Comment on “Absolute stability analysis for negative-imaginary systems”

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Abstract

This note provides the connection between the paper “Absolute stability analysis for negative-imaginary systems” and classical results in absolute stability. Strictly negative-imaginary systems satisfy the Aizerman conjecture.

Key words: Absolute stability, Negative Imaginary Systems, Popov criterion, Aizerman conjecture, stability criteria.

1 Introduction

The main result (Theorem 9) in (Dey et al., 2016) is a sufficient stability condition for strictly proper and strongly strict negative-imaginary (SSNI) systems in positive feedback with a diagonal, memoryless, slope-restricted nonlinearity. In classical language the result for single-input single-output (SISO) systems may be stated succinctly as “SSNI systems satisfy the Kalman conjecture with positive feedback”; Theorem 9 in (Dey et al., 2016) also provides the natural generalization of this statement to multivariable systems. Dey et al. (2016) prove their result via a Lur’e-Postnikov type Lyapunov function, Popov multipliers and loop transformation. In fact it can be shown via simple application of the Popov criterion. It follows immediately that the memoryless, diagonal nonlinearity need only be sector-bounded, not slope-restricted (under usual assumptions of well-posedness). In addition the condition is both necessary and sufficient for the absolute stability of strictly proper strictly negative-imaginary (SNI) systems. A similar result also follows immediately for negative feedback, where absolute stability can be shown for any diagonal, memoryless, sector-bounded nonlinearity. In short, SISO SNI systems satisfy the Aizerman conjecture; the natural generalization of this statement to multivariable systems also holds.

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2 Technical development

The result is a straightforward application of standard results. In particular, similar arguments are well-known in the literature, for example showing that second order plants hold the Aizerman conjecture (Vidyasagar, 1993). Nevertheless we provide technical details for completeness.

2.1 Preliminary definitions and results

Lur’e system: We are concerned with the absolute stability of a Lur’e system consisting of a linear time-invariant $m \times m$ SNI system G in feedback (positive or negative) with a diagonal, memoryless, sector-bounded nonlinearity Φ. Hence

$$y = Gu, \tag{1}$$

$$u(t) = \begin{cases} \Phi(y(t)) & \text{(positive feedback)}, \\ -\Phi(y(t)) & \text{(negative feedback)}. \end{cases} \tag{2}$$

NI and SNI systems: Classes of negative-imaginary systems are defined in (Lanzon and Petersen, 2008; Petersen and Lanzon, 2010); for a recent overview see (Ferrante et al., 2016). A linear time-invariant system is NI if all the poles of its transfer function matrix $G(s)$ lie in the open left-half plane and

$$j\left[G(j\omega) - G^*(j\omega)\right] \geq 0 \text{ for all } \omega \in (0,\infty). \tag{3}$$

The system $G(s)$ is SNI if in addition it satisfies

$$j\left[G(j\omega) - G^*(j\omega)\right] > 0 \text{ for all } \omega \in (0,\infty). \tag{4}$$

As in (Lanzon and Petersen, 2008; Dey et al., 2016), we have restricted our attention to stable systems with rational transfer function matrices with no poles on the imaginary axis. See (Xiong et al., 2010; Ferrante and Ntogramatzidis, 2013) for further extensions to transfer function matrices that are analytic only in the open right half plane and irrational transfer function matrices, respectively.

As is standard in absolute stability (e.g. (Brogliato et al., 2007), and as in (Dey et al., 2016)) we will only be concerned with systems that are strictly proper; that is to say $G(\infty) = 0$. We assume this condition tacitly from now on. It follows immediately (Lemma 2 in (Lanzon and Petersen, 2008)) that

$$G(0) = G(0)^\top > 0. \tag{5}$$

Corresponding stability results for biproper systems can be derived by loop transformation, but further care must be taken over well-posedness conditions.

Stability: We will say a system is stable if the origin is globally asymptotically stable.

Static Linear Feedback: A general result for the stability of the feedback interconnection between a SNI system and NI system is given in (Lanzon and Petersen, 2008); for the reader's ease of reference, we include here a simple proof when the NI system is a static gain.

Theorem 1 (special case of Theorem 5 in (Lanzon and Petersen, 2008)): *The positive feedback interconnection between a strictly proper SNI system G and $K = K^T$ is stable if and only if*

$$G(0)^{-1} - K > 0. \tag{6}$$

Proof: Let $G(s) = C(sI - A)^{-1}B$ be a minimal realization. Then A is Hurwitz and there exists a real matrix $Y > 0$ such that $AY + YA^* \leq 0$ and $B = -AYC^*$. Further, $CYC^* = G(0)$. Define $\Psi = AY$ and $T = Y^{-1} - C^*KC$. Then the positive feedback interconnection between $G(s)$ and K is stable if and only if $A + BKC = \Psi T$ is Hurwitz. Following the same argument as in (Lanzon and Petersen, 2008), this is in turn equivalent to the condition $T > 0$. Since K and C^*YC are both symmetric, the eigenvalues of $KC^*YC = KG(0)$ are real. If $\bar{\lambda}[X]$ denotes the maximum eigenvalue of X we can say

$$\begin{aligned} T > 0 &\Leftrightarrow \bar{\lambda}[KC^*YC] < 1, \\ &\Leftrightarrow G(0)^{-1} - K > 0. \end{aligned}$$

■

The following statements follow from Theorem 1:

Corollary 1: *The positive feedback interconnection between a strictly proper SNI system G and $K = K^T > 0$ is stable if and only if*

$$G(0) < K^{-1}. \tag{7}$$

Corollary 2: *The negative feedback interconnection between a strictly proper SNI system G and $K = K^T$ is stable if and only if*

$$G(0)^{-1} + K > 0. \tag{8}$$

Corollary 3: *The negative feedback interconnection between a strictly proper SNI system G and $K = K^T > 0$ is stable.*

Memoryless sector-bounded nonlinearity: The nonlinearity is characterised by the map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $u(t) = \Phi(y(t))$, $\Phi(y(t))_i = \phi_i(y_i(t))$ for some $\phi_i : \mathbb{R} \to \mathbb{R}$ and

$$\Phi(y(t))^\top [M^{-1}\Phi(y(t)) - y(t)] \leq 0, \tag{9}$$

with diagonal $M > 0$. We say Φ is sector-bounded on $[0, M]$.

We assume well-posedness of the closed-loop system (1), (2). This can be guaranteed under mild conditions, for example that Φ is Lipschitz continuous (Khalil, 2002).

Absolute stability: We say the closed-loop system (1), (2) is absolutely stable for a specific plant G if stability is guaranteed for a class of nonlinearity: in our case all memoryless Φ sector-bounded on $[0, M]$ satisfying conditions for well-posedness.

Here we state the modern form of the Aizerman conjecture (e.g. Brogliato et al., 2007).

Aizerman's conjecture: If the negative feedback interconnection between a strictly proper linear SISO system G and any $\phi(y) = ky$ is asymptotically stable for all $k \in [a, b]$, then the negative feedback interconnection between G and any memoryless nonlinearity $\phi(\cdot)$ in the sector $[a, b]$ is also stable.

Aizerman's conjecture in this form is true for first- and second-order continuous-time systems (Brogliato et al., 2007; Vidyasagar, 1993). Counterexamples are known for third-order continuous-time systems (Fitts, 1966) and second-order discrete-time systems (Carrasco et al., 2015; Heath et al., 2015). Second-order counterexamples are also known (Narendra and Taylor, 1973) for early statements of the conjecture with open rather than closed intervals. A useful account of the historical development can be found in (Bragin et al., 2011).

The Popov criterion for SISO and multivariable systems is well-known (Popov, 1961; Yakubovich, 1967); however textbooks often only cover particular cases. If the nonlinearity is diagonal, we can state the following result:

Popov criterion: *The negative feedback interconnection between a strictly proper plant G and a nonlinearity Φ sector-bounded on $[0, M]$ is absolutely stable if there exists a Popov* *multiplier $Z(s) = I + s\Gamma$, where Γ is a diagonal matrix, such that*

$$H(s) = M^{-1} + Z(s)G(s), \tag{10}$$

is strictly positive real.

Although textbooks such as (Brogliato et al., 2007; Khalil, 2002) restrict $\Gamma > 0$ this condition is not required; see (Yakubovich et al., 2004). The result is straightforward in the IQC formulation of Megretski and Rantzer (1997) but in its embryonic form was already known in the sixties (Yakubovich, 1967). Some more recent papers cover multivariable cases of the Popov criterion, e.g. (Park, 1997; Heath and Li, 2009). Our absolute stability proof in the sequel follows the argument of Vidyasagar (1993) for second order SISO systems. In fact we only require $\Gamma = \gamma I$ for some $\gamma \in \mathbb{R}$. If the nonlinearity is, in addition, slope-restricted then the existence of a Popov multiplier allows stronger input-output stability results (Carrasco et al., 2013).

It is stated by Dey et al. (2016) that the "use of positive feedback in absolute stability framework makes this present work fundamentally distinct from most of the extensive literature available on absolute stability for slope-restricted nonlinearities." However, absolute stability results can be expressed for either negative or positive feedback interconnections without loss of generality. Specifically, the negative feedback interconnection between G and ϕ is equivalent to the positive feedback interconnection between $-G$ and ϕ. In particular, the Popov criterion can be applied to systems with positive feedback by substituting $-G$ for G.

2.2 Main results

In this section, we derive stronger results than Theorem 9 in (Dey et al., 2016) by using the Popov criterion and Corollary 1.

Theorem 2: *The positive feedback interconnection between a strictly proper stable SNI system G and a memoryless nonlinearity in the sector $[0, M]$ is absolutely stable if and only if $G(0) < M^{-1}$.*

Proof: Sufficiency follows via construction of a Popov multiplier for $-G$ with the form

$$Z(s) = I(1 + \gamma s) \text{ with } \gamma < 0. \tag{11}$$

so that H in (10) can be written

$$H(s) = M^{-1} - G(s) - \gamma s G(s). \tag{12}$$

Immediately we have the relation

$$H(0) = M^{-1} - G(0), \tag{13}$$

and hence the condition on $G(0)$. From (4) we can say

$$H(j\omega) + H(j\omega)^* \geq 2M^{-1} - G(j\omega) - G(j\omega)^* \text{ for all } \omega. \tag{14}$$

Hence, and by the continuity of eigenvalues, there is some $\omega_1 > 0$ independent of γ such that

$$H(j\omega) + H(j\omega)^* > 0 \text{ for all } \omega \in [0, \omega_1]. \tag{15}$$

Similarly, since $G(\infty) = 0$, from (14) and by continuity of the eigenvalues we have

$$H(\infty) + H(\infty)^* > 0, \tag{16}$$

and there is some $\omega_2 > 0$ independent of γ such that

$$H(j\omega) + H(j\omega)^* > 0 \text{ for all } \omega \in [\omega_2, \infty). \tag{17}$$

Finally, given the compact interval $[\omega_1, \omega_2]$, there exist some $\varepsilon > 0$ and $\delta > 0$ such that

$$2M^{-1} - G(j\omega) - G^*(j\omega) > -\varepsilon I, \tag{18}$$

and

$$(j\omega G(j\omega) + (j\omega G(j\omega))^*) > \delta I. \tag{19}$$

for all $\omega \in [\omega_1, \omega_2]$. It suffices to choose $\gamma < -\varepsilon/\delta$ to ensure $H(s)$ is strictly positive real.

Necessity follows from Corollary 1. ■

Theorem 3: *The negative feedback interconnection between a strictly proper stable SNI system G and a memoryless nonlinearity in the sector $[0, M]$ is absolutely stable.*

Proof: Similar to Theorem 2, but we construct a Popov multiplier for $+G$ with $\gamma > 0$. The steady state condition becomes

$$G(0) + M^{-1} > 0, \tag{20}$$

which, by (5), is true for all strictly proper SNI systems. ■

It follows as an immediate corollary that the Aizerman conjecture is true for SISO SNI systems. Similarly a natural generalization of the Aizerman conjecture is true for multivariable SNI systems.

3 Conclusion

Classical methods can deal with multivariable systems, positive/negative feedback and positive/negative multipliers (Desoer and Vidyasagar, 1975; Yakubovich et al., 2004; Megretski and Rantzer, 1997). They can be used to show that SISO SNI systems satisfy the Aizerman conjecture, and that a similar statement is true for multivariable systems. The result is considerably stronger than the main result of Dey et al. (2016) in that: the conditions for absolute stability are both necessary and sufficient; the memoryless nonlinearity need not be slope-restricted, only sector-bounded and satisfying conditions for well-posedness; the system need only be SNI, not necessarily SSNI; the result is valid for both positive and negative feedback.

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