} u \tag{2.19}$$

$$y = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2.20}$$

and it block diagram is given in Fig. 2.1. Note that each mode evolves regardless of the rest of modes.

Exercise 2.1.3. Use the command `ss2ss` to find the transformation of the systems defined by $T = V^{-1}$. The result should correspond with the system given in (2.19) and (2.20).

Exercise 2.1.4. Use the command `canon` to find the modal form of the system. Does this modal form correspond with the modal form in (2.19) and (2.20)? If not, why?

2.2 Solution of a state-space representation

2.2.1 Exponential matrix

Before dealing with the solution of a system, we need to introduce the concept of the exponential of a matrix.

Definition 2.2.1. Given a square matrix $A \in \mathbb{R}^{n\times n}$, we define the exponential of the matrix A, henceforth, e^A as follows

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \tag{2.21}$$

The above definition is the straightforward generalization of the exponential function from numbers to matrices. Note that students should be able to carry out all operations on the right-hand side, but computing this infinite sum could be somehow tedious. We will be particularly interested in the exponential matrix e^{At}, since it turns up in the solution of LTI systems.

The exponential matrix has some properties:

$$e^{0} = I \tag{2.22}$$

$$e^{(a+b)A} = e^{aA}e^{bA} \tag{2.23}$$

$$e^{A}e^{-A} = I \tag{2.24}$$

$$e^{\Lambda} = \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \quad \text{where} \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{2.25}$$

$$\frac{d}{dt}e^{At} = Ae^{At} \tag{2.26}$$

Exercise 2.2.2. Using Definition 2.2.1, show that $\frac{d}{dt}e^{At} = Ae^{At}$.

Result 2.2.3. *Given a square matrix $A \in \mathbb{C}^{n\times n}$, with $\Lambda = V^{-1}AV$ where Λ is diagonal, then*

$$e^{A} = Ve^{\Lambda}V^{-1}. \tag{2.27}$$

Proof. Using the definition of the exponential matrix

$$e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!}A^{k} \tag{2.28}$$

Let us pre-multiply by V^{-1} and post-multiply by V

$$V^{-1}e^{A}V = \sum_{k=0}^{\infty} \frac{1}{k!}(V^{-1}A^{k}V) = \sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^{k} = e^{\Lambda}, \tag{2.29}$$

hence

$$e^{A} = Ve^{\Lambda}V^{-1}. \tag{2.30}$$

■

Exercise 2.2.4. During the above proof, we have used that $\Lambda^{k} = VA^{k}V^{-1}$ for any $k \in \mathbb{N}$. Prove by induction this result. Hint: show that it is true for $k = 1$, then assume that it is true for $k = n - 1$, and show that it is true for $k = n$.

2.2.2 Worked Example: Computing exponential matrix e^{At}

The computation of exponential matrix using (2.21) is difficult. The use of Result 2.2.3 and property 2.25 is simpler. Let us consider

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}. \tag{2.31}$$

Then, we have show that $\Lambda = V^{-1}AV$ with $\Lambda = \text{diag}(-1, -4)$ and

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } V^{-1} = \frac{-1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{2.32}$$

Then, it is trivial to check that $\Lambda t = V^{-1}(At)V$. Therefore, the application of Result 2.2.3 yields

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \frac{-1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} =$$
$$\frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix}. \quad (2.33)$$

Exercise 2.2.5. Using symbolic variable (`syms`) and exponential matrix(`expm`) in MATLAB, reproduce the above result.

2.2.3 Autonomous case

Let us start with the autonomous case.

Theorem 2.2.6. *The matrix differential equation*

$$\dot{x} = Ax, \quad x(0) = x_0; \quad (2.34)$$

has a unique solution given by $x(t) = e^{At}x_0$ *for all* $t > 0$.

Proof. Let us assume that the solution of the system is $x(t) = e^{At}x_0$. Using (2.22), it is clear that $x(0) = e^0 x_0 = x_0$. Moreover, using (2.26)

$$\dot{x}(t) = \frac{d(e^{At})}{dt}x_0 = Ae^{At}x_0 = Ax(t) \quad (2.35)$$

Therefore, the proposed solution $x(t) = e^{At}x_0$ satisfies both condition in (2.34). Theorems about existence and uniqueness of the solution of a ODE gives the desired result. Interested students can go to any textbook on differential equations to find such a theorem. ■

Definition 2.2.7 (The state-transition matrix)**.** From the solution of the matrix ODE (2.34), the matrix e^{At} is referred to as the state-transition matrix. From any instant t_0 up to the instant $t_0 + t$, the states are related by

$$x(t) = e^{A(t-t_0)}x(t_0). \quad (2.36)$$

Worked example: Solution in the state-space

Let us consider the dynamical system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (2.37)$$

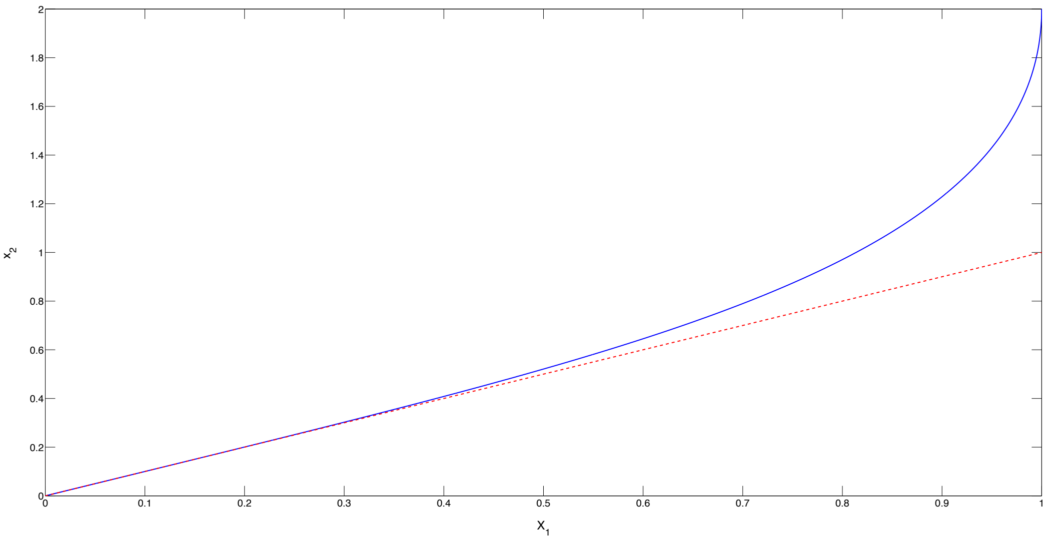


Figure 2.2: In solid blue, first 10 seconds of the trajectory of the autonomous system (2.37). In dashed red, direction of the "slower" mode. The initial value is $(1, 2)$ and the system evolves towards the point $(0, 0)$. From all the possible directions to approach zero, the trajectory chooses the direction given by the eigenvector associated with the eigenvalue whose real part is greater.

Then, the solution of the system is given by $x(t) = e^{At}x(0)$. As we have previously computed the exponential matrix, then the solution of the trajectory of the state is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{-4t} \\ 4e^{-t} + 2e^{-4t} \end{bmatrix} \tag{2.38}$$

Exercise 2.2.8. Plot Fig. 2.2 using MATLAB and `expm`.

Exercise 2.2.9. Plot Fig. 2.2 using Simulink and State-Space block.

2.2.4 Stability of an LTI system

The previous example has shown that the solution of the dynamical system (2.34) is given by a linear combination of e^{λ_i}, where λ_i are the eigenvalues of A. In short, the eigenvalues of the matrix A contains a very important piece of information. Since the representation of the dynamical system is not unique, we should expect that all the possible realisations of the system provide the same information.

Result 2.2.10. *If A is similar to B, i.e. there exists an nonsingular matrix V such that $A = V^{-1}BV$, then the eigenvalues of A and B are the same.*

The above result ensures that the dynamical behaviour defined by the eigenvalues of the matrix A will remain unmodified if we carry out a transformation, as we should expect.

A general definition of stability may be complicated, but let us restrict our attention to linear systems in this chapter. Then we can propose the following definition:

Definition 2.2.11 (Stability)**.** An autonomous system $\dot{x} = Ax$ is said to be asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all initial conditions $x_0 \in \mathbb{R}^n$. It is said to be marginally stable if for any initial condition $x_0 \in \mathbb{R}^n$ such that $\|x_0\| < \delta$, there exists $M \in \mathbb{R}$ such that $\|x(t)\| < M$ for all $t > 0$, but $\lim_{t\to\infty} x(t) \neq 0$. Finally, if the system is neither asymptotically stable nor marginal stable, it said to be unstable.

General definitions for nonlinear systems will be covered in Chapter 3. For further discussion, read Chapter 4 in "Nonlinear Systems" by H. K. Khalil, but it is beyond the scope of this unit.

Result 2.2.12. *An autonomous system $\dot{x} = Ax$ is asymptotically stable if all the eigenvalues of A have strictly negative real part. The system is marginally stable if it has one or more distinct poles on the imaginary axis, and any remaining poles have negative real part. Finally, the system is unstable either if any pole has a positive real part, or any repeated poles on the imaginary axis.*

Stability is a very important concept, so we have a special name for the matrices with this property in their eigenvalues.

Definition 2.2.13 (Hurwitz)**.** A square matrix is said to be Hurwitz if all its eigenvalues have strictly negative real part.

We are not going to develop a formal proof, but the argument is along these lines: if all eigenvalues are strictly negative, then

$$\lim_{t\to\infty} e^{At} = 0, \tag{2.39}$$

since all elements of this matrix are linear combination of $e^{\lambda_i t}$ for all $i = 1, \ldots, n$ and these exponentials approach zero as t goes to infinity. Therefore

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At} x_0 = 0, \tag{2.40}$$

Exercise 2.2.14. If the eigenvalues provide information about the stability of the system, other information is provided by the eigenvectors (See Fig. 2.2). When t approaches infinity, then $x(t)$ approaches the direction of the eigenvalue with maximum real part. Show this property. Hint: Transform the dynamical system and the initial conditions to the modal form and evolve the state.

Non-autonomous case

Once we have addressed the autonomous case, we can propose the solution for the non-autonomous case. By analogy, the solution of a non-homogeneous ODE is given by the addition of the solution of the homogeneous case, i.e. the autonomous case, plus the particular solution.

Theorem 2.2.15 (General Case)**.** *Consider the matrix differential equation*

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{2.41}$$

For $t > 0$ the state $x(t)$ of the above system is given by

$$x(t) = e^{tA}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.42}$$

Proof. We can write (2.41) as

$$\dot{x}(t) - Ax(t) = Bu(t). \tag{2.43}$$

Multiplying by e^{-tA} yields

$$e^{-tA}[\dot{x}(t) - Ax(t)] = e^{-tA}Bu(t) \tag{2.44}$$

which is equivalent to

$$\frac{d}{dt}[e^{-tA}x(t)] = e^{-tA}Bu(t). \tag{2.45}$$

Finally, integrating over $[0, t]$ it follows

$$e^{-tA}x(t) = e^{-0A}x_0 + \int_0^t e^{-\tau A}Bu(t)d\tau. \tag{2.46}$$

Hence, multiplying by e^{tA} the desired result is obtained. ■

Once the evolution of the state is known, the output of the system can be trivially computed as

$$y(t) = Cx(t) + Du(t) \tag{2.47}$$

for any $t > 0$.

When the system has inputs and outputs, we need to define stability properties carefully. Nevertheless, for linear systems, both definitions, autonomous stability and input-output stability, are equivalent. Therefore, we will say that the system 2.41 is stable if the autonomous system associated with the system, i.e. $u(t) = 0$ for all $t \in \mathbb{R}$, is stable.

2.3 Solution using Laplace transform

2.3.1 Laplace transform

In the previous chapter, we discussed the state-space representation of systems which are defined via ODE or transfer functions. Now we are going to derive the solution of the differential equation via Laplace transform. Instead of using transfer functions with high order, we are going to use a first order transfer function but with matrices instead of numbers.

Definition 2.3.1. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, its Laplace transform is defined by

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt. \tag{2.48}$$

Moreover, we say that $f(t)$ is the inverse Laplace transform of $F(s)$.

See your notes on Control Fundamentals by W. P. Heath for further discussion.

We are not going to deal with the properties of this transformation here, but students should be familiar with them. For instance:

$$L[\dot{f}(t)] = sF(s) - f(0^-) \tag{2.49}$$

$$L[\ddot{f}(t)] = s^2F(s) - sf(0^-) - f(0^-) \tag{2.50}$$

$$L[f(t - t_0)] = e^{-st_0}F(s) \tag{2.51}$$

$$L[f(at)] = \frac{1}{a}F(s/a) \quad \forall a > 0 \tag{2.52}$$

$$L[f(t) * g(t)] = F(s)G(s) \tag{2.53}$$

All these properties make the Laplace transform a very useful tool for solving ODEs. In particular, the last item is what control engineers use every day.

Result 2.3.2. *Let $h(t)$ be the impulse response of an LTI system with zero initial conditions. Then the output of the system can be computed as*

$$y(t) = u(t) * h(t) = \int_0^t u(\tau)h(t - \tau)d\tau. \tag{2.54}$$

By using (2.53)*, the above integral becomes a product*

$$Y(s) = L[y(t)] = L[u(t)]L[h(t)] = H(s)U(s)$$

. ■

For the rest of the world, the Laplace transform is "just" a way for solving a differential equation. Our world is transformed by Laplace!

2.3.2 Solution of the system: The autonomous case

Given the system defined by

$$\dot{x} = Ax, \quad x(0) = x_0; \tag{2.55}$$

the solution of the system can be computed using the Laplace domain

$$sX(s) - x_0 = AX(s). \tag{2.56}$$

Then the solution in the frequency-domain of the system (2.55) is

$$X(s) = (sI - A)^{-1}x_0. \tag{2.57}$$

But we must go back to the time domain. To this end, let us expand the above expression as follows

$$(sI - A)^{-1} = s^{-1}(I - A/s)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}(A/s)^k = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \dots + \frac{A^{n-1}}{s^n} + \dots \tag{2.58}$$

Then, we need to use the inverse laplace transformation tables to find that

$$L[t^n] = \frac{n!}{s^{n+1}} \tag{2.59}$$

which is equivalent to

$$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}. \tag{2.60}$$

Moreover, if we are considering a matrix form, then

$$L^{-1}\left[\frac{X}{s^n}\right] = \frac{Xt^{n-1}}{(n-1)!}. \tag{2.61}$$

where X is a matrix.

Now we can translate the frequency-domain expression (2.57) into the time-domain

$$x(t) = L^{-1}[X(s)] = \left(L^{-1}\left[\frac{I}{s}\right] + L^{-1}\left[\frac{A}{s^2}\right] + L^{-1}\left[\frac{A^2}{s^3}\right] + \dots + L^{-1}\left[\frac{A^{n-1}}{s^n}\right] + \dots\right)x_0. \tag{2.62}$$

And using (2.61), then the final result is achived

$$x(t) = \left(I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^{n-1}t^{n-1}}{(n-1)!} + \dots\right)x_0 = e^{At}x_0. \tag{2.63}$$

As expected, the result is the same as developed in previous section. But this offers an alternative method of computing the solution. Moreover, let us state the above result properly

Result 2.3.3. *Let A be a square matrix, the inverse Laplace transform of $(sI - A)^{-1}$ is given by e^{At}, in short*

$$L^{-1}[(sI - A)^{-1}] = e^{At} \tag{2.64}$$

■

Worked example: Solution in the state-space

Let us consider the dynamical system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.65}$$

Then let us use (2.56)

$$X(s) = \begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{2.66}$$

hence a new posiblity of solving the problem turns up. Let us invert the above matrix and then carry out the inverse Laplace transformation.

$$\begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+5s+4} \begin{bmatrix} s+3 & 1 \\ 2 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{2}{(s+1)(s+4)} & \frac{s+2}{(s+1)(s+4)} \end{bmatrix}. \tag{2.67}$$

As an intermediate step, we need to apply the partial fraction decomposition to these fractions. After simple algebra, it follows that

$$\frac{s+3}{(s+1)(s+4)} = \frac{a_1}{(s+1)} + \frac{b_1}{(s+4)} = \frac{2/3}{(s+1)} + \frac{1/3}{(s+4)}; \tag{2.68}$$

$$\frac{1}{(s+1)(s+4)} = \frac{a_2}{(s+1)} + \frac{b_2}{(s+4)} = \frac{1/3}{(s+1)} + \frac{-1/3}{(s+4)}; \tag{2.69}$$

$$\frac{2}{(s+1)(s+4)} = \frac{a_3}{(s+1)} + \frac{b_3}{(s+4)} = \frac{2/3}{(s+1)} + \frac{-2/3}{(s+4)}; \tag{2.70}$$

$$\frac{s+2}{(s+1)(s+4)} = \frac{a_4}{(s+1)} + \frac{b_4}{(s+4)} = \frac{1/3}{(s+1)} + \frac{2/3}{(s+4)}. \tag{2.71}$$

Then,

$$L^{-1}[(sI-A)^{-1}] = \begin{bmatrix} L^{-1}\left[\frac{s+3}{(s+1)(s+4)}\right] & L^{-1}\left[\frac{1}{(s+1)(s+4)}\right] \\ L^{-1}\left[\frac{2}{(s+1)(s+4)}\right] & L^{-1}\left[\frac{s+2}{(s+1)(s+4)}\right] \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t}+e^{-4t} & e^{-t}-e^{-4t} \\ 2e^{-t}-2e^{-4t} & e^{-t}+2e^{-4t} \end{bmatrix}. \tag{2.72}$$

As a result, we have been able to compute the exponential matrix as

$$e^{At} = L^{-1}[(sI-A)^{-1}] = \frac{1}{3} \begin{bmatrix} 2e^{-t}+e^{-4t} & e^{-t}-e^{-4t} \\ 2e^{-t}-2e^{-4t} & e^{-t}+2e^{-4t} \end{bmatrix}. \tag{2.73}$$

So the solution of the system is

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t}+e^{-4t} & e^{-t}-e^{-4t} \\ 2e^{-t}-2e^{-4t} & e^{-t}+2e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4e^{-t}-e^{-4t} \\ 4e^{-t}+2e^{-4t} \end{bmatrix} \tag{2.74}$$

2.3.3 Transfer function of a state-space representation

Let us consider the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{2.75}$$

$$y(t) = Cx(t) + Du(t). \tag{2.76}$$

Firstly, let us apply the Laplace transform to the state equation as previously

$$sX(s) - x_0 = AX(s) + BU(s), \tag{2.77}$$

grouping terms

$$(sI - A)X(s) = BU(s) + x_0, \tag{2.78}$$

hence the state trajectory in the frequency-domain is

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x_0. \tag{2.79}$$

We need to keep in mind that we are working with matrices so the order is important when we move terms from one side to the other.

Exercise 2.3.4. Use (2.79) to obtain (2.42). Hint: You will need to use Result 2.3.3 and property (2.53).

Secondly, let us apply the Laplace tranform to the output equation and use the above result

$$Y(s) = CX(s) + DU(s) = C((sI - A)^{-1}BU(s) + (sI - A)^{-1}x_0) + DU(s), \tag{2.80}$$

thus it follows

$$Y(s) = (C(sI - A)^{-1}B + D)U(s) + C(sI - A)^{-1}x_0. \tag{2.81}$$

Finally, we need to consider that the transfer function of a system assumes null initial conditions, and thus we reach the desired result

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D. \tag{2.82}$$

Worked Example: Transfer function of a state-space representation

Let us find the transfer function of the system given by the state-space representation

$$A = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix}, \; B = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \text{ and } D = 0. \tag{2.83}$$

The first step is to compute the inverse of $(sI - A)$.

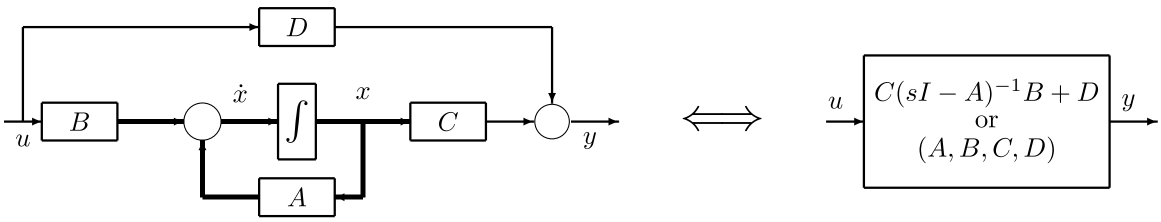


Figure 2.3: Block diagram with matrices and its transfer function representation.

Result 2.3.5. *The inverse of a matrix of any order can be computed as the adjoint over the determinant, i.e.*

$$A^{-1} = \frac{\text{Adj}(A)}{\det(A)} \tag{2.84}$$

where the adjoint of a matrix is the matrix of cofactors of the transpose matrix. Finally, the cofactor of the element (i, j) is the minor of A without the row i and the column j times $(-1)^{i+j}$.

■

Thus, we need to write first the transpose of the matrix that we want to invert

$$(sI - A)^\top = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix} \tag{2.85}$$

Let us compute some cofactors of the transpose matrix.

Element $(1, 1)$ **of the adjoint of** $(sI - A)$ It is defined by the minor of $(sI - A)^\top$ without fisrt column and row times $(-1)^{1+1}$, i.e.

$$\text{Adj}(sI - A)_{1,1} = (-1)^2 \begin{vmatrix} s & -1 \\ 26 & s+9 \end{vmatrix} = s^2 + 9s + 26. \tag{2.86}$$

Element $(2, 1)$ **of the adjoint of** $(sI - A)$ It is defined by the minor of $(sI - A)^\top$ without fisrt column and second row times $(-1)^{2+1}$, i.e.

$$\text{Adj}(sI - A)_{2,1} = (-1)^3 \begin{vmatrix} -1 & 0 \\ 26 & s+9 \end{vmatrix} = s + 9. \tag{2.87}$$

Element $(3, 1)$ **of the adjoint of** $(sI - A)$ It is defined by the minor of $(sI - A)^\top$ without fisrt column and third row times $(-1)^{3+1}$, i.e.

$$\text{Adj}(sI - A)_{3,1} = (-1)^4 \begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} = 1 \tag{2.88}$$

Following this procedure

$$\text{Adj}(sI - A) = \begin{bmatrix} 9s + s^2 + 26 & -24 & -24s \\ 9 + s & 9s + s^2 & -26s - 24 \\ 1 & s & s^2 \end{bmatrix} \tag{2.89}$$

and applying the above trick

$$\begin{bmatrix} s & 0 & -24 \\ -1 & s & 26 \\ 0 & -1 & s+9 \end{bmatrix}^{-1} = \frac{1}{s^3 + 9s^2 + 26s + 24} \begin{bmatrix} 9s + s^2 + 26 & -24 & -24s \\ 9 + s & 9s + s^2 & -26s - 24 \\ 1 & s & s^2 \end{bmatrix}, \tag{2.90}$$

hence we have all elements to carry out the matrix product $G(s) = C(sI - A)^{-1}B$

$$G(s) = \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \frac{1}{s^3 + 9s^2 + 26s + 24} \begin{bmatrix} 9s + s^2 + 26 & -24 & -24s \\ 9 + s & 9s + s^2 & -26s - 24 \\ 1 & s & s^2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} =$$

$$\frac{1}{s^3 + 9s^2 + 26s + 24} \begin{bmatrix} 1 & s & s^2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} = \frac{3s^2 + 6s + 4}{s^3 + 9s^2 + 26s + 24} \tag{2.91}$$

2.3.4 Inverse “à la Rosenbrock”

In his book, H. H. Rosenbrock provides an interesting way of computing the most common inverse in state-space representation. This approach is based in the Cayley-Hamilton theorem. Several results in state-space make use of this theorem, so let us state this relevant result.

Let Δ be the determinant of the matrix $sI - A$, known as the characteristic polynomial of A, and given by

$$\Delta = \left| sI - A \right| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{2.92}$$

and the values s that satisfies this characteristic equation are called eigenvalues. For sake of convenience, we have replaced the symbol λ by s, but this definition is the same as the one given at the beginning of this Chapter.

Theorem 2.3.6 (Cayley-Hamilton Theorem)**.** *Every square matrix satisfies its own characteristic equation, i.e.*

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 = 0 \tag{2.93}$$

Then the inverse of the matrix $sI - A$ is given as follows:

Result 2.3.7. *The inverse of $sI - A$ is given by*

$$(sI - A)^{-1} = \frac{p_{n-1}(s)I + p_{n-2}(s)A + \cdots + p_1(s)A^{n-2} + A^{n-1}}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{2.94}$$

where the polynomials p_k are given by

$$p_k(s) = \sum_{j=0}^{k} a_{n-k+j} s^j \tag{2.95}$$

with $a_n = 1$.

This result is insightful since it allows us to write the transfer function of a system as frequency dependent linear combination of relevant matrices

$$G(s) = D + \frac{p_{n-1}(s)}{\Delta(s)}CB + \frac{p_{n-2}(s)}{\Delta(s)}CAB + \cdots + \frac{p_1(s)}{\Delta(s)}CA^{n-2}B + \frac{1}{\Delta(s)}CA^{n-1}B. \tag{2.96}$$

The matrices $\{CA^kB\}$ for $k = 0, 1, \ldots$ are refereed to as the system Markov parameters. They are widely used in MPC (Model Predictive Control) and system identification, where they are usually obtained from the discrete-time state space form. The inverse "à la Rosenbrock" shows the meaning of these parameters in the continuous-time domain. Let us consider $D = 0$, then we can rewrite (2.96) as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{2.97}$$

where the coefficients b_k are given by

$$b_{n-1} = CB \tag{2.98}$$

$$b_{n-2} = a_{n-1}CB + CAB \tag{2.99}$$

$$b_{n-3} = a_{n-2}CB + a_{n-1}CAB + CA^2B \tag{2.100}$$

$$\vdots = \vdots \tag{2.101}$$

$$b_1 = a_2CB + a_3CAB + \cdots + a_{n-1}CA^{n-3}B + CA^{n-2}B \tag{2.102}$$

$$b_0 = a_1CB + a_2CAB + \cdots + a_{n-2}CA^{n-2}B + a_{n-1}CA^{n-2}B + CA^{n-1}B \tag{2.103}$$

2.4 Learning Outcomes

The learning outcomes of this chapter can be summarised as follows:

- The modal canonical form splits the system into independent modes via a diagonalisation of the matrix A.
- Given an (A, B, C, D) representation with A diagonalisable and state x, the modal canonical form is obtained by applying the transformation $q = V^{-1}x$, where the columns of V are the eigenvectors of the matrix A. The set of coordinates q are modes of the system.
- Some matrices cannot be diagonalised, but the modal form can be obtained via Jordan forms (more details about Jordan forms are beyond the scope of this course).
- The exponential of a square matrix A is defined by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{2.104}$$

  and $\frac{d}{dt}e^{At} = Ae^{At}$

- The matrix e^{At} is referred to as state transition and the evolution of the autonomous system between the instant t_0 and t_1 is given by $x(t_1) = e^{A(t_1-t_0)}x(t_0)$.
- The state transition matrix is a linear combination of the exponential of the eigenvalues of the matrix A.
- If all eigenvalues of A have strictly negative real part, A is said to be Hurwitz and $\lim_{t\to\infty} e^{At} = 0$. This property ensures that the system is stable.
- The Laplace transform of the exponential matrix provides an alternative method to compute it with finite number of terms:

$$L(e^{At}) = (sI - A)^{-1} \text{ or } e^{At} = L^{-1}((sI - A)^{-1}). \tag{2.105}$$

- The transfer function of a system with a state-space representation (A, B, C, D) is $G(s) = C(sI - A)^{-1}B + D$.

2.5 Further examples

Let us find the inverse Laplace transform of the transfer function:

$$G(s) = \frac{1}{s^3 + 2s^2 + 101s}. \tag{2.106}$$

The first step is decompose this fraction in partial fractions:

$$\frac{1}{s^3 + 2s^2 + 101s} = \frac{\alpha}{s} + \frac{\beta s + \gamma}{s^2 + 2s + 101}. \tag{2.107}$$

After some simple algebraic manipulation, it follows that

$$1 = \alpha s^2 + 2\alpha s + 101\alpha + \beta s^2 + \gamma s. \tag{2.108}$$

Then, we need to find the solution of the simultaneous equations

$$101\alpha = 1, \tag{2.109}$$

$$2\alpha + \gamma = 0, \tag{2.110}$$

$$\alpha + \beta = 0. \tag{2.111}$$

It is trivial to show that the solution is

$$\alpha = 1/101, \tag{2.112}$$

$$\gamma = -2/101, \tag{2.113}$$

$$\beta = -1/101, \tag{2.114}$$

hence

$$\frac{1}{s^3 + 2s^2 + 101s} = \frac{1/101}{s} - \frac{1}{101}\frac{s+2}{(s+1)^2 + 100}. \tag{2.115}$$

The inverse Laplace transform of $1/s$ is 1 for all $t > 0$. We need further manipulations with the second fraction. In particular, we need to use:

$$\mathcal{L}^{-1}\left(\frac{s+a}{(s+a)^2 + b^2}\right) = e^{-at}\cos(bt) \tag{2.116}$$

$$\mathcal{L}^{-1}\left(\frac{b}{(s+a)^2 + b^2}\right) = e^{-at}\sin(bt) \tag{2.117}$$

so let us rewrite this fraction to be able to use this expression:

$$\frac{s+2}{(s+1)^2 + 100} = \frac{s+1}{(s+1)^2 + 10^2} + \frac{1}{10}\frac{10}{(s+1)^2 + 10^2} \tag{2.118}$$

Then, it follows that

$$\mathcal{L}^{-1}\left(\frac{s+2}{s+2s+101}\right) = e^{-t}\cos(10t) + \frac{1}{10}e^{-t}\sin(10t) \tag{2.119}$$

As a result, the inverse Laplace transform of $G(s)$ is given by

$$\mathcal{L}^{-1}(G(s)) = \frac{1}{101} - \frac{1}{101}\left(e^{-t}\cos(10t) + \frac{1}{10}e^{-t}\sin(10t)\right) \tag{2.120}$$

Chapter 3

Nonlinear systems

The state-space representation of a system can be used for both linear and nonlinear systems. In this chapter, we shall only briefly deal with nonlinear systems. Students will be able to test their endurance with Nonlinear Systems within the unit Nonlinear and Adaptive Control in the second semester. This chapter will focus on linearisation of nonlinear systems around an equilibrium point or around an operating point.

3.1 Linearisation of nonlinear systems

3.1.1 Equilibrium point of a nonlinear system

Let us consider the autonomous dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{3.1}$$

where the vectorial function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ has an appropriate continuity condition such that the above differential equation has a unique solution.

The mathematical definition of an equilibrium point is the following:

Definition 3.1.1 (Equilibrium point)**.** The point $x_e \in \mathbb{R}^n$ is an equilibrium point of the system (3.1) if $f(x_e) = 0$, i.e.

$$\dot{x}(t)\Big|_{x=x_e} = f(x_e) = 0. \tag{3.2}$$

Now, we can introduce a more sophisticated definition of stability.

Definition 3.1.2. An equilibrium point x_e is said to be stable if for any initial condition around the equilibrium point, then the distance between the solution of (3.1) at any instant and the equilibrium point is bounded. Mathematically, for any $\epsilon > 0$ there exists δ such that

$$\|x(0) - x_e\| < \delta \text{ then } \|x(t) - x_e\| < \epsilon \quad \forall t \geq 0. \tag{3.3}$$

Moreover, it is said to be asymptotically stable if $\lim_{t\to\infty} x(t) = 0$. Finally, if the equilibrium point x_e is not stable, it is said to be unstable.

Loosely speaking, an equilibrium point is stable if the trajectory of the state tries to recover or stay close to the equilibrium point when the system is slightly disturbed. So it is able to keep this position even though small disturbances can disturb the system. When the equilibrium point is unstable, any small perturbation will lead to the loss of equilibrium and the equilibrium position will not be recovered. Further classifications of stability, such as globally, locally, exponentially; can be found in classical nonlinear literature (for futher discussion read Khalil).

In the linear case, we are only interested in the trivial equilibrium point $x_e = 0$, so the definition of stability is given for the system. However, this definition is more fundamental since a system such as a pendulum has two equilibrium points, where one is stable and the other unstable. Therefore, the concept of stability is inherent to the equilibrium point.

Exercise 3.1.3. Give an example of a linear system with several equilibrium points.

3.1.2 Linearisation around equilibrium points

The nonlinear system behaves approximately as a linear system near an equilibrium point (assuming some nice properties such as as continuity in f, etc.). This fact is supported by the Taylor expansion[1] of f around the equilibrium point

$$f(x) \simeq f(x_e) + J_f(x_e)(x - x_e), \tag{3.4}$$

where J_f is the Jacobian matrix. Now it clear that if $f(x_e) = 0$ and we define a new state as $\Delta x = x - x_e$, then the $f(x)$ becomes the matrix J_f times the vector Δx if $x - x_e$ is close to zero, hence the nonlinear system (3.1) becomes linear

$$\frac{d}{dt}(\Delta x) \approx A\Delta x \tag{3.5}$$

where $A = J_f(x_e)$ and we have used that if $\Delta x = x - x_e$ then $\frac{d}{dt}(\Delta x) = \dot{x}$.

We still need to define the Jacobian matrix of f:

[1] Nothing to do with Taylor Swift.

Definition 3.1.4. Given a vectorial function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, then the Jacobian matrix $J_f \in \mathbb{R}^{n \times n}$ is defined by

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{3.6}$$

We can analyse the equilibrium point by performing a linearisation and studying the properties of the Jacobian matrix at the equilibrium point.

Result 3.1.5. *The stability of the equilibrium point x_e of the system* (3.1) *is equivalent to the stability of the linear system defined by $\dot{x} = Ax$, where $A = J_f(x_e)$.*

3.1.3 Worked example: simple pendulum

A simple pendulum (Fig. 3.1) is described by the equation

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{3.7}$$

where θ is the angle between the vertical position and the position of the pendulum and l is the length of the pendulum. The state-space representation of this second order differential equation can be deduced as follows.

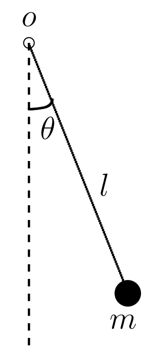


Figure 3.1: A rigid and massless bar of length l joins the pivot o and the mass m. Let θ be the angle between the vertical position and the bar.

Let us define the states

$$x_1 = \theta; \tag{3.8}$$

$$x_2 = \dot{\theta}. \tag{3.9}$$

Then $\dot{x}_2 = -\frac{g}{l}\sin(\theta)$, so the state-space representation of this system is

$$\dot{x}_1 = x_2; \tag{3.10}$$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1). \tag{3.11}$$

Hence the vectorial function $f(x_1, x_2)$ is given by

$$f(x_1, x_2) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}, \tag{3.12}$$

and the equilibrium points satisfies $f(x) = 0$, i.e.

$$x_2 = 0; \tag{3.13}$$

$$\sin(x_1) = 0. \tag{3.14}$$

As a result, the equilibrium points of the system 3.7 are $x_{e1} = (0, 0)$ and $x_{e2} = (\pi, 0)$, if we restrict our attention to $\theta \in (-\pi, \pi]$. Before analysing the equilibrium point let us find the Jacobian matrix:

$$J_f(x_1, x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}(x_2) & \frac{\partial}{\partial x_2}(x_2) \\ \frac{\partial}{\partial x_1}(-\frac{g}{l}\sin(x_1)) & \frac{\partial}{\partial x_2}(-\frac{g}{l}\sin(x_1)) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (-\frac{g}{l}\cos(x_1)) & 0 \end{bmatrix} \tag{3.15}$$

Equilibrium point $x_{e1} = (0, 0)$ At the equilibrium point $x_{e1} = (0, 0)$, the system can be linearised as

$$\dot{x} = J_f(0, 0)x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} x \tag{3.16}$$

The eigenvalues of $J_f(0, 0)$ are given by

$$\det(J_f(0, 0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{l} = 0 \text{ and so } \lambda = \pm\sqrt{\frac{g}{l}}j \tag{3.17}$$

Hence, the real part of the eigenvalues is 0 and they are at different localisations, therefore the equilibrium point x_{e1} is marginally stable.

Equilibrium point $x_{e2} = (\pi, 0)$ At the equilibrium point $x_{e2} = (\pi, 0)$, we need to use a translation of the space since the equilibrium point is not the origin, hence $\Delta x = x - x_{e2}$. Thus the system can be linearised as

$$\dot{\Delta x} = J_f(\pi, 0)\Delta x = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \Delta x \tag{3.18}$$

The eigenvalues of $J_f(\pi, 0)$ are given by

$$\det(J_f(\pi, 0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{l} = 0 \text{ and so } \lambda = \pm\sqrt{\frac{g}{l}} \tag{3.19}$$

Hence, the real part of one eigenvalue is greater than 0; therefore the equilibrium point x_{e2} is unstable.

3.1.4 Linearisation around an operating point

When the nonlinear system is required to be operating with an input that is different to zero, then we can no longer refer to as an equilibrium point of the system since this concept is related with the autonomous system. In this case, we use the concept of operating point.

Let us consider the nonlinear system given by

$$\dot{x} = f(x,u); \tag{3.20}$$

$$y = h(x,u); \tag{3.21}$$

then if $f(x_o,u_o)=0$, the point (x_o,u_o) is referred to as an operating point. Under this condition, we can perform a linearisation of the system as follows. Let us define the new input, state and output as the variation around u_o, x_o, and y_o

$$\Delta u = u - u_o, \tag{3.22}$$

$$\Delta x = x - x_o, \tag{3.23}$$

$$\Delta y = y - y_o. \tag{3.24}$$

Carrying out a Taylor expansion of $f(x,u)$ and $h(x,u)$ around the point (x_o,u_o), it follows that

$$f(x,u) \simeq f(x_o,u_o) + J_f^x(x_o,u_o)(x-x_o) + J_f^u(x_o,u_o)(u-u_o); \tag{3.25}$$

$$h(x,u) \simeq h(x_o,u_o) + J_h^x(x_o,u_o)(x-x_o) + J_h^u(x_o,u_o)(u-u_o); \tag{3.26}$$

where $f(x_o,u_o)=0$ and $h(x_o,u_o)=y_o$. Finally, the linearised system is given by

$$\frac{d}{dt}(\Delta x) \simeq J_f^x(x_o,u_o)\Delta x + J_f^u(x_o,u_o)\Delta u; \tag{3.27}$$

$$\Delta y \simeq J_h^x(x_o,u_o)\Delta x + J_h^u(x_o,u_o)\Delta u; \tag{3.28}$$

where the superscripts x and u in the Jacobian matrices indicate the parameter that is considered as a variable.

In order to compare both systems, we should implement both systems as depicted in Fig. 3.2. One could wonder about what happens if the plant is linear. In this case, the implementation of the operating point is not so critical since the system is the same at any point (x,u) and we can always replace variables by increments of the variables.

3.1.5 Worked example: The quadruple-tank process

This problem was introduced by Johansson in his paper: "The quadruple-tank process: a multivariable laboratory process with an adjustable zero". This process is interesting since it

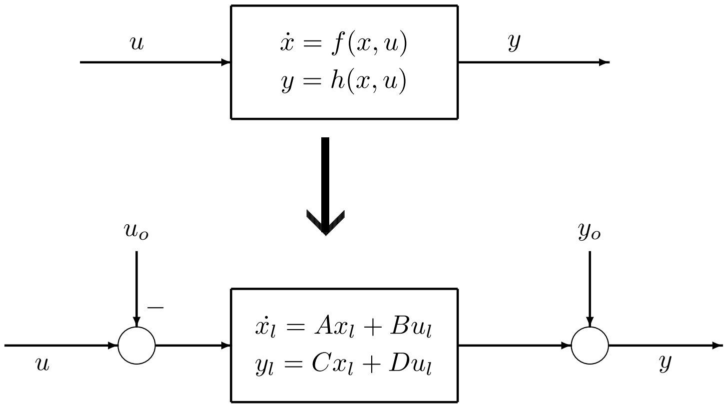


Figure 3.2: Implementation of an operating poing in a nonlinear plant. In the figure, Δx, Δu and Δy have been replaced by x_l, u_l and y_l.

provides an academic example of a plant with multivariable zero and directionality. Moreover the process can be set up in such a way that the zero can be either in the left-half plane or in the right-half plane. This system will be discussed in the last part of this unit.

The experiment consists in 4 tanks as in Fig. 3.3. The inputs of the system are the voltages applied to two pumps. These pump water from the bottom basin into the four tanks, in particular, Pump 1 into Tank 1 and Tank 4; and Pump 2 into Tank 2 and Tank 3. It is assumed that the water flow is proportional to the voltage in the pump. The flow through each pump is split into two subflows, outputs 1 and 2 at the top of the Fig 3.3. If the total flow is f_i for $i = 1, 2$, then the flow through Output 1 is $f_{i,1} = \gamma_i f_i$; and the flow through Output 2 is $f_{i,2} = (1 - \gamma_i) f_i$ where $0 \leq \gamma_1 \leq 1$. It is assumed that each tank has the same section A.

The state of the system is defined by the four levels of the tanks: L_1, L_2, L_3, and L_4. The dynamics of each tank is given by the law of conservation of the mass for a liquid with constant density

$$\frac{d}{dt}\text{volume of water} = \text{water flow in} - \text{water flow out} \tag{3.29}$$

and Bernoulli's equation provides that the discharge flow f_{di} at a Tank i with a hole in the bottom of the cylinder with a section D_{oi} is given by

$$f_{di} = D_{oi}\sqrt{2gL_i}, \tag{3.30}$$

where we have assumed a perfect discharge.

Tank 1 Let us apply the law of conservation of the mass to Tank 1

$$\frac{d}{dt}(AL_1) = (f_{1,2} + f_{d3}) - f_{d1} = ((1 - \gamma_1)kV_{p1} + D_{o3}\sqrt{2gL_3}) - D_{o1}\sqrt{2gL_1} \tag{3.31}$$

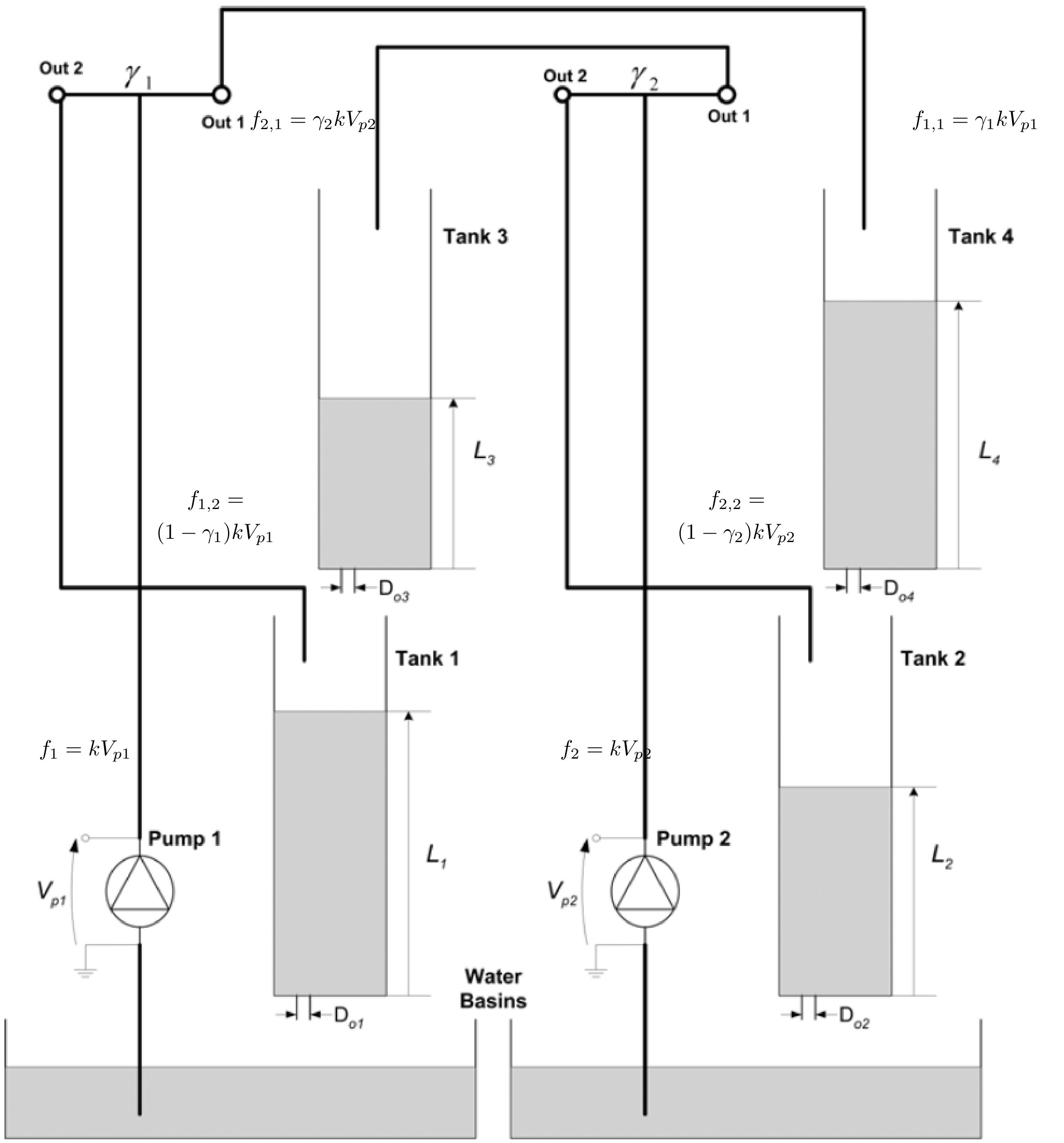


Figure 3.3: The quadruple tank. Adapted from Quanser manual.

and we obtain the dynamics of the state L_1 as

$$\dot{L}_1 = -\frac{D_{o1}}{A}\sqrt{2gL_1} + \frac{D_{o3}}{A}\sqrt{2gL_3} + \frac{(1-\gamma_1)k}{A}V_{p1}. \tag{3.32}$$

This equation is nonlinear in the states L_1 and L_3.

Tank 2 Let us apply the law of conservation of the mass to Tank 2

$$\frac{d}{dt}(AL_2) = (f_{2,2} + f_{d4}) - f_{d2} = ((1-\gamma_2)kV_{p2} + D_{o4}\sqrt{2gL_4}) - D_{o2}\sqrt{2gL_2} \tag{3.33}$$

and we obtain the dynamics of the state L_2 as

$$\dot{L}_2 = -\frac{D_{o2}}{A}\sqrt{2gL_2} + \frac{D_{o4}}{A}\sqrt{2gL_4} + \frac{(1-\gamma_2)k}{A}V_{p2}. \tag{3.34}$$

Tank 3 Let us apply the law of conservation of the mass to Tank 3

$$\frac{d}{dt}(AL_3) = f_{2,1} - f_{d3} = \gamma_2 kV_{p2} - D_{o3}\sqrt{2gL_3} \tag{3.35}$$

and we obtain the dynamics of the state L_3 as

$$\dot{L}_3 = -\frac{D_{o3}}{A}\sqrt{2gL_3} + \frac{\gamma_2 k}{A}V_{p2}. \tag{3.36}$$

Tank 4 Finally, let us apply the law of conservation of the mass to Tank 4

$$\frac{d}{dt}(AL_4) = f_{1,2} - f_{d4} = \gamma_1 kV_{p1} - D_{o4}\sqrt{2gL_4} \tag{3.37}$$

and we obtain the dynamics of the state L_3 as

$$\dot{L}_4 = -\frac{D_{o4}}{A}\sqrt{2gL_4} + \frac{\gamma_1 k}{A}V_{p1}. \tag{3.38}$$

As a result, the state-space representation of the quadruple tank process is

$$\dot{L}_1 = f_1(L_1, L_2, L_3, L_4, V_{p1}, V_{p2}) = -\frac{D_{o1}}{A}\sqrt{2gL_1} + \frac{D_{o3}}{A}\sqrt{2gL_3} + \frac{(1-\gamma_1)k}{A}V_{p1}, \tag{3.39}$$

$$\dot{L}_2 = f_2(L_1, L_2, L_3, L_4, V_{p1}, V_{p2}) = -\frac{D_{o2}}{A}\sqrt{2gL_2} + \frac{D_{o4}}{A}\sqrt{2gL_4} + \frac{(1-\gamma_2)k}{A}V_{p2}, \tag{3.40}$$

$$\dot{L}_3 = f_3(L_1, L_2, L_3, L_4, V_{p1}, V_{p2}) = -\frac{D_{o3}}{A}\sqrt{2gL_3} + \frac{\gamma_2 k}{A}V_{p2}, \tag{3.41}$$

$$\dot{L}_4 = f_4(L_1, L_2, L_3, L_4, V_{p1}, V_{p2}) = -\frac{D_{o4}}{A}\sqrt{2gL_4} + \frac{\gamma_1 k}{A}V_{p1}. \tag{3.42}$$

The output of the quadruple tank process is given by the levels of Tank 1 and Tank 2, so it can be written in our classical linear form

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}. \tag{3.43}$$

Now, let us assume that we want to operate the plant around the input voltages V_{p1}^o and V_{p2}^o. Then, from the last two equations, there exist unique levels L_3^o and L_4^o such that

$$0 = -\frac{D_{o3}}{A}\sqrt{2gL_3^o} + \frac{\gamma_2 k}{A}V_{p2}^o, \tag{3.44}$$

$$0 = -\frac{D_{o4}}{A}\sqrt{2gL_4^o} + \frac{\gamma_1 k}{A}V_{p1}^o. \tag{3.45}$$

Once L_3^o and L_4^o have been determined, it is clear that there exist unique levels L_1^o and L_2^o such that

$$0 = -\frac{D_{o1}}{A}\sqrt{2gL_1^o} + \frac{D_{o3}}{A}\sqrt{2gL_3^o} + \frac{(1-\gamma_1)k}{A}V_{p1}^o, \tag{3.46}$$

$$0 = -\frac{D_{o2}}{A}\sqrt{2gL_2^o} + \frac{D_{o4}}{A}\sqrt{2gL_4^o} + \frac{(1-\gamma_2)k}{A}V_{p2}^o; \tag{3.47}$$

hence setting the voltage of the pumps, the operating point is determined by $(L_1^o, L_2^o, L_3^o, L_4^o, V_{p1}^o, V_{p2}^o)$, in short, (L^o, V^o).

Exercise 3.1.6. Find the expression of $(L_1^o, L_2^o, L_3^o, L_4^o)$ as a function of (V_{p1}^o, V_{p2}^o).

Let us define the new set of coordinates as follows

$$x_1 = L_1 - L_1^o, \tag{3.48}$$

$$x_2 = L_2 - L_2^o, \tag{3.49}$$

$$x_3 = L_3 - L_3^o, \tag{3.50}$$

$$x_4 = L_4 - L_4^o. \tag{3.51}$$

and the new set of inputs and outputs

$$u_1 = V_{p1} - V_{p1}^o, \tag{3.52}$$

$$u_2 = V_{p2} - V_{p2}^o, \tag{3.53}$$

$$y_1 = L_1 - L_1^o, \tag{3.54}$$

$$y_2 = L_2 - L_2^o. \tag{3.55}$$

The linearisation around this point is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial L_1}(L^o,V^o) & \frac{\partial f_1}{\partial L_2}(L^o,V^o) & \frac{\partial f_1}{\partial L_3}(L^o,V^o) & \frac{\partial f_1}{\partial L_4}(L^o,V^o)\\ \frac{\partial f_2}{\partial L_1}(L^o,V^o) & \frac{\partial f_2}{\partial L_2}(L^o,V^o) & \frac{\partial f_2}{\partial L_3}(L^o,V^o) & \frac{\partial f_2}{\partial L_4}(L^o,V^o)\\ \frac{\partial f_3}{\partial L_1}(L^o,V^o) & \frac{\partial f_3}{\partial L_2}(L^o,V^o) & \frac{\partial f_3}{\partial L_3}(L^o,V^o) & \frac{\partial f_3}{\partial L_4}(L^o,V^o)\\ \frac{\partial f_4}{\partial L_1}(L^o,V^o) & \frac{\partial f_4}{\partial L_2}(L^o,V^o) & \frac{\partial f_4}{\partial L_3}(L^o,V^o) & \frac{\partial f_4}{\partial L_4}(L^o,V^o)\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}\frac{\partial f_1}{\partial V_{p1}}(L^o,V^o) & \frac{\partial f_1}{\partial V_{p2}}(L^o,V^o)\\ \frac{\partial f_1}{\partial V_{p1}}(L^o,V^o) & \frac{\partial f_2}{\partial V_{p2}}(L^o,V^o)\\ \frac{\partial f_3}{\partial V_{p1}}(L^o,V^o) & \frac{\partial f_3}{\partial V_{p2}}(L^o,V^o)\\ \frac{\partial f_4}{\partial V_{p1}}(L^o,V^o) & \frac{\partial f_4}{\partial V_{p2}}(L^o,V^o)\end{bmatrix} \begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{3.56}$$

Now we need to find the derivatives of the functions given in (3.39), (3.40), (3.41), and (3.42), and evaluate them at the operating point (L^o, V^o). Then it follows

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-\frac{D_{o1}}{2A}\sqrt{\frac{2g}{L_1^o}} & 0 & \frac{D_{o3}}{2A}\sqrt{\frac{2g}{L_3^o}} & 0\\ 0 & -\frac{D_{o2}}{2A}\sqrt{\frac{2g}{L_2^o}} & 0 & \frac{D_{o4}}{2A}\sqrt{\frac{2g}{L_4^o}}\\ 0 & 0 & -\frac{D_{o3}}{2A}\sqrt{\frac{2g}{L_3^o}} & 0\\ 0 & 0 & 0 & -\frac{D_{o4}}{2A}\sqrt{\frac{2g}{L_4^o}}\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}\frac{(1-\gamma_1)k}{A} & 0\\ 0 & \frac{(1-\gamma_2)k}{A}\\ 0 & \frac{\gamma_2 k}{A}\\ \frac{\gamma_1 k}{A} & 0\end{bmatrix} \begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{3.57}$$

The state-space representation of the dynamical system is completed with the output equation

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}. \tag{3.58}$$

Exercise 3.1.7. Given the quadruple plant with values $k = 0.5\cdot 10^{-4}/60$ m^{3}s^{-1}V^{-1}, $A = 0.03^2\pi$ m^{-2}, $D_{oi} = 4\pi \cdot 10^{-6}$m^{-2} for $i = 1, 2$, $D_{oi} = \pi \cdot 10^{-6}$m^{-2} for $i = 3, 4$, and $\gamma_i = 0.25$ for $i = 1, 2$. Carry out a simulation of the nonlinear system in Simulink and compare the nonlinear system with the linearisation of the operating point given by the pump 1 working at 6 V and the pump 2 at 4 V. Data: $g = 9.81$ ms^{-2}. Levels at operating point: $L_1^o = 6.7802 \cdot 10^{-3}$ m; $L_2^o = 4.5388 \cdot 10^{-3}$ m; $L_3^o = 3.5862 \cdot 10^{-3}$ m; and $L_4^o = 8.0690 \cdot 10^{-3}$ m.

3.2 Introduction to Lyapunov Stability

In the previous section, we have studied the stability of a linear system when it is linearised around an equilibrium point. However, it just provided information of a very narrow region of the state-space. In this section we are going to introduce a result that will be able to study wider region of the state-space.

We are going to reduce our attention to the autonomous system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0; \tag{3.59}$$

and we assume that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ has nice properties in such a way that the above equation has one and only one solution and $f(0) = 0$. The latter assumption ensures that there is at least one equilibrium point, which is assumed without loss of generality to be the origin of the state space.

Theorem 3.2.1 (Lyapunov Theorem)**.** *Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R}^n - \{0\}$. If*

$$\dot{V}(x) \leq 0 \tag{3.60}$$

for all $x \in \mathbb{R}^n$ then the equilibrium point $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \tag{3.61}$$

for all $x \in \mathbb{R}^n - \{0\}$, then the equilibrium point is asymptotically stable.

The application of this theorem will be covered in the second semester in Nonlinear Control and Optimal and Robust Control.

3.3 Learning Outcomes

The learning outcomes of this chapter can be summarised as follows:

- The state-space representation is able to describe nonlinear systems.
- The point x_e of the dynamical system $\dot{x} = f(x)$ is said to be an equilibrium point if $f(x_e) = 0$.
- The linearisation around an equilibrium point is always possible. The dynamics of the system is described by the Jacobian matrix of f evaluated at the equilibrium point x_e.
- The stability of an equilibrium point x_e is given by the eigenvalues of the Jacobian matrix evaluated at x_e as any linear system.
- When we have a nonlinear system operating with a constant input different to zero, then the linearisation is still possible. This linerisation is referred to as linearisation around an operating point. The matrices of the state-space representation are given by different Jacobian matrices, all of them evaluated at the operating point.
- Stability analysis of nonlinear systems is a beautiful and fun topic. One of the most important approaches is the Lyapunov theorem, which will be widely studied in the second semester.

Chapter 4

Controllability and Observability

This chapter deals with two important concepts for state-space control: controllability and observability. The former concerns the ability of the input to "move" the system from one state to another state. The latter concerns the ability of the output to "reveal" the state of the system. Both concepts were developed in the fifties and sixties of the last Century when control engineers started to use state-space techniques instead of classical transfer functions. Since then, controllability and observability have appeared in modern control techniques using the state-space representation.

4.1 Controllability

4.1.1 Definition

Let us start with the mathematical definition.

Definition 4.1.1 (Controllability)**.** The pair $[A, B]$ is said to be controllable if for any $x_0,\ x_1 \in \mathbb{R}^n$ and any $t_1 > 0$, there exists a control action $u : [0, t_1] \mapsto \mathbb{R}^m$ such that the solution of

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{4.1}$$

is x_1 at $t = t_1$, i.e. $x(t_1) = x_1$.

We will say that a state-space representation is controllable if its pair $[A, B]$ is controllable. Loosely speaking, given two points of the state space, the controllability of the system ensures that we will be able to move between them by a correct design of the input. In the opposite case, i.e. if the pair $[A, B]$ is not controllable; the input will be not enough to determine a desired trajectory of the system.

If a system is in its modal form, then it is very easy to check the controllability of the pair $[A, B]$, where A is diagonal. We need to check if all elements of B are non-zero. If so, the pair $[A, B]$ is controllable. Otherwise, the mode corresponding to the zero element of B will not be affected by the input at all, and its fate will be fixed by the matrix A itself, i.e., as in the autonomous mode.

4.1.2 Test for controllability

This section presents a more elegant method to test controllability than using the modal form. The following theorem provides an essential tool to check the controllability of the pair $[A, B]$.

Result 4.1.2. *The pair $[A, B]$ is controllable if and only if the matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{4.2}$$

is full rank.

This result introduces an important matrix in our lives, so let us define it properly.

Definition 4.1.3 (Controllability matrix)**.** Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, the controllability matrix associated with the pair $[A, B]$, henceforth, $\mathcal{C}(A, B) \in \mathbb{R}^{n\times nm}$ is given by

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{4.3}$$

For the case of a SISO system, i.e., $m = 1$, the controllability matrix is square and full rank is reduced to nonsingular.

Result 4.1.4. *The pair $[A, B]$ with $B \in \mathbb{R}^{n\times 1}$, is controllable if and only if the matrix $\mathcal{C}(A, B)$ is nonsingular, i.e.*

$$\det\left(\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}\right) \neq 0 \tag{4.4}$$

Once again, we should expect that such an important property does not depend on the basis that we choose to express A and B. The following result ensures that this property does not change due to a transformation.

Result 4.1.5. *Given the pair $[A, B]$ and a nonsingular matrix T, the pair $[TAT^{-1}, TB]$ is controllable if and only if $[A, B]$ is controllable.*

Exercise 4.1.6. Show that $\mathcal{C}(TAT^{-1}, TB) = T\mathcal{C}(A, B)$. What conclusion can be drawn? Hint: Given a nonsingular matrix T, then $\text{rank}(X) = \rho$ if and only if $\text{rank}(TX) = \rho$.

Finally, now we can understand why the control canonical form has such a name.

Result 4.1.7. *If the pair $[A, B]$ is controllable, then there exists a nonsingular matrix T such that the state space representation $(TAT^{-1}, TB, CT^{-1}, D)$ is in the control canonical form.*

We have introduced the control canonical form in Chapter 1. It is related to the controllability of the pair $[A, B]$ by the following result:

Result 4.1.8. *The pair $[A, B]$ is controllable if and only if there exist a nonsingular matrix T and the set of numbers a_i with $i = 0, n-1$ such that*

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and } TB = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4.5}$$

Finally, the following result shows that any SISO control canonical form of a transfer function with no pole-zero cancellation is controllable:

Result 4.1.9. *A pair $[A, B]$ is uncontrollable if and only if there is a the left eigenvector of A, v, such that $v^* B = 0$.*

Whereas the eigenvalues that we have used previously are also referred to as right-eigenvalues, the definition of left eigenvalues is similar:

Definition 4.1.10. The vector v is said to be a left eigenvalue of A if there exists λ such that $v^* A = v^* \lambda$

4.1.3 Proof of the main result

Students interested in controllability and observability will find more information in Chapter 2 in Kailath and Chapters 5 and 6 in Antsaklis and Michel. The aim of this Section is to show a simplified version of the proof of Result 4.1.2.

The concept of controllability of the pair $[A, B]$ required that given an initial state x_0, we must find an input $u(t)$ such that we reach the state x_1 after some period of time. To simplify our development, we are going to try to find very powerful input which is able to move the state instantaneously.

The first initial input that we can find with this property is the dirac function. Let us consider $u_1(t) = \alpha_1 \delta(t)$ for any real value α_1, then applying equation (2.42)[1], the new state x_1 will be reached at time 0^+, where 0^+ is the "following instant" after 0, with definition $0^+ = \lim_{\epsilon \to 0^+} 0 + \epsilon$. Under this conditions, the new state will be given by

$$x_1 = x(0^+) = e^{A0^+} x_0 + \int_{0^-}^{0^+} e^{A(0^+ - \tau)} B(\alpha_1 \delta(\tau)) d\tau, \tag{4.6}$$

where $e^{A0^+} = I$ and using the Dirac function property (or definition)

$$\int_{0^-}^{0^+} f(\tau)\delta(\tau) d\tau = f(0), \tag{4.7}$$

it thus follows that

$$x_1 = x_0 + B\alpha_1. \tag{4.8}$$

So if we want to reach the state x_1 with n coordinates, the question results in a set of n simultaneous equations (one for each state) with 1 unknown α_1. Evidently, we cannot find one α_1 for each state x_1. However, it does not mean that the system is uncontrollable since we can propose a more sophisticated input.

Let us consider $u_2(t) = \alpha_1 \delta(t) + \alpha_2 \delta^{(1)}(t)$, where we are using the derivative of the Dirac delta. It comes from generalised function (or distribution) theory, and you may not be familiar with it, do not panic! The only property that we want to use is that

$$\int_{0^-}^{0^+} f(\tau)\delta^{(1)}(\tau) dt = (-1) \left. \frac{df(\tau)}{d\tau} \right|_{\tau=0}. \tag{4.9}$$

Then, using(2.42), we obtain the expression of the new state as follows:=

$$x_1 = x(0^+) = e^{A0^+} x_0 + \int_{0^-}^{0^+} e^{A(0^+ - \tau)} B(\alpha_1 \delta(\tau) + \alpha_2 \delta^{(1)}(\tau)) d\tau. \tag{4.10}$$

Thus, using (4.7) and (4.9), it follows

$$x_1 = x_0 + B\alpha_1 + (-1) \left. (-A) e^{-A\tau} \right|_{\tau=0} = x_0 + B\alpha_1 + AB\alpha_2. \tag{4.11}$$

As a result, given x_1 we can find the required input by solving the set of simultaneous equation given by

$$\begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = (x_1 - x_0) \tag{4.12}$$

Once again, we have n equations but only 2 knowns (α_1 and α_2), hence we cannot find a solution for any x_1.

[1]We should have used 0^- in the lower limit of the integral in our development. If after the following development, you still have questions, please, let me know!

Now, we can propose the following input

$$u_n(t) = \alpha_1\delta(t) + \alpha_2\delta^{(1)}(t) + \cdots + \alpha_n\delta^{(n-1)}(t) \tag{4.13}$$

In general, the k-th derivative of the Dirac delta holds the following property

$$\int_{0^-}^{0^+} f(t)\delta^{(k)}(t)dt = (-1)^k \left.\frac{d^k f(t)}{dt^k}\right|_{t=0}. \tag{4.14}$$

A similar development yields to

$$x_1 = x_0 + B\alpha_1 + AB\alpha_2 + \cdots + A^{n-1}B\alpha_n. \tag{4.15}$$

Finally, we have obtained an set of simultaneous equations with the same number of equations as unknowns

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = (x_1 - x_0), \tag{4.16}$$

and the controllability matrix turns up. Then, it will have a solution for any value of x_1 (and x_0) if and only if the controllability matrix has rank n. As this is the maximum rank that it can have, then we say it is full rank.

If the student is not familiar with the Cayley-Hamilton theorem, they could be tempted to introduce a new derivative to obtain more freedom if the controllability matrix is not full rank. However, students who are familiar with the Cayley-Hamilton theorem, will know that it is useless since this theorem state that A^n is a linear combination of A^0, A^1, ..., and A^{n-1}. As a result, we will only be able to reach any state x_1 if the controllability matrix is full rank.

The above development show the sufficiency of Result 4.1.2, i.e. the pair $[A, B]$ is controllable if the controllability matrix is full rank. To prove the necessity of Result 4.1.2 it is required to show that if the controllability matrix is not full rank, then we cannot reach the output with any other input. It goes beyond the scope of this unit, but this part of the proof can be found in the reading list.

4.1.4 Worked example

Let us consider the pair

$$A = \begin{bmatrix} -2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.17}$$

Then the controllability matrix is

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 4 & 0 & -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -8 & 0 & 8 & 0 \\ 0 & 1 & 0 & -5 & 0 & 21 & 0 & -85 \\ 0 & 0 & 0 & 2 & 0 & -10 & 0 & 42 \end{bmatrix} \tag{4.18}$$

Trick. The rank of a matrix $X^{n\times m}$ is between 0 and $\min\{n, m\}$. The matrix $X^{n\times m}$ has rank larger than or equal to k, in short, $\text{rank}(X) \geq k$, if there is one minor of order k different to zero. A minor of a matrix is the determinant of a square submatrix, roughly speaking the intersection between the same number of rows and columns. ■

With this information, let us start considering 1 as a possible rank of the matrix. Then it is clear that the minor that intersects the row 1 and column 1 is different to zero. As a result, the rank of the matrix is 1 or larger than 1. The next step is to find a minor of order 2 different to zero. For instance, the minor intersecting rows 1 and 3 and columns 1 and 2 is

$$\begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} = 4 \neq 0. \tag{4.19}$$

As a result, the rank of the matrix is 2 or greater than 2. Following this procedure, the minor intersecting rows 1, 3 and 4 and columns 1, 2 and 4 is

$$\begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 2 \end{vmatrix} = 8 \neq 0. \tag{4.20}$$

As a result, the rank of the matrix is 3 or greater than 3. Finally, let us try with minors of order 4. The minor with first 4 rows and columns is

$$\begin{vmatrix} 4 & 0 & -8 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 2 \end{vmatrix} \tag{4.21}$$

Brute-force method can be applied, but the following tricks make of this determinant a straightforward computation.

Trick. Swapping two rows or two columns changes the sign of the determinant. The determinant of a triangular matrix is the product of the diagonal elements. ■

Hence

$$\begin{vmatrix} 4 & 0 & -8 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 4 & 0 & -8 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -32 \neq 0. \tag{4.22}$$

where we have swapped rows 2 and 3 first and after applied properties of triangular matrices.

Exercise 4.1.11. Use the MATLAB command `ctrb` to find the controllability matrix of the pair (A, B). Use the MATLAB command `rank` to find the rank of this matrix.

4.2 Stabilizability

Stability is a relaxation of the controllability concept. Loosely speaking, any controllable pair $[A, B]$ will be stabilizable, but not any stabilizable pair $[A, B]$ will be controllable.

Definition 4.2.1 (Stabilizability)**.** The pair $[A, B]$ is said to be stabilizable if there exists $K \in \mathbb{R}^{m\times n}$ such that the matrix $(A - BK)$ is Hurwitz.

At first glance, it is difficult to find the connection between controllability and stabilizability. The concept of controllability informs us about the ability of controlling the modes of the system. The concept of stabilizability informs us about the possibility of stabilizing the system through a state feedback interconection (Chapter 5). Therefore, if there are modes that cannot be controlled, i.e. our pair $[A, B]$ is not controllable, we need to ensure that these modes are stable.

We can state out test for checking if the pair $[A, B]$ is stabilizable as follows:

Result 4.2.2. *Let us assume that the system A is given in the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{4.23}$$

where the pair $[A_{11}, B_1]$ is controllable. Then the system is stabilizable if and only if A_{22} is Hurwitz.

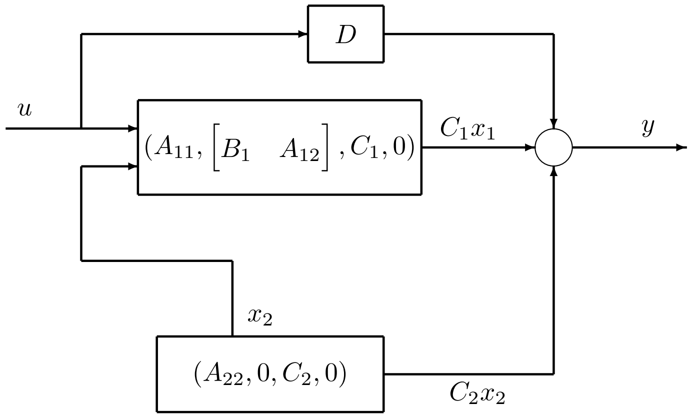


Figure 4.1: In the representation (4.23), the uncontrollable system can be expressed as two subsystems, one where the states x_1 can be controlled from u, and another where u does not have any ability to modify the trajectory of the state x_2.

Two straightforward conclusions can be drawn:

1. As commented, if $[A, B]$ is controllable, then the system is stabilizable.
2. If A is Hurwitz, then the pair $[A, B]$ is stabilizable for any B (by choosing $K = 0$).

A natural question arises: what happens if the system is not in the above form? Then, it is complicated... The full solution of this problem is far beyond the scope of these notes, but Section 2.4 in Linear Systems by T. Kailath and Chapter 18 in Linear System Theory by W. J. Rugh are recommended for daredevil students. Here we will just state that it is possible always possible to transform the pair $[A, B]$ into the above desired form.

Result 4.2.3. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. If rank($\mathcal{C}(A, B)$) $= \rho < n$, then there exists an nonsingular matrix T such that*

$$TAT^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}, TB = \begin{bmatrix} Y_1 \\ 0 \end{bmatrix} \quad (4.24)$$

where $X_{11} \in \mathbb{R}^{\rho\times\rho}$ and $Y_{11} \in \mathbb{R}^{\rho\times m}$ and the pair $[X_{11}, Y_{11}]$ controllable.

In this course, we will transform the system into its modal form.

4.2.1 Worked example: Jan 2013 Exam

Q4 (Jan 2013 Exam) requires to decide whether the pair

$$A = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad (4.25)$$

is stabilizable.

The controllability matrix is

$$\mathcal{C}(A,B) = \begin{bmatrix} -2 & 20 \\ 1 & -10 \end{bmatrix}, \tag{4.26}$$

and $\det(\mathcal{C}(A,B)) = 0$, hence there is an uncontrollable mode and we need to determine if it is stable or unstable.

To do so, we will transform this pair into the modal form. The characteristic equation of $\det(A - \lambda I) = 0$ is

$$\lambda^2 + 5\lambda - 50 = 0. \tag{4.27}$$

Hence the eigenvalues are 5 and -10. Now we need to find the eigenvectors.

$\lambda = 5$: We need to find a non-trivial solution of the simultaneous equation

$$\begin{bmatrix} -7-5 & 6 \\ 6 & 2-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.28}$$

i.e.

$$-12x + 6y = 0, \tag{4.29}$$
$$6x - 3y = 0. \tag{4.30}$$
$$\tag{4.31}$$

Hence, the solution is any vector with $y = 2x$, for instance, $v_1 = (1, 2)$.

$\lambda = -10$: We need to find a non-trivial solution of the simultaneous equation

$$\begin{bmatrix} -7-(-10) & 6 \\ 6 & 2-(-10) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.32}$$

i.e.

$$3x + 6y = 0, \tag{4.33}$$
$$6x + 12y = 0. \tag{4.34}$$
$$\tag{4.35}$$

Hence, the solution is any vector with $-2x = y$, for instance, $v_1 = (-2, 1)$.

Using these two vectors

$$V = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{4.36}$$

where $\text{diag}(5, -10) = V^{-1}AV$; and

$$V^{-1}B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.37}$$

As a result, the uncontrollable mode is the mode $\lambda = 5$, hence the uncontrollable mode is unstable; therefore the pair $[A, B]$ is not stabilizable.

4.3 Observability

4.3.1 Definition

Again, let us start with the mathematical definition.

Definition 4.3.1 (Observability)**.** The pair $[A, C]$ is said to be observable if for any $t_1 > 0$, the initial condition x_0 of

$$\dot{x}(t) = Ax(t); \tag{4.38}$$

$$y(t) = Cx(t); \tag{4.39}$$

can be determined from the time history of the output in $[0, t_1]$.

The concept of observability may less be intuitive since you may not be familiar with the concept of observer. Loosely speaking, if a system is observable then we can "guess" what is happening with the state of the system from output information. This concept is independent of the input; hence it is only determined by the pair $[A, C]$. We will explain why we do not need to use the input in Chapter 5.

Similarly, if the state-space representation is in its modal form, to check that the pair $[A, C]$ is observable, just check that all elements of C are non-zero. If one of them is zero, this mode is said to be unobservable. Once again, this is not a really nice form for testing observability, since there are systems that cannot be expressed in the modal form.

4.3.2 Derivation of the Observability matrix

We are going to say that the state $x_{\bar{o}} \neq 0$ is unobservable[2] if the output of the system $\dot{x}(t) = Ax(t)$, and $y(t) = Cx(t)$ is null for all $t \geq 0$ when $x(0) = x_{\bar{o}}$.

[2]Only controllable and unobservable states can be properly defined form a mathematical point of view as they belong to subspaces. The sets of uncontrollable states and observable states are not subspaces.

Following Theorem 2.2.6, the solution of the autonomous systems is given by $x(t) = e^{At}x_{\bar{o}}$, hence

$$y = Ce^{At}x_{\bar{o}} = C\sum_{k=0}^{\infty}\frac{1}{k!}(At)^k x_{\bar{o}}.$$

Applying the Cayley-Hamilton theorem, we can write the exponential matrix in terms of a finite addition as the rest are linear combination of the first terms[3]:

$$e^{At} = \alpha_1(t)I + \alpha_2(t)A + \alpha_3(t)A^2 + \cdots + \alpha_n(t)A^{(n-1)};$$

then the expression of the output of the system is

$$y(t) = Ce^{At}x_{\bar{o}} = C(\alpha_1(t)I + \alpha_2(t)A + \alpha_3(t)A^2 + \cdots + \alpha_n(t)A^{(n-1)})x_{\bar{o}}.$$

or in matrix form

$$y = \begin{bmatrix}\alpha_1(t) & \alpha_2(t) & \alpha_3(t) & \dots & \alpha_n(t)\end{bmatrix}\begin{bmatrix}C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1}\end{bmatrix} x_{\bar{o}} = 0.$$

As the above expression is true for any value of t, it implies that

$$\begin{bmatrix}C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1}\end{bmatrix} x_{\bar{o}} = \mathcal{O}[A, C]x_{\bar{o}} = 0,$$

hence the unobservable states belong to the kernel (or null space) of the observability matrix, i.e. $x_{\bar{o}} \in \ker(\mathcal{O}[A, C])$. A second property that this expression gives us is that if $x_{\bar{o}} \in \ker(\mathcal{O}[A, C])$, then $e^{At}x_{\bar{o}} \in \ker(\mathcal{O}[A, C])$ for all $t \geq 0$. So if you start in the subspace of unobservable states, then you will stay there when you evolve the system. This property is called invariance with respect to the evolution $\dot{x} = Ax$. The final conclusion is that if $\mathcal{O}[\mathcal{A}, \mathcal{C}])$ is full rank, then the rank-nullity theorem ensure that the $\ker(\mathcal{O}[A, C]) = \{0\}$.

For completeness, let us mention that the subspace of controllable states is given by the column space of the controllability matrix, see Equation (4.16).

[3]Method 5 in goo.gl/2MKMux

4.3.3 Test for observability

In this section, we present a test for observability which can be used independently of the realization of $[A, C]$.

Result 4.3.2. *The pair $[A, C]$ is observable if and only if the matrix*

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4.40}$$

is full rank.

Another important matrix has been introduced in our lives, so let us define it properly.

Definition 4.3.3 (Observability matrix)**.** Given $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$, the observability matrix associated with the pair $[A, C]$, henceforth, $\mathcal{O}(A, C) \in \mathbb{R}^{nm\times n}$ is given by

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4.41}$$

For the case of a SISO system, i.e., $m = 1$, the observability matrix is square and full rank is reduced to nonsingular.

Result 4.3.4. *The pair $[A, C]$ with $C \in \mathbb{R}^{n\times 1}$, is observable if and only if the observability matrix is nonsingular, i.e.*

$$\det\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) \neq 0 \tag{4.42}$$

Once again, we should expect that such an important property does not depend on the basis that we choose to express A and C. The following result ensures that this property does not change due to a transformation.

Result 4.3.5. *Given the pair* $[A, C]$ *and a nonsingular matrix* T*, the pair* $[TAT^{-1}, CT^{-1}]$ *is observable if and only if* $[A, C]$ *is observable.*

Exercise 4.3.6. Show that $\mathcal{O}(TAT^{-1}, CT^{-1}) = \mathcal{O}(A, C)T^{-1}$. What conclusion can be drawn?

Finally, now we can understand why the observer canonical form has such a name.

Result 4.3.7. *If the pair* $[A, C]$ *is observable, then there exists a nonsingular matrix* T *such that the state space representation* $(TAT^{-1}, TB, CT^{-1}, D)$ *is in the observer canonical form.*

Finally, a result that we will use in Chapter 6 is given as follows:

Result 4.3.8. *A pair* $[A, C]$ *is unobservable if and only if* $Cv = 0$ *where* v *is a eigenvector of* A*.*

4.3.4 Worked example

Let us determine whether $[A, C]$ is observable where

$$A = \begin{bmatrix} -2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.43}$$

Then the observability matrix is

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -8 & -8 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 8 & 16 & 0 & 0 \\ 0 & 0 & 21 & 10 \\ 0 & -16 & 0 & 0 \\ 0 & 0 & -85 & -42 \end{bmatrix}. \tag{4.44}$$

Moreover, it is easy to check that

$$\begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -8 & -8 & 0 & 0 \\ 0 & 0 & -5 & -2 \end{vmatrix} = - \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -8 & 0 & -8 & 0 \\ 0 & -5 & 0 & -2 \end{vmatrix} = -64 \neq 0 \tag{4.45}$$

hence the rank of $\mathcal{O}(A,C) = 4$, i.e. $\mathcal{O}(A,C)$ is full-rank. As a result, the pair $[A,C]$ is observable.

Exercise 4.3.9. Use the MATLAB command `obsv` to find the observability matrix of the pair (A,C). Use the MATLAB command `rank` to find the rank of this matrix.

4.4 Detectability

Detectability is a relaxation of the observability concept just as stabilizability is a relaxation of controllability. Indeed, any observable pair $[A,C]$ will be detectable, but not any detectable pair $[A,C]$ will be observable.

Definition 4.4.1 (Detectability)**.** The pair $[A,C]$ is said to be detectable if there exists $L \in \mathbb{R}^{n\times m}$ such that the matrix $(A - LC)$ is Hurwitz.

The concept of detectability informs us about the possibility of designing a observer where the error between real state and observer state approaches zero as t goes to infinity (see Chapter 4). Therefore, if there are modes that cannot be observed, i.e. our pair $[A,C]$ is not observable, we need to ensure that these modes are stable.

We can state our test for checking if the pair $[A,C]$ is detectable as follows:

Result 4.4.2. *Let us assume that the system A is given in the form*

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, C = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{4.46}$$

where the pair $[A_{11}, C_1]$ is observable. Then the system is detectable if and only if A_{22} is Hurwitz.

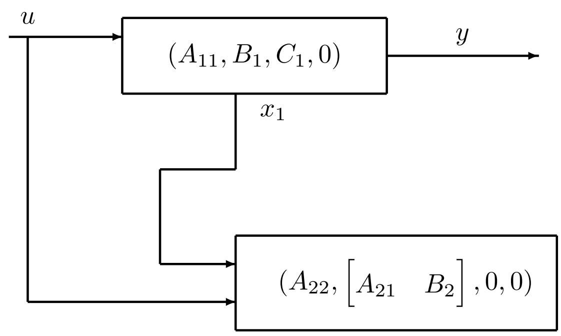


Figure 4.2: The unobservable system can be expresed as two subsystems: one where the state x_1 can be determined from the history of y, and another where the state x_2 does not have any ability to modify the output y.

Two straightforward conclusions can be drawn:

1. As commented, if $[A, C]$ is observable, then the system is detectable.

2. If A is Hurwitz, then the pair $[A, C]$ is detectable for any C.

A similar question arises: What happens if the system is not in the above form? The answer is the same: it is complicated but the same literature can be consulted if you want to have some fun. Once again, here we will just state that it is always possible to transform the pair $[A, C]$ into the above desired form.

Result 4.4.3. *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. If rank$(\mathcal{O}(A, C)) = \rho < n$, then there exists an nonsingular matrix T such that*

$$TAT^{-1} = \begin{bmatrix} X_{11} & 0 \\ X_{12} & X_{22} \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} Y_1 & 0 \end{bmatrix} \tag{4.47}$$

where $X_{11} \in \mathbb{R}^{\rho\times\rho}$ and $Y_1 \in \mathbb{R}^{m\times\rho}$ and the pair $[X_{11}, Y_1]$ observable.

In this course, we will transform the system into its modal form.

4.4.1 Worked example: Jan 2013 Exam

Q4 (Jan 2013 Exam) requires to decide whether the pair

$$A = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix} \tag{4.48}$$

is detectable.

The observability matrix is

$$\mathcal{O}(A, C) = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}, \tag{4.49}$$

and $\det(\mathcal{O}(A, C)) = 0$. Hence there is an unobservable mode and we need to determine if it is stable or unstable.

Using the previous result of Section 4.2.1

$$V = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{4.50}$$

where

$$V^{-1}AV = \begin{bmatrix} 5 & 0 \\ 0 & -10 \end{bmatrix}; \text{ and } CV = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix} \tag{4.51}$$

As a result, the unobservable mode is the mode $\lambda = -10$. Hence the unobservable mode is stable, and therefore the pair $[A, C]$ is detectable.

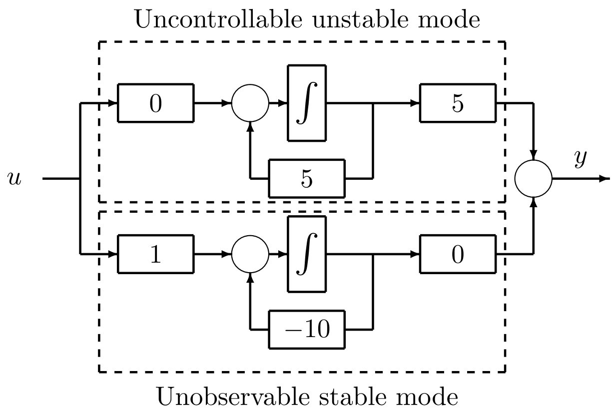


Figure 4.3: Block diagram of the modal form in Q4 (Jan 2013) and Exercise 4.4.4. It is clear that there is no possible connection between the input and the output.

If we sketch the block diagram of the modal form, it is very easy to understand what is happening in the system, see Fig 4.3.

Exercise 4.4.4. Show that the transfer function of the state-space representation

$$A = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad D = 0 \tag{4.52}$$

is zero, see Fig. 4.3. How many zeros has this system? Where are the zeros?

4.5 Final remarks

4.5.1 Duality

The student can check that my development of both previous sections is very similar and I hope there are no typos due to copy and paste. The symmetry between controllability and observability is known as Duality and it can be formally stated as follows:

Result 4.5.1 (Duality)**.** *Let $A \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times m}$. The pair $[A, X]$ is controllable if and only if the pair $[A^\top, X^\top]$ is observable.*

In the same way, stabilizability and detectability are also dual concepts.

Result 4.5.2. *Let $A \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times m}$. The pair $[A, X]$ is stabilizable if and only if the pair $[A^\top, X^\top]$ is detectable.*

This duality will also turn up in the design of controllers and observers (pole placement in Chapter 5 and Optimal Control and Estimation in Optimal and Robust Control unit, Semester 2)

4.5.2 Kalman's decomposition

The combination of Result 4.2.3 and Result 4.4.3 leads to the following result. This is the formal procedure that we use to obtain the controllability and observability of a system. However, it may be somehow difficult.

Result 4.5.3. *Any state-space representation can be transformed into a state-space representation in the following form*

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \tag{4.53}$$

$$y = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \tag{4.54}$$

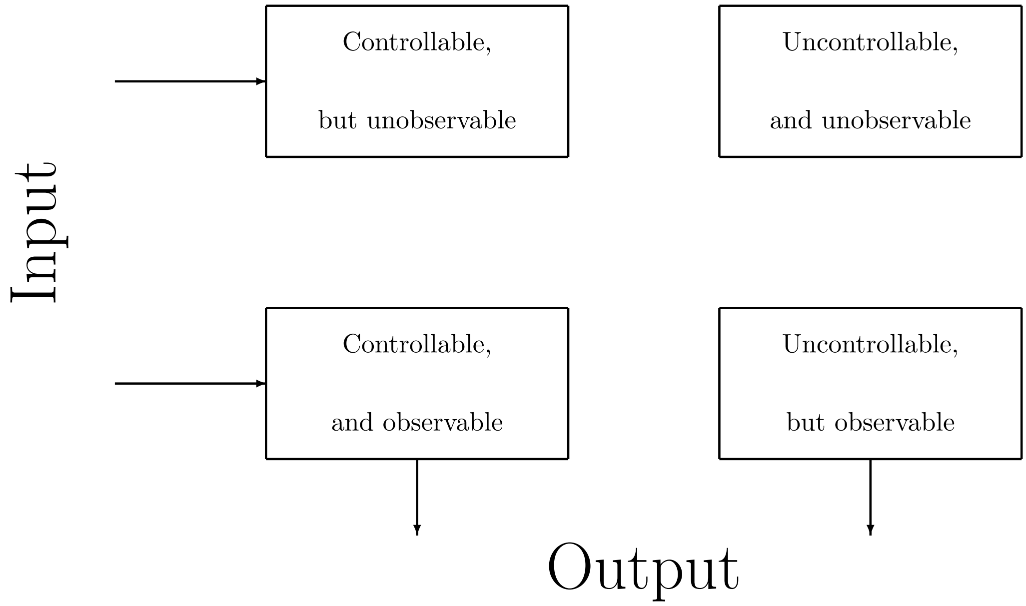
where the pair $\left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right]$ *is controllable and the pair* $\left[\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} C_2 & C_4 \end{bmatrix}\right]$ *is observable. Moreover, the tranfer function* $G(s) = C(sI - A)^{-1}B + D$ *is reducible to* $G(s) = C_2(sI - A_{22})^{-1}B_2 + D$.

Exercise 4.5.4. Given the system

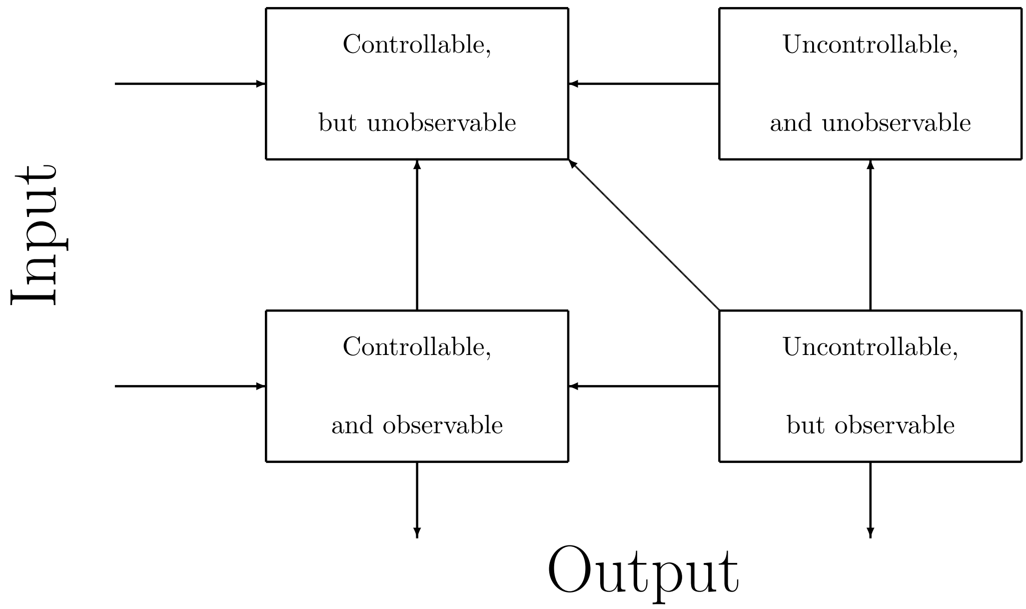
$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \text{ and } D = 0. \tag{4.55}$$

use the MATLAB command `minreal` to find the Kalman's decomposition of this system.

A formal procedure for this decomposition can be found in Chapter 6 in *A Linear Systems Primer* by Antsaklis and Michel and is highly recommended. A more advanced reference is Chapter 3 in *Robust and Optimal Control* by Zhou, Doyle, and Glover, where results can be found with formal proofs.



(a) Diagonal Kalman's decomposition. In this case there is no interaction between the four part of the system



(b) General Kalman's decomposition: Some interaction between the block.

Figure 4.4: Kalman's decomposition. The MATLAB function `minreal` eliminates all states that are not in the controllable and observable block. It is stated that `minreal` can also provide Kalman's decompostion, but I recomend some critical thinking.

4.6 Learning Outcomes

The learning outcomes of this chapter can be summarised as follows:

- Definition of controllability of a linear system or the pair $[A, B]$.
- The pair $[A, B]$ is controllable if a only if the controllability matrix

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{4.56}$$

  is full rank.
- When the pair $[A, B]$ is not controllable but all the uncontrollable modes are stable, then we say the pair $[A, B]$ is stabilizable. The modal form of A can be used to test this.
- Definition of observability of a linear system or the pair $[A, C]$.
- The pair $[A, C]$ is observable if a only if the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{4.57}$$

  is full rank.
- When the pair $[A, C]$ is unobservable but all the unobservable modes are stable, then we said the pair $[A, C]$ to be detectable. The modal form of A can be used to test this.
- There exists a duality between controllability and observability: The pair $[A, X]$ is controllable if and only if the pair $[A^\top, X^\top]$ is observable.
- Kalman's decomposition splits the system into four blocks: a part that can be controlled and observed, a part that just can be controlled, a part that just can be observed and a part that cannot be either observer or controlled.

Chapter 5

Design in the state-space

Until now, the unit has covered the underpinning principles that will be needed to design controllers and observers. From now on, we will be focused on using the previous chapters from a control engineering point of view.

At first, we will assume that we can look inside the system, and all states can be perfectly known. With this information, we will design a controller that modifies the original dynamics of the system in a predefined manner. However, this assumption is not very realistic in most systems. Hence, secondly, we will need to study how to discover what is happening inside the system using just "input and output information". Finally, we will combine both designs in order to obtain a realistic controller.

A very important part of control, tracking control, will not be covered in this unit. State-space control can cope with set-points, and we will study this issue in the second semester.

5.1 State-feedback controller

This section will propose a problem, the design of a control gain in order to have an appropriate behaviour of the closed-loop system, see Fig 5.1. The problem is rewritten in a mathematical fashion. As with many mathematical problems, we will find situations where the problem is correctly defined and we have at least one solution, but we will also easily find situations where the problem is ill-defined and no solution can be found. When we are able to guarantee the existence of a solution, we will study the solution of the problem.

5.1.1 Design problem

The design of a state feedback controller can be stated as the following design problem:

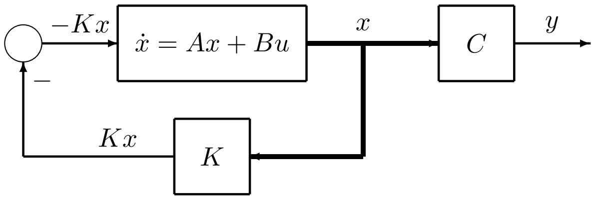


Figure 5.1: State feedback problem.

Problem 5.1.1 (Pole placement)**.** Given the state-space equation

$$\dot{x} = Ax + Bu, \tag{5.1}$$

find the state feedback law $u = -Kx$ such that the poles of the dynamical system

$$\dot{x} = (A - BK)x \tag{5.2}$$

are placed at $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

As we are considering only the input $u = -Kx$ we assume that the reference signal will be always null, and we focus our attention on achieving the steady state smoothly. This control design problem is just a mathematical problem that can be stated as follows.

Problem 5.1.2. Given the pair $[A, B]$ where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, find $K \in \mathbb{R}^{m\times n}$ such that the eigenvalues of $A - BK$ are given by $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,

Remark 5.1.3. Note that we could ask for complex eigenvalues $\alpha_i \in \mathbb{C}$, but we should include their complex conjugate, i.e., there exist $1 \leq j \leq n$ such that $\alpha_j = \alpha_i^*$, if we want $K \in \mathbb{R}^{m\times n}$. If we do not include the complex eigenvalues in pairs, then $K \in \mathbb{C}^{m\times n}$. We will always include complex conjugate pairs in the desired values α_i, but interested students are welcome to study other cases.

Example 5.1.4. Let us consider $A = 5$ and $B = 1$, find $K \in \mathbb{R}$ such that the eigenvalues of $(A - BK) = -2$, i.e.

$$(5 - 1K) = -2. \tag{5.3}$$

Then the solution of the problem is $K = 7$. △

Example 5.1.5. Let us consider $A = 5$ and $B = 0$, find $K \in \mathbb{R}$ such that the eigenvalues of $(A - BK) = -2$, i.e.

$$(5 - 0K) = -2. \tag{5.4}$$

In this example, we would like to stabilise the mode of this unstable system. However, it is trivial that it is impossible. For any value of K, the pole of the system will at 5. △

5.1.2 Existence of solutions

The above example has provided a trivial situation where we cannot find a solution of the control design problem 5.1.1 or the mathematical problem 5.1.2. So the first thing that we need to figure out is when the problem is well defined.

Result 5.1.6. *Problem 5.1.1 has a solution for any $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ if and only if the pair $[A, B]$ is controllable.* ■

The mathematical notion of controllability that we have studied in Chapter 4, is the key point to be able to solve the control problem 5.1.1. If the pair $[A, B]$ contains a mode that is uncontrollable, then this mode will not be modified by the selection of K. As a result, the eigenvalues of $A - BK$ will always contain the value of the uncontrollable mode.

Example 5.1.7. Given the pair $A = 5$ and $B = 0$, then the controllability matrix is given by $\mathcal{C}(A, B) = 0$. △

We can place the modes that are controllable but we must to preserve the uncontrollable modes in our design.

Result 5.1.8. *Let the pair $[A, B]$ to be uncontrollable and let $\{\lambda_1, \lambda_2, \ldots, \lambda_\rho)\}$ be the eigenvalues of the uncontrollable modes. Then, Problem 5.1.1 has a solution if and only if the desired set of eigenvalues of $A - BK$ is of the form $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-\rho}, \lambda_1, \lambda_2, \ldots, \lambda_\rho\}$.* ■

Example 5.1.9. Let us consider $A = 5$ and $B = 0$, find $K \in \mathbb{R}$ such that the eigenvalues of $(A - BK) = 5$, i.e.

$$(5 - 0K) = 5. \tag{5.5}$$

Hence the above problem has a solution, in fact, has infinite number of solutions, any $K \in \mathbb{R}$ is a solution of the problem. △

Finally, one could wonder if we could solve Problem 5.1.1 but instead of any location of the poles, we are just interested in placing the poles in the LHP, i.e. given the pair $[A, B]$, we want to find K such that $A - BK$ is Hurwitz. Now Definition 4.2.1 seems natural. Controllability offer us the possibility of designing the placement of all the poles, whereas stabilizability ensures that we will be able to place all poles in the LHP.

5.1.3 Worked example

Let us consider the system

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u,$$

and design a state feedback controller such that the poles of the closed-loop systems are located at $\{-1, -2, -3\}$, assuming that we have access to all states.

It should not be strange that the control canonical form of a system is desired to solve the control design. The control canonical form of the above transfer function is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \tag{5.6}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \tag{5.7}$$

Let us consider the control action $u = -Kx$, the closed-loop system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+2) & -(k_2+3) & -(k_3-5) \end{bmatrix} x; \tag{5.8}$$

and we need to choose k_1, k_2, and k_3 to fulfil the design specification. To this end, the characteristic equation is computed

$$\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ (k_1+2) & (k_2+3) & \lambda + (k_3+5) \end{bmatrix} = (\lambda^3 + (k_3+5)\lambda^2 + (k_2+3)\lambda + (k_1+2), \tag{5.9}$$

and compared with the desired

$$(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6. \tag{5.10}$$

Then, $k_3 = 1$, $k_2 = 8$, and $k_1 = 4$. In summary, the designed gain is

$$K = \begin{bmatrix} 4 & 8 & 1 \end{bmatrix} \tag{5.11}$$

Note that this values depends on the realisation of the system, so if we apply a transformation $z = Tx$, K in the new basis will be different.

5.1.4 Worked example: Jan 2013 exam

Q4 (Jan 2013 Exam) requires to decide whether the poles of $(A - BK)$ can be placed at $-1 \pm j$, with

$$A = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{5.12}$$

This example has been studied in Chapter 4, where we have found that the system is not fully controllable. Since poles of A are in $\{5, -10\}$, applying Result 5.1.8 we conclude that it is not possible to place the poles of $A - BK$ at this location since either 5 or -10 must be included in the desired location. Here we are going to show that it is actually true.

Let us analyse the eigenvalues of $A - BK$:

$$A - BK = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -7 + 2k_1 & 6 + 2k_2 \\ 6 - k_1 & 2 - k_2 \end{bmatrix} \tag{5.13}$$

The eigenvalues of $A - BK$ satisfy

$$\det(\lambda I - (A - BK)) = \begin{vmatrix} \lambda + 7 - 2k_1 & -6 - 2k_2 \\ -6 + k_1 & \lambda - 2 + k_2 \end{vmatrix} = 0. \tag{5.14}$$

Computing this determinant, we find that the characteristic equation is given by

$$\lambda^2 + (5 - 2k_1 + k_2)\lambda + (-50 + 10k_1 - 5k_2) = 0 \tag{5.15}$$

Then, solving this second order equation and using $k = -2k_1 + k_2$

$$\lambda = \frac{-(5+k) \pm \sqrt{(5+k)^2 - 4(-50-5k)}}{2} = \frac{-(5+k) \pm \sqrt{k^2 + 30k + 225}}{2} = \frac{-(5+k) \pm (k+15)}{2} \tag{5.16}$$

As a result, the eigenvalues of $A + BK$ are given by

$$\lambda_1 = \frac{-(5+k) + (k+15)}{2} = 5; \tag{5.17}$$

$$\lambda_2 = \frac{-(5+k) - (k+15)}{2} = -10 + 2k_1 - k_2. \tag{5.18}$$

In summary, we have demonstrated that Result 5.1.8 is satisfied for this example. Furthermore, this has provided a less elegant procedure to find that the system is not stabilizable, since $A - BK$ will have a pole in the RHP for any values (k_1, k_2).

5.1.5 Solution of the Pole Placement Problem: Ackermann's formula

If the state space representation is in the control canonical form, we have shown that it is straightforward to find the solution of the problem. However, the previous method could be somewhat complex in other representations. The general solution to the Pole Placement problem was given by Juergen Ackermann in 1972 for SISO systems. In this case, if the system is controllable, there is one and only one solution.

Result 5.1.10. *Given the pair* $[A, B]$ *where* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times 1}$*, then the matrix* $A - BK$ *has eigenvalues at* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *if* $K \in \mathbb{R}^{1\times n}$ *is given by Ackermann's formula:*

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}(A, B)\beta(A) \tag{5.19}$$

where $\beta(A) = A^n + \beta_{n-1}A^{n-1} + \beta_{n-2}A^{n-2} + \cdots + \beta_1 A + \beta_0 I$ *with* β_i *the coefficients of the desired characteristic polynomial, i.e.*

$$\beta(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\ldots(\lambda - \alpha_{n-1})(\lambda - \alpha_n). \tag{5.20}$$

■

Once again, we need to highlight that the coefficient of the characteristic polynomial will be real if poles are specified including complex conjugate pairs. If not, we will able to find a solution, but $K \in \mathbb{C}^{1\times n}$.

MATLAB command is `acker`. For MIMO systems, it is possible that the simultaneous equation to be solved be underdetermined, i.e. there are infinite solutions to the Pole Placement problem. Moreover, Ackermann's formula will have issues related to inverting high order matrices. To solve both problems, other algorithms has been proposed and MATLAB offers an alternative command (`place`). These algorithms are beyond the scope of these notes but, for further discussion, see J. Kautsky, N. K. Nichols, and P. Van Dooren, "Robust Pole Assignment in Linear State Feedback," International Journal of Control, 41 (1985), pp. 1129-1155.

5.1.6 Worked Example: Q3 Jan 2013 Exam

Given the system

$$\dot{x} = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u, \tag{5.21}$$

design a state-feedback controller such that the poles of the closed-loop system are located at $\{-1 \pm j\}$.

The first step is to check if the problem will have a solution, so the controllability matrix $\mathcal{C}(A, B)$ need to be invertible:

$$\mathcal{C}(A, B) = \begin{bmatrix} 3 & -6 \\ 1 & -10 \end{bmatrix} \text{ and so } \begin{vmatrix} 3 & -6 \\ 1 & -10 \end{vmatrix} = -30 + 6 = -24 \neq 0 \tag{5.22}$$

hence this matrix is nonsingular. Moreover, we will need to find its inverse

$$\mathcal{C}^{-1}(A, B) = \frac{-1}{24} \begin{bmatrix} -10 & 6 \\ -1 & 3 \end{bmatrix}. \tag{5.23}$$

We need the desired characteristic polynomial, it is given by

$$\beta(\lambda) = (\lambda - (-1 + j))(\lambda - (-1 + j)) = \lambda^2 + 2\lambda + 2. \tag{5.24}$$

Then,

$$\beta(A) = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}^2 + 2\begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} + \begin{bmatrix} -2 & -6 \\ -6 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \tag{5.25}$$

Now we are able to find K by applying Ackermann's formula as follows

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1}(A, B)\beta(A) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{-1}{24} \begin{bmatrix} -10 & 6 \\ -1 & 3 \end{bmatrix} 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{5}{12} \begin{bmatrix} 1 & -3 \end{bmatrix}. \tag{5.26}$$

Exercise 5.1.11. Use Ackermann's formula to reproduce the results found in Section 5.1.3. Use also the command `acker`.

5.1.7 Final discussion

Even though we have solved the design problem in a very nice fashion, the real problem remains. What is a good location of the poles of the system? It will be the work of the control engineer to understand the problem and design a correct location of the poles. If poles are located in such a way that the state reaches zero very fast, it is nice as long as my actuator is able to cope with the "energy" that the control action will require to modify the state of the system. A less demanding controller can be designed by choosing the poles location in a slower region, i.e. closer to the imaginary axis, but always in the LHP, evidently!

5.2 Observer design

Duality between control and observation or estimation has been commented at the end of Chapter 4. Once again, we are going to exploit this duailty here, and we will reproduce the same notions as in the previous section. As some students may not be familiar with the concept of observer, we are going to start by explaining the notion and usefulness of observers.

5.2.1 Introduction to the concept of observer

As previously mentioned, the availability of all states in the feedback is a very strong assumption. In most of the cases, some states of the plant will not be accessible. The target of an

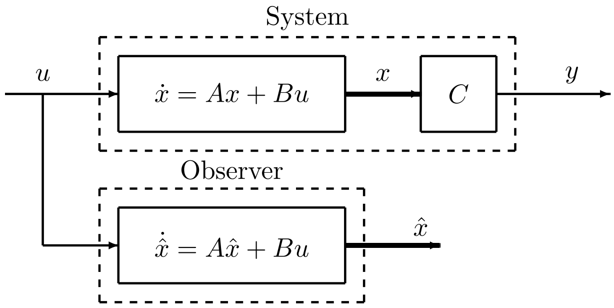


Figure 5.2: Observer design assuming that $x(0) = \hat{x}(0)$.

observer is to provide the information of what is happening inside the system by the use of a virtual system, which will be referred to as an observer or estimator. This virtual system will modify its state in such a way that, after some time, the states of the system and the state of the observer match.

What virtual system are we going to use as an observer? Firstly, let us assume that if both systems have the same state at some instant and if the dynamics of the states and inputs are the same for both systems, then the states will match at any future instant. So it seems natural to use an artificial system with the same dynamic.

Let us consider the strictly proper system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0; \tag{5.27}$$

$$y(t) = Cx(t) \tag{5.28}$$

and let observer be given by the same dynamic

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad \hat{x}(0) = x_0 \tag{5.29}$$

then, it is trivial to see that $\hat{x}(t) = x(t)$ for all $t \in \mathbb{R}$. The state $\hat{x}$ is referred to as estimation of the state x. As a result, the use of the observer allows us to see what is happening inside the system; one can think of the observer as "virtual sensor" that is measuring x.

Evidently, our assumption of $x(0) = \hat{x}(0)$ is totally unrealistic. Hence, if it does not hold, we need to inform the observer that something is wrong. As the only information about the state x is given through the output $y = Cx$, the feedback to the observer must be given as the difference between the real output of the system and the estimated output $\hat{y} = C\hat{x}$. Hence, the observer will need to modify its dynamics as a function of the difference $y - \hat{y}$, so the final expression for the observer is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}), \tag{5.30}$$

where $L \in \mathbb{R}^{n \times n_y}$. Then, let us define the error state as:

$$e(t) = x(t) - \hat{x}(t); \tag{5.31}$$

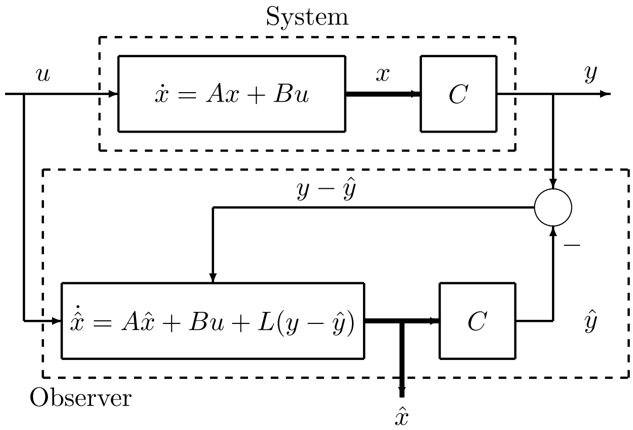


Figure 5.3: Observer design where we inform to the observer about a possible error between x and $\hat{x}$.

and its dynamics equation is

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) + Bu(u) - (A\hat{x} + Bu(t) + LCx - LC\hat{x}) = (A - LC)e(t). \quad (5.32)$$

Therefore, by construction, the observer results in a "magical system" called the error system, which is an autonomous dynamical system governed by the matrix $A - LC$. We say this system is "magical" in the sense that it cannot be realised. It would require the information x that is unavailable. Note the difference with the observer, which may only exist inside a computer or PLC, but can be realised.

In the following subsection, we are going to follow the same flow as Section 5.1.

5.2.2 Observer design

Similar to the case of state-feedback design where we need to place the poles of $A - BK$ in a desired location, we can state a new design problem where we need to place the poles of the error system

Problem 5.2.1 (Observer design)**.** Given the strictly proper system

$$\dot{x} = Ax + Bu, \quad (5.33)$$

$$y = Cx, \quad (5.34)$$

and the observer system

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly; \quad (5.35)$$

find the value of L such that the poles of the error system

$$\dot{e} = (A - LC)e \quad (5.36)$$

are placed at $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Once again, the observer design problem can be translated into a mathematical problem. Although we have included the matrix B, the solution will be independent of B.

Problem 5.2.2. Given the pair $[A, C]$ where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$, find $L \in \mathbb{R}^{n\times m}$ such that the eigenvalues of $A - LC$ are given by $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,

Remark 5.2.3. The same comments about the inclusion of complex conjugate poles hold here as in the control design. Moreover, we have included B in Problem 5.2.1, but as our design is just interested in $A - LC$, the solution will be independent of B. Since the observer design is independent of B, students can understand now why we defined observability using the autonomous system, i.e., $B = 0$.

The student will be able to find examples where the pair $[A, C]$ does not allow us to fix the poles of the error system at a desired location. Hence, the first step is to find conditions to ensure the existence of a solution.

5.2.3 Existence of solutions

Result 5.2.4. *Problem 5.2.1 has a solution for any $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ if and only if the pair $[A, C]$ is obsevable.* ■

The mathematical notion of observability that we have studied in Chapter 4, is the key point to be able to solve the observer desing problem. If the pair $[A, C]$ contains a mode that is unobservable, then the error system will contain this mode regardless of the selection of L. As a result, the eigenvalues of $A - LC$ will always contain the value of the unobservable mode.

Exercise 5.2.5. Design an uncontrollable pair $[A, C]$ and then choose randomly L in MATLAB, with adequate dimensions. Check that at least one eigenvalue of $[A, C]$ corresponds with an eigenvalue of A. △

We can place the modes that are observable but we must preserve the unobservable modes in our design.

Result 5.2.6. *Let the pair $[A, C]$ to be unobservable and let $\{\lambda_1, \lambda_2, \ldots, \lambda_\rho)\}$ be the eigenvalues of the unobservable modes. Then, Problem 5.1.1 has a solution if and only if the desired set of eigenvalues of $A - LC$ is of the form $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-\rho}, \lambda_1, \lambda_2, \ldots, \lambda_\rho\}$.* ■

Finally, one could wonder if we could solve Problem 5.2.1 but instead of any location of the poles, we are just interested in placing the poles in the LHP, i.e. given the pair $[A, C]$, we want to find L such that $A - BK$ is Hurwitz. This will ensure that after some time, the state of the error system will approach zero, or equivalenty, the state $\hat{x}$ will approach the state x, with is the main target of the observer.

Now, Definition 4.4.1, the definition of detectability, seems natural. Observability allows us the possibility of designing the error system with special dynamics, whereas detectability just ensures that we will be able to place all poles of the error system in the LHP. The consequence of the lack of observability is the lack of freedom in the location of the poles. This lack of freedom becomes an issue when the poles of the observer stay in the RHP, then the state x will never approach $\hat{x}$, hence we cannot design a suitable observer.

5.2.4 Worked example

Let us consider the system

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u,$$

design an observer such that the poles of the error systems are located at $\{-1, -1+j, -1-j\}$.

In this case, it should not be strange that the observer canonical form of a system is desired to solve the observer design problem. The observer canonical form of the above transfer function is

$$\dot{x} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{5.37}$$

and the output is given by

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \tag{5.38}$$

Then, we need to design L such that the eigenvalues of $A - LC$ are $\{-1, -1+j, -1-j\}$. Firstly, let us analyse the location of the eigenvalues of

$$A - LC = \begin{bmatrix} 0 & 0 & -2 - l_1 \\ 1 & 0 & -3 - l_2 \\ 0 & 1 & -5 - l_3 \end{bmatrix}; \tag{5.39}$$

Then, the characteristic equation is computed

$$\det(I\lambda - (A - LC)) = \begin{vmatrix} \lambda & 0 & (2 + l_1) \\ -1 & \lambda & (3 + l_2) \\ 0 & -1 & \lambda + (5 + l_3) \end{vmatrix} = \lambda^3 + (l_3 + 5)\lambda^2 + (l_2 + 3)\lambda + (l_1 + 2), \tag{5.40}$$

and compared with the desired

$$(\lambda + 1)(\lambda + 1 + j))(\lambda + 1 - j) = \lambda^3 + 3\lambda^2 + 4\lambda + 2. \tag{5.41}$$

Then, $l_3 = -2$, $l_2 = 1$, and $l_1 = 0$. In summary, the designed gain is

$$L = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}. \tag{5.42}$$

5.2.5 Solution of the problem

Until now, we have exploited the duality property just to follow the same steps. Now we are going to use this duality for solving the problem. The following result is based in the fact that a matrix and its transpose have the same eigenvalues.

Result 5.2.7. *The eigenvalues of $A - LC$ are the same as the eigenvalues of $A^\top - C^\top L^\top$.*

Then, the problem of designing a observer for the pair $[A, C]$ becomes the problem of designing a controller K for the pair $[A^\top, C^\top]$ and setting $L = K^\top$. Hence the same commands can be used in MATLAB, `L=acker(A',C', α)'` or `L=place(A',C', α)'`, where α is the desired characteristic polynomial.

Exercise 5.2.8. Use Result 5.2.7 to show that the Observer version of the Ackermann's formula is given by

$$L = \beta(A)\mathcal{O}^{-1}(A, C) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5.43}$$

Exercise 5.2.9. Use Ackermann's formula to reproduce the results in Section 5.2.4. Use also the MATLAB command `acker`

5.2.6 Final discussion

As in the previous discussion, the main problem remains: What is a good location of the poles of the error system? It will depend on the confidence that we have in our model and the noise of the sensor. One could think that a slow design for the observer will demand low confidence in model and output, whereas a fast design will requires a high fidelity in the model and low noise ratio. If we have a good model but noisy output, some states could be estimated faster than others. However, it is very difficult to quantify the above statement.

These questions and issues will be tackled in Optimal and Robust Control in the second semester.

5.3 Output feedback design

The last section of this chapter combines the two elements that we have presented previously. On the one hand, we will design an observer as a virtual sensor of all the states of the system. On the other hand, we will use the estimated states to develop a control action, see Fig. 5.4.

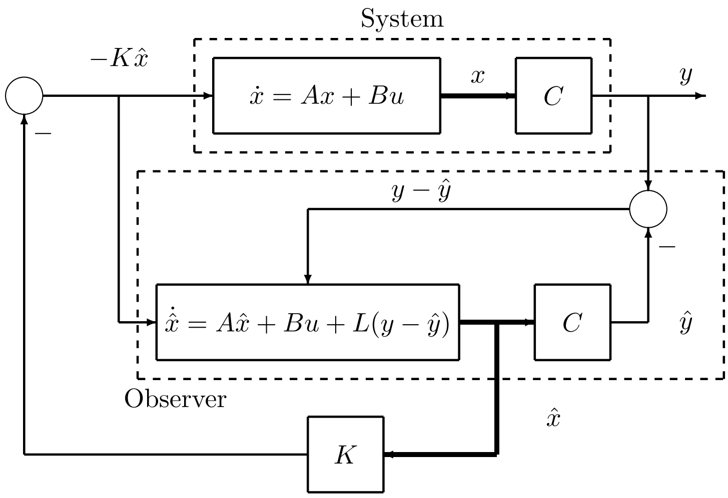


Figure 5.4: Output feedback controller. The closed-loop system between u and y contain two set of states, the state of the plant and the state of the observer.

5.3.1 Separation principle

The separation principle states that the design of an output-feedback controller with the poles of the state-feedback at the location $\{\alpha_i^c\}$ and the error system at the location $\{\alpha_i^e\}$ can be designed independently.

On the one hand, design a state-feedback gain K such that the eigenvalues of $A - BK$ are placed at $\{\alpha_i^c\}$ and, on the other hand, design observer gain L such that the eigenvalues of

$A - LC$ are at $\{\alpha_i^e\}$. Then, the system

$$\begin{aligned} \dot{x} &= Ax + Bu, && (5.44)\\ y &= Cx, && (5.45)\\ \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}), && (5.46)\\ u &= -K\hat{x} && (5.47) \end{aligned}$$

will have the desired properties.

To this end, let us consider the state $x_{cl} = (x, e)$, where $e = x - \hat{x}$. It follows

$$\dot{x} = Ax - BK\hat{x} = Ax - BK(x - e) = Ax - BKx + BKe = (A - BK)x + BKe, \qquad (5.48)$$

and

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax - BK\hat{x} - A\hat{x} + BK\hat{x} + L(Cx - C\hat{x}) = Ae - LCe = (A - LC)e. \qquad (5.49)$$

As a result the closed-loop systems behave as the autonomous system

$$\frac{d}{dt}\left(\begin{bmatrix} x \\ e \end{bmatrix}\right) = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}. \qquad (5.50)$$

The result is obtained applying a well-known result in matrix algebra:

Result 5.3.1. *The set of eigenvalues of the matrix*

$$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \qquad (5.51)$$

is the union of the set of eigenvalues of X and the set of eigenvalues of Z

As a result, the eigenvalues of the matrix in the right-hand side of (5.50) is the union of the eigenvalues of $A - BK$ and $A - LC$. These problems are identical to Problem 5.1.1 and 5.2.1, whose solutions have been presented in previous sections.

5.3.2 Design considerations

The design specification of the controller are straightforward: fast poles will reject disturbances quickly, but the actuator will suffer to provide this control action. Evidently, the effort of the actuator can be reduced by slowing the poles down. Pole placement is a very primitive method, and more sophisticated methods will be considered in the second semester.

However, the observer design can be slightly more difficult. The control engineer will need to understand the limitation of the bandwidth of the system. The design of the observer can be as fast as we wish, since there is not a physical actuation. Once we have understood how fast the observer can be, the controller should be designed to be 5-10 times slower.

5.4 Learning outcomes

The learning outcomes of this chapter can be summarised as follows:

- State-feedback design provides a gain K such that the poles of $A - BK$ are located at some predetermined location of the complex plane.
- If the pair $[A, B]$ is controllable, then we will be able to find a solution of the state-feedback problem. If the pair $[A, B]$ has uncontrollable modes, we will not be able to modify these modes in $A - BK$ regardless of the selection of the gain K.
- Ackermann's formula provides a sophisticated manner to find the solution for SISO systems.
- The observer of a system is a "virtual sensor" that measures the state of the system. The error between the state x and the estimated state $\hat{x}$ behaves as an autonomous dynamical system $\dot{e} = (A - LC)e$.
- The observer design provides a gain L such that the poles of $A - LC$ are located at some predetermined location of the complex plane.
- If the pair $[A, C]$ is observable, then we will be able to find a solution of the observer design problem.
- When the pair $[A, C]$ is unobservable but all the unobservable modes are stable, then we can still design an observer but we cannot "move" the unobservable modes.
- There exists a duality between state-feedback control and observer design: the matrices $A - XY$ and $A^\top - Y^\top X^\top$ have the same eigenvalues.
- The output feedback design is the combination of the state-feedback and observer designs.
- The separation principle ensures that they can be independently designed, and we will keep their original properties when combined.
- The designer must understand the limitations of actuators and sensors to produce a sound design. Pole placement is a very rudimentary design technique but straightforward.

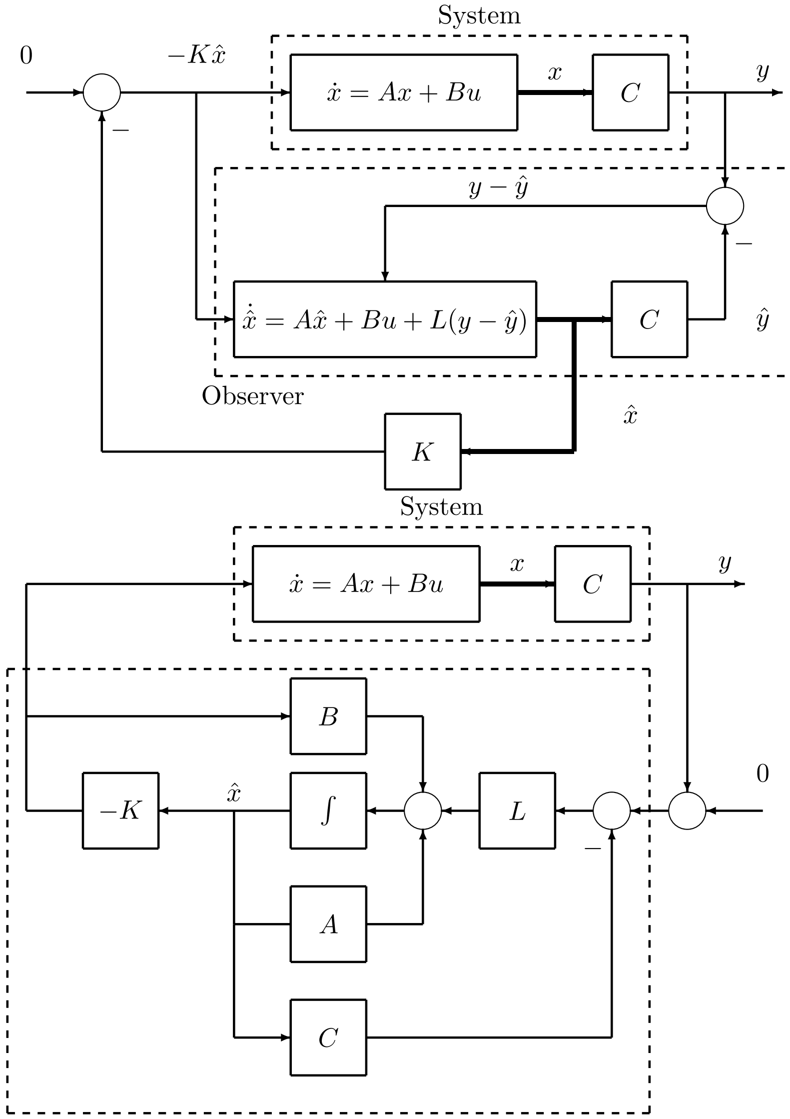


Figure 5.5: Transformation Step 1

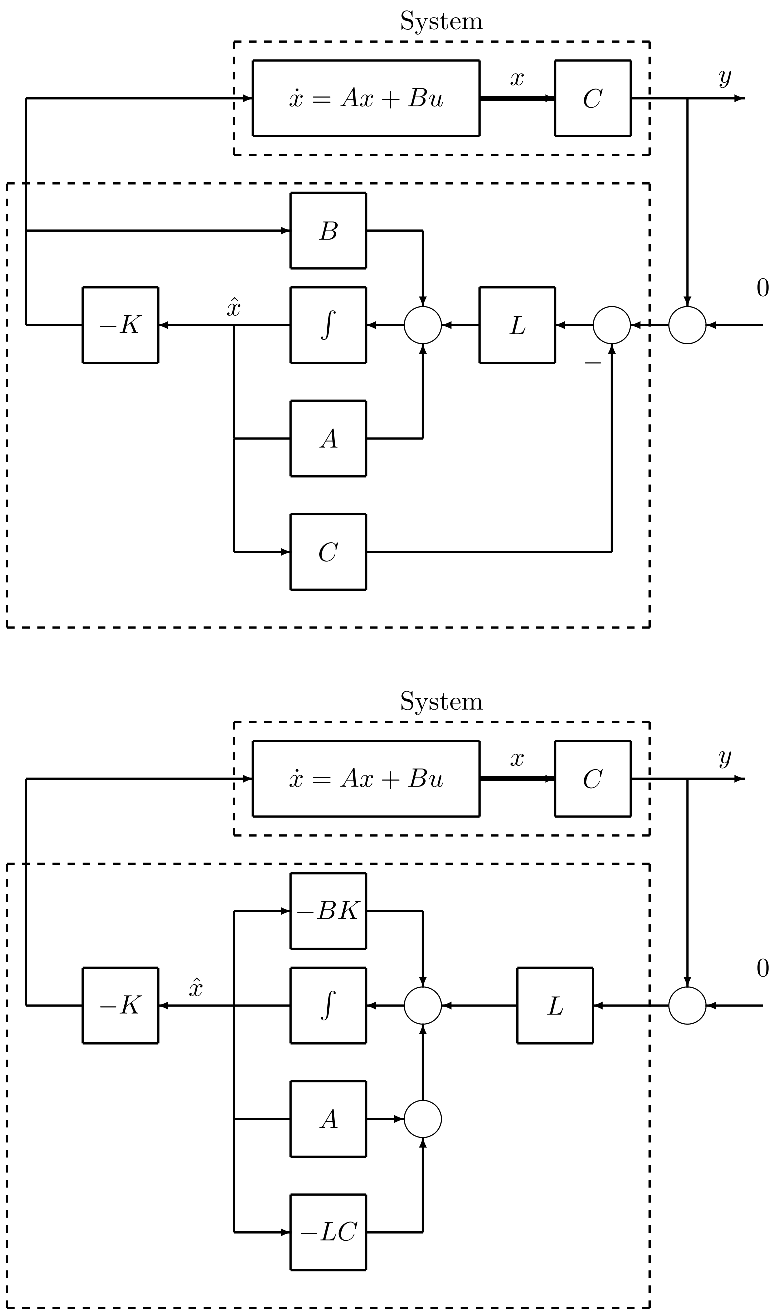


Figure 5.6: Transformation Step 2

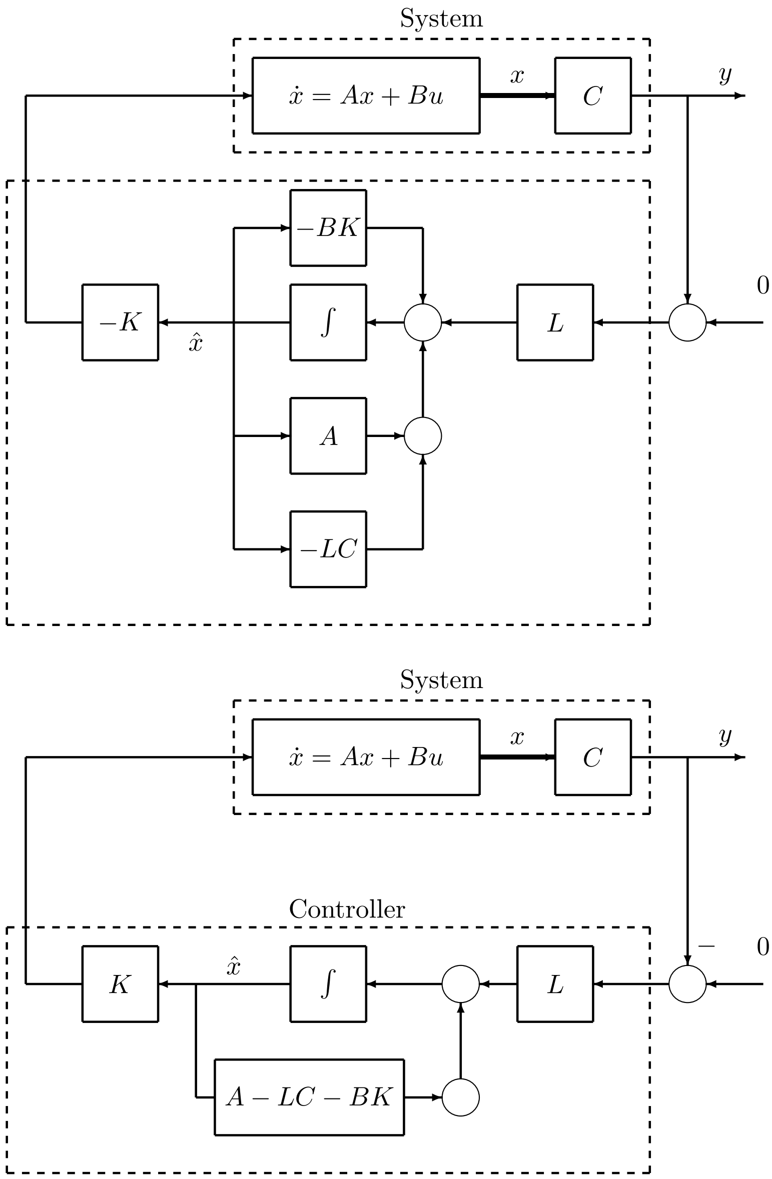


Figure 5.7: Transformation Step 3. As a result, $C(s) = K(sI - (A - LC - BK))^{-1}L$

Chapter 6

Realisation of MIMO transfer functions

Undergraduate Control Engineering textbooks introduce transfer functions from ODEs. However, they restrict their attention to SISO systems. On the other hand, Advanced Control Engineering books use transfer function matrices with no introduction. The following sections provide an introduction to MIMO transfer function.

6.1 MISO systems: transfer function column-vector

A general differential equation with k inputs and one output is very similar to the SISO version:

$$\phi(\{y^{(i)}\}_{i=0}^{n}, \{u_1^{(i)}\}_{i=0}^{m_1}, \{u_2^{(i)}\}_{i=0}^{m_2}, \dots, \{u_k^{(i)}\}_{i=0}^{m_k}) = 0, \tag{6.1}$$

where $m_i < n$ for all $1 < i < k$ and $z^{(i)}$ means $\frac{d^{(i)}z}{dt^{(i)}}$. Undoubtedly we have more freedom in the forced component of the equations, but solving this differential equation is equivalent to solving a SISO system once all inputs have been decided.

If the system is linear, equation (6.1) becomes an ODE

$$\begin{aligned}
&\frac{d^n}{dt^n}y + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + a_{n-2}\frac{d^{n-2}}{dt^{n-2}}y + \dots + a_2\frac{d^2}{dt^2}y + a_1\frac{d}{dt}y + a_0 y = \\
&\quad b^1_{m_1}\frac{d^{m_1}}{dt^{m_1}}u_1 + b^1_{m_1-1}\frac{d^{m_1-1}}{dt^{m_1-1}}u_1 + b^1_{m_1-2}\frac{d^{m_1-2}}{dt^{m_1-2}}u_1 + \dots + b^1_1\frac{d}{dt}u_1 + b^1_0 u_1 + \\
&\quad b^2_{m_2}\frac{d^{m_2}}{dt^{m_2}}u_2 + b^2_{m_2-1}\frac{d^{m_2-1}}{dt^{m_2-1}}u_2 + b^2_{m_2-2}\frac{d^{m_2-2}}{dt^{m_2-2}}u_2 + \dots + b^2_1\frac{d}{dt}u_1 + b^2_0 u_2 + \dots \\
&\quad b^k_{m_k}\frac{d^{m_k}}{dt^{m_k}}u_k + b^k_{m_k-1}\frac{d^{m_k-1}}{dt^{m_k-1}}u_k + b^k_{m_k-2}\frac{d^{m_k-2}}{dt^{m_k-2}}u_k + \dots + b^k_1\frac{d}{dt}u_k + b^k_0 u_k.
\end{aligned} \tag{6.2}$$

Using the Laplace transform, then we obtain

$$\begin{aligned} s^nY + a_{n-1}s^{n-1}Y + a_{n-2}s^{n-2}Y + \cdots + a_2s^2Y + a_1sY + a_0Y = \\ b^1_{m_1}s^{m_1}U_1 + b^1_{m_1-1}s^{m_1-1}U_1 + b^1_{m_1-2}s^{m_1-2}U_1 + \cdots + b^1_1sU_1 + b^1_0U_1 + \\ b^2_{m_2}s^{m_2}U_2 + b^2_{m_2-1}s^{m_2-1}U_2 + b^2_{m_2-2}s^{m_2-2}U_2 + \cdots + b^2_1sU_2 + b^2_0U_2 + \ldots \\ b^k_{m_k}s^{m_k}U_k + b^k_{m_k-1}s^{m_k-1}U_k + b^k_{m_k-2}s^{m_k-2}U_k + \cdots + b^k_1sU_k + b^k_0U_k. \end{aligned} \quad (6.3)$$

Finally, we take common factors Y, U_1, U_2, …, and U_k, so it follows that

$$\begin{aligned} (s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0)Y = \\ (b^1_{m_1}s^{m_1} + b^1_{m_1-1}s^{m_1-1} + b^1_{m_1-2}s^{m_1-2} + \cdots + b^1_1s + b^1_0)U_1 + \\ (b^2_{m_2}s^{m_2} + b^2_{m_2-1}s^{m_2-1} + b^2_{m_2-2}s^{m_2-2} + \cdots + b^2_1s + b^2_0)U_2 + \ldots \\ (b^k_{m_k}s^{m_k} + b^k_{m_k-1}s^{m_k-1} + b^k_{m_k-2}s^{m_k-2} + \cdots + b^k_1s + b^k_0)U_k. \end{aligned} \quad (6.4)$$

which provides the desired result

$$\begin{aligned} Y = \frac{b^1_{m_1}s^{m_1} + b^1_{m_1-1}s^{m_1-1} + b^1_{m_1-2}s^{m_1-2} + \cdots + b^1_1s + b^1_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0}U_1 + \\ \frac{b^2_{m_2}s^{m_2} + b^2_{m_2-1}s^{m_2-1} + b^2_{m_2-2}s^{m_2-2} + \cdots + b^2_1s + b^2_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0}U_2 + \ldots \\ \frac{b^k_{m_k}s^{m_k} + b^k_{m_k-1}s^{m_k-1} + b^k_{m_k-2}s^{m_k-2} + \cdots + b^k_1s + b^k_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0}U_k. \end{aligned} \quad (6.5)$$

or,

$$Y(s) = \begin{bmatrix} \frac{\text{num}_1(s)}{\text{den}(s)} & \frac{\text{num}_2(s)}{\text{den}(s)} & \cdots & \frac{\text{num}_k(s)}{\text{den}(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_k(s) \end{bmatrix} \quad (6.6)$$

where we have included the dependence on s to highlight that it is a transfer function vector and

$$\begin{aligned} \text{num}_1(s) &= b^1_{m_1}s^{m_1} + b^1_{m_1-1}s^{m_1-1} + b^1_{m_1-2}s^{m_1-2} + \cdots + b^1_1s + b^1_0, & (6.7) \\ \text{num}_2(s) &= b^2_{m_2}s^{m_2} + b^2_{m_2-1}s^{m_2-1} + b^2_{m_2-2}s^{m_2-2} + \cdots + b^2_1s + b^2_0, & (6.8) \\ \vdots \quad & \quad \vdots \\ \text{num}_k(s) &= b^k_{m_k}s^{m_k} + b^k_{m_k-1}s^{m_k-1} + b^k_{m_k-2}s^{m_k-2} + \cdots + b^k_1s + b^k_0, & (6.9) \\ \text{den}(s) &= s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0. & (6.10) \end{aligned}$$

On the other hand, if the differential equation 6.1 is nonlinear, then we need to apply state-space control techniques that we have learn, where the trivial states $x_1 = y$, $x_2 = \frac{dy}{dt}$,…,$x_n = \frac{d^{n-1}y}{dt^{n-1}}$ may be the simplest way of obtaining the state-space representation.

6.1.1 Worked example

Consider the differential equation

$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = \ddot{u}_1 + 14\dot{u}_1 + 40u_1 + 2\dot{u}_2 + 4u_2 \tag{6.11}$$

then we can apply the Laplace transform as follows

$$(s^3 + 7s^2 + 14s + 8)Y(s) = (s^2 + 14s + 40)U_1(s) + (2s + 4)U_2(s). \tag{6.12}$$

Rearranging the equation, it follows that

$$Y(s) = \begin{bmatrix} \left(\frac{s^2+14s+40}{s^3+7s^2+14s+8}\right) & \left(\frac{2s+4}{s^3+7s^2+14s+8}\right) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \tag{6.13}$$

and simplifying yields

$$Y(s) = \begin{bmatrix} \left(\frac{s+10}{s^2+3s+2}\right) & \left(\frac{2}{s^2+5s+4}\right) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \tag{6.14}$$

6.2 MIMO systems: transfer function matrix

Once we know how to tackle a system with several inputs and one output, we can think in the most general case, where we have several outputs and several inputs. We will need as many equations as outputs. A natural assumption, it is that we will obtain independently the dynamics for every output, i.e.

$$\phi(\{y_1^{(i)}\}_{i=0}^{n_1}, \{u_1^{(i)}\}_{i=0}^{m_1}, \{u_2^{(i)}\}_{i=0}^{m_2}, \ldots, \{u_k^{(i)}\}_{i=0}^{m_k}) = 0, \tag{6.15}$$

$$\phi(\{y_2^{(i)}\}_{i=0}^{n_2}, \{u_1^{(i)}\}_{i=0}^{m_1}, \{u_2^{(i)}\}_{i=0}^{m_2}, \ldots, \{u_k^{(i)}\}_{i=0}^{m_k}) = 0, \tag{6.16}$$

$$\vdots \qquad \vdots \tag{6.17}$$

$$\phi(\{y_l^{(i)}\}_{i=0}^{n_l}, \{u_1^{(i)}\}_{i=0}^{m_1}, \{u_2^{(i)}\}_{i=0}^{m_2}, \ldots, \{u_k^{(i)}\}_{i=0}^{m_k}) = 0. \tag{6.18}$$

Then, we have again two cases. If the system is linear, we will be able to write every equation as a transfer function vector matrix, and combine all these column vectors in a matrix as follows

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_l(s) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{num}_{1,1}(s)}{\mathrm{den}_1(s)} & \frac{\mathrm{num}_{1,2}(s)}{\mathrm{den}_1(s)} & \cdots & \frac{\mathrm{num}_{1,k}(s)}{\mathrm{den}_1(s)} \\ \frac{\mathrm{num}_{2,1}(s)}{\mathrm{den}_2(s)} & \frac{\mathrm{num}_{2,2}(s)}{\mathrm{den}_2(s)} & \cdots & \frac{\mathrm{num}_{2,k}(s)}{\mathrm{den}_2(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{num}_{l,1}(s)}{\mathrm{den}_l(s)} & \frac{\mathrm{num}_{l,2}(s)}{\mathrm{den}_l(s)} & \cdots & \frac{\mathrm{num}_{l,k}(s)}{\mathrm{den}_l(s)} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_k(s) \end{bmatrix} \tag{6.19}$$

On the other hand, if these equations are nonlinear, once again the state space representation of the system could be obtained by using the states $x_1 = y_1$, $x_2 = \frac{dy_1}{dt}$,...,$x_{n_1} = \frac{d^{n_1-1}y_1}{dt^{n_1-1}}$, $x_{n_1+1} = y_2$, $x_{n_2+1} = \frac{dy_2}{dt}$,...,$x_{n_1+n_2} = \frac{d^{n_2-1}y_1}{dt^{n_2-1}}$, and so on.

One could think of the most general case where all the outputs are in all the equations. Then the solution of this system is a simultaneous differential equation. In the linear case, extra conditions will be required to obtain a solution in a similar way to standard simultaneous equations. In the nonlinear case, we can either perform a linearization and recover the previous case or develop a state-space representation as discussed previously. However, it may not be straightforward.

Exercise 6.2.1. Obtain the transfer function associated with the quadruple tank process using the linearisation in Exercise 3.1.7 as

$$G(s) = C(sI - A)^{-1}B + D. \tag{6.20}$$

Write the two differential equations associated with this transfer function.

6.3 Rosenbrock system matrix

The state-space methods are usually motivated to study MIMO systems in contrast with transfer function methods. The Control Systems Centre at The University of Manchester was internationally recognised by the development of frequency methods for MIMO systems such as the inverse Nyquist array design technique. Howard. H. Rosenbrock and Alistair G. J. MacFarlane were the leading researchers in this development.

The Rosenbrock system matrix of a state space representation is defined as follows:

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \tag{6.21}$$

It provides a link between both representation, state-space and transfer function. The transfer function between the input i and output j is given by

$$g_{ij} = \frac{\begin{vmatrix} sI - A & b_i \\ -c_j & d_{ij} \end{vmatrix}}{|sI - A|} \tag{6.22}$$

where b_i is the column i of B and c_j is the row j of C.

6.4 Trivial realisation of a MIMO system

A realisation of an MIMO system can be found trivially by using the realisation of each element. For example, let us consider a 2-by-2 system as follows

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{bmatrix} \tag{6.23}$$

where $G_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$, for $i = 1, 2, 3, 4$. Then a state-space realisation of the system is given by

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ B_3 & 0 \\ 0 & B_4 \end{bmatrix} \tag{6.24}$$

$$C = \begin{bmatrix} C_1 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & C_4 \end{bmatrix} \quad D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

Exercise 6.4.1. Extend the above expression to a 3-by-3 transfer function matrix.

Worked example: Realisation of a MIMO system

Consider the differential equation given (6.11), then we have shown that the relationship between inputs and output in the Laplace domain is given by

$$G(s) = \begin{bmatrix} \frac{s+10}{s^2+3s+2} & \frac{2}{s^2+5s+4} \end{bmatrix} \tag{6.25}$$

Then a state space realisation of the above plant can be found as follows:

Element $(1, 1)$ The transfer function

$$G_{11} = \frac{s + 10}{s^2 + 3s + 2} \tag{6.26}$$

We can, for instance, use the control canonical form to represent this transfer function

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1; \tag{6.27}$$

$$y = \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{6.28}$$

Element $(1,2)$ The transfer function

$$G_{12} = \frac{2}{s^2 + 5s + 4} \tag{6.29}$$

Again, we can use the controllability canonical form to represent this transfer function

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2; \tag{6.30}$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}. \tag{6.31}$$

With these two realisations we can conclude that the state-space realisation of (6.25) is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \tag{6.32}$$

$$y = \begin{bmatrix} 10 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{6.33}$$

Exercise 6.4.2. Show in MATLAB that the state-space representation of this system corresponds with the transfer function matrix (6.25).

6.5 Minimal realisation

6.5.1 Definition

When we find a state space realisation of SISO transfer functions using the canonical forms, it seems intuitive that all states are controllable and observable if there is no pole-zero cancellation. This kind of realisation is called a minimal realisation.

Definition 6.5.1. The state-space representation (A, B, C, D) is said to be a minimal realisation if and only if the pair $[A, B]$ is controllable and the pair $[A, C]$ is observable.

Result 6.5.2. *All minimal realisations of a given transfer function are similar to each other.*

Loosely speaking, if we have two minimal realisations of the same system, then there exists a transformation matrix T such that one transforms into the other.

6.5.2 SISO

For SISO systems, the minimality can be easily associated with pole-zero cancellations.

Result 6.5.3. *Any canonical form of a SISO transfer function with no pole-zero cancellation is minimal.*

Proof. Let us show that the controller canonical form is minimal. Given a general transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{n(s)}{d(s)}; \tag{6.34}$$

where $b_i \neq 0$ for at least one $0 \leq i < n$. Then its controller canonical form is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6.35}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}. \tag{6.36}$$

The controllability of this form is trivial, however its observability is not straightforward. Let us denote α a root of the numerator polynomial, i.e.

$$n(\alpha) = b_{n-1}\alpha^{n-1} + \cdots + b_1\alpha + b_0 = 0. \tag{6.37}$$

We can rewrite this equation as follows

$$\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{bmatrix} = C \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{bmatrix} = 0. \tag{6.38}$$

On the other hand, let us denote β a root of the denominator polynomial $d(s)$, which corresponds with the characteristic polynomial associated with the matrix A, hence β is also a eigenvalue of A. Further the eigenvectors of the matrix A have a very particular structure:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{n-2} \\ \beta^{n-1} \end{bmatrix} = \beta \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{n-2} \\ \beta^{n-1} \end{bmatrix}; \tag{6.39}$$

By applying Result 4.3.8, the system is unobservable if and only if there exist a root of $n(s)$, α, and a root of $d(s)$, β, such that $\alpha = \beta$. Therefore, the controller canonical form is observable if and only if there is no pole-zero cancellation. ■

6.5.3 MIMO

However, when we find the realisation of a MIMO transfer function matrix, the complexity increases. If we transform each element of the transfer function matrix, then it is possible to find that some states are uncontrollable or unobservable. In MIMO, the dynamics of a output can be different for every input, but they share the same equation, hence cancellation due to shared common states is very common as we have shown in Section 6.1.1. When a MIMO transfer function matrix is realised without considering this possibility of sharing states, then nonminimal realisations are obtained. Let us consider an example.

Worked example: Lack of minimality

Let us consider the realisation from previous section

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \tag{6.40}$$

$$y = \begin{bmatrix} 10 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{6.41}$$

As there is no pole-zero cancellation in the transfer function, one could assume that this is a minimal realisation, but it is easy to show that it is not.

The controllability matrix is

$$\mathcal{C}(A,B) = \begin{bmatrix} 0 & 0 & 1 & 0 & -3 & 0 & 7 & 0 \\ 1 & 0 & -3 & 0 & 7 & 0 & -15 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 & 0 & 21 \\ 0 & 1 & 0 & -5 & 0 & 21 & 0 & -85 \end{bmatrix}. \tag{6.42}$$

It is easy to check that this matrix has rank 4, i.e. it is full rank. As result, the pair $[A, B]$ is controllable.

On the other hand, the observability matrix is

$$\mathcal{O}(A,C) = \begin{bmatrix} 10 & 1 & 2 & 0 \\ -2 & 7 & 0 & 2 \\ -14 & -23 & -8 & -10 \\ 46 & 55 & 40 & 42 \end{bmatrix}. \tag{6.43}$$

The student can check that $\det(\mathcal{O}(A,C)) = 0$; hence there is one state that cannot be observed. As a result, the developed (A, B, C, D) representation of (6.25) using minimal representations of each component is not a minimal realisation.

Exercise 6.5.4. Use the command `minreal` to find the transformation T to obtain a system with the structure given by Kalman's decomposition.

6.6 Gilbert's realisation

6.6.1 Procedure

With this example, we can deduce an interesting property of MIMO systems. If we construct the realisation of a system by using controllability canonical forms or observability canonical forms of the individual elements, the minimality of the realisation is no longer necessarily preserved. Therefore, it is natural to ask whether we can realise the MIMO transfer function in such a way that we obtain a minimal realisation.

The answer to this question is given by Gilbert's realisation. It provides a procedure to obtain a minimal representation. The first step is to write the transfer function (l-by-k)-matrix $G(s)$ as follows:

$$G(s) = D + \frac{W(s)}{p(s)}, \tag{6.44}$$

where $p(s)$ is a scalar polynomial and $W(s)$ is a matrix whose elements are polynomial. Let us denote λ_i for $i = 1, \ldots, n$, the roots of $p(s)$ and, for simplicity's sake, assume that they are real and distinct. The method can be generalised to more general cases. Once we have found all the poles, then we decompose the fraction (6.44) in partial fractions as follows

$$G(s) = D + \sum_{i=0}^{n} \frac{W_i}{(s - \lambda_i)}. \tag{6.45}$$

Note that W_i is a constant matrix. Let us denote the rank of W_i by ρ_i, then it is possible to write these matrices as:

$$W_i = C_i B_i, \tag{6.46}$$

where $B_i \in \mathbb{R}^{\rho_i \times k}$ and $C_i \in \mathbb{R}^{l \times \rho_i}$.

Then, Gilbert's realisation of the transfer function $G(s)$ is given by

$$A = \begin{bmatrix} \lambda_1 I_{\rho_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n I_{\rho_n} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \tag{6.47}$$

$$C = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \qquad D$$

6.6.2 Worked example

Once again, let us consider the transfer function

$$G(s) = \begin{bmatrix} \frac{s+10}{s^2+3s+2} & \frac{2}{s^2+5s+4} \end{bmatrix} \tag{6.48}$$

The first step is to decompose each transfer function in partial fractions. After some algebra, it follows

$$\frac{s+10}{s^2+3s+2} = \frac{-8}{s+2} + \frac{9}{s+1} \tag{6.49}$$

$$\frac{2}{s^2+5s+4} = \frac{-2/3}{s+4} + \frac{2/3}{s+1} \tag{6.50}$$

Then, transfer function (6.48) can be written as

$$G(s) = \begin{bmatrix} \left(\frac{-8}{s+2} + \frac{9}{s+1} + \frac{0}{s+4}\right) & \left(\frac{-2/3}{s+4} + \frac{2/3}{s+1} + \frac{0}{s+2}\right) \end{bmatrix} \tag{6.51}$$

where all poles are introduced in each transfer function. Now, we can rewrite the transfer function matrix as in (6.45) as follows

$$G(s) = \frac{\begin{bmatrix} -8 & 0 \end{bmatrix}}{s+2} + \frac{\begin{bmatrix} 0 & -2/3 \end{bmatrix}}{s+4} + \frac{\begin{bmatrix} 9 & 2/3 \end{bmatrix}}{s+1} \tag{6.52}$$

where $W_1 = \begin{bmatrix} -8 & 0 \end{bmatrix}$, $W_2 = \begin{bmatrix} 0 & -2/3 \end{bmatrix}$, and $W_3 = \begin{bmatrix} 9 & 2/3 \end{bmatrix}$. Then it is trivial that the rank of each matrix is one. Finally, we need to decompose this matrices as in (6.46), where $B_i \in \mathbb{R}^{1\times 2}$ and $C_i \in \mathbb{R}^{1\times 1}$ for all $i = 1, 2, 3$. Finally, a trivial but correct solution is considering $B_i = W_i$ and $C_i = 1$, for all $i = 1, 2, 3$.

Finally, the Gilbert's realisation of the system is given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -8 & 0 \\ 0 & -2/3 \\ 9 & 2/3 \end{bmatrix} \tag{6.53}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Exercise 6.6.1. Show in MATLAB that this state-space representation corresponds with the transfer function (6.48).

Exercise 6.6.2. Show that the representation (6.53) is minimal.

6.7 Some operations with systems

The last part of the unit summarises some operations with state-space realisations. Let us denote

$$[A, B, C, D] := C(sI - A)^{-1}B + D; \tag{6.54}$$

then the following operations with systems can be defined:

Transformation Given a nonsingular matrix $V = T^{-1}$, a transformation of the system will produce the same system, i.e.

$$[A, B, C, D] = [T^{-1}AT, T^{-1}B, CT, D]. \tag{6.55}$$

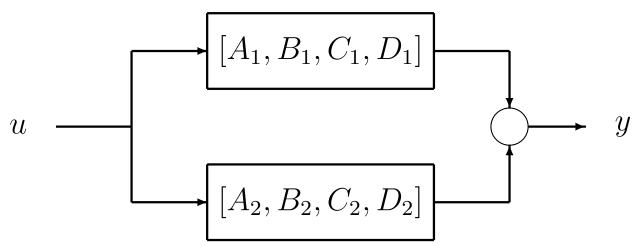


Figure 6.1: Addition of two system.

Addition The addition of two system is given by

$$[A_1, B_1, C_1, D_1] + [A_2, B_2, C_2, D_2] = \left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_1 + D_2 \right] \tag{6.56}$$

Product The product of two system is defined

$$[A_1, B_1, C_1, D_1] \times [A_2, B_2, C_2, D_2] = \left[\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}, D_1D_2 \right] \tag{6.57}$$

or

$$[A_1, B_1, C_1, D_1] \times [A_2, B_2, C_2, D_2] = \left[\begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix}, \begin{bmatrix} D_1C_2 & C_1 \end{bmatrix}, D_1D_2 \right] \tag{6.58}$$

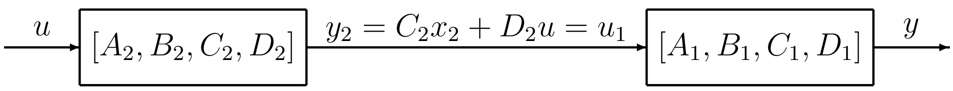


Figure 6.2: Product of two systems.

The dynamics of the state x_2 is straightforward

$$\dot{x}_2 = A_2x_2 + B_2u. \tag{6.59}$$

But the dynamics of the state x_1 requires some treatment

$$\dot{x}_1 = A_1x_1 + B_1u_1 = A_1x_1 + B_1(C_2x_2 + D_2u) = A_1x_1 + B_1C_2x_2 + B_1D_2u \tag{6.60}$$

If we consider the total as $x = (x_1, x_2)$, both equations lead to

$$\frac{d}{dt}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} u. \tag{6.61}$$

Finally, the output is given by

$$y = C_1x_1 + D_1u_1 = C_1x_1 + D_1(C_2x_2 + D_2u) = C_1x_1 + D_1C_2x_2 + D_1D_2u \tag{6.62}$$

or

$$y = \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1D_2u. \tag{6.63}$$

As a result, we have found the four matrices given in (6.57). The student can find the matrices in (6.58) by choosing $x = (x_2, x_1)$

As the order in the product does not match with the order of the block diagram (see Fig 6.2), H_∞ guys have proposed to change the direction of the arrows (see Fig. 6.3).

Inverse If D is a square nonsingular matrix, then

$$([A, B, C, D])^{-1} = [A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1}]. \tag{6.64}$$

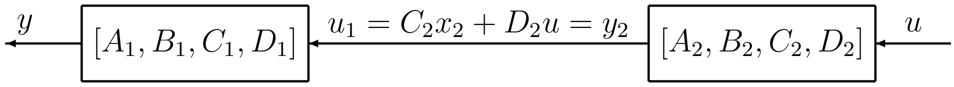


Figure 6.3: A reason for changing the direction of the arrows.

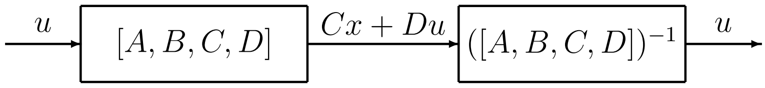


Figure 6.4: Inverse system.

6.7.1 Worked example

Consider the product of a system with it inverse system given in Fig. 6.4. Let us show that this product is equal to the identity system as defined in (6.64). Using (6.61) and the systems in Fig. 6.4, i.e.

$$[A_1, B_1, C_1, D_1] = [A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1}], \tag{6.65}$$

$$[A_2, B_2, C_2, D_2] = [A, B, C, D]; \tag{6.66}$$

it follows that

$$\frac{d}{dt}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} A - BD^{-1}C & -BD^{-1}C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -BD^{-1}D \\ B \end{bmatrix} u = \begin{bmatrix} A - BD^{-1}C & -BD^{-1}C \\ 0 & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -B \\ B \end{bmatrix} u \tag{6.67}$$

and

$$y = \begin{bmatrix} D^{-1}C & D^{-1}C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D^{-1}Du = \begin{bmatrix} D^{-1}C & D^{-1}C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Iu, \tag{6.68}$$

where it is easy to see that there is a very singular structure in the matrices, for example, the input affect is exactly opposite manner to both states, whereas the output depend on both states in the same way. Let us transform the system into the new coordinates $z_1 = x_1 + x_2$ and $z_2 = x_2$, then it follows that

$$\dot{z}_1 = \dot{x}_1 + \dot{x}_2 = (A - BD^{-1}C)x_1 - Bu + (A - BD^{-1}C)x_2 + Bu = (A - BD^{-1}C)(x_1 + x_2) = (A - BD^{-1}C)z_1; \tag{6.69}$$

and

$$y = D^{-1}C(x_1 + x_2) + Iu = D^{-1}Cz_1 + Iu. \tag{6.70}$$

As a result, we obtain the diagonal form of the system that corresponds with its Kalman's decomposition, i.e.

$$\frac{d}{dt}\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u \tag{6.71}$$

and

$$y = \begin{bmatrix} D^{-1}C & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + Iu, \tag{6.72}$$

where the state z_1 is observable but uncontrollable and z_2 is controllable but unobservable. Therefore, from an input-ouput point of view, both states can be eliminated and only I must be considered, i.e.

$$([A, B, C, D])^{-1} \times [A, B, C, D] = I. \tag{6.73}$$

Exercise 6.7.1. Derive the state space representation of the unit feedback of the system $[A, B, C, D]$.

Exercise 6.7.2. Derive the state space representation of the feedback interconnection between the system $[A_1, B_1, C_1, D_1]$ and $[A_2, B_2, C_2, D_2]$.

6.8 Learning outcomes

- A state-space realisation is minimal if and only if the pair $[A, B]$ is controllable and the pair $[A, C]$ is observable.
- All minimal representations of a system are equivalents, hence they have the same number of states.
- For multivariable systems, then minimality of each element does not ensure minimality of the whole transfer function.
- The minimal realisation of a system can be obtained using Gilbert's realisation.
- Obtain a minimal represetation using Gilbert's realisation.
- Inverses, additions and product can be defined in the state-space representation.