

Second-order counterexamples to the discrete-time Kalman conjecture

W. P. Heath^aJ. Carrasco^aM. de la Sen^b

^aControl Systems Centre, School of Electrical and Electronic Engineering, The University of Manchester, Manchester M13 9PL, UK

^bInstitute of Research and Development of Processes, Faculty of Science and Technology, University of the Basque Country, PO Box 644, Bilbao, Spain

Abstract

The Kalman conjecture is known to be true for third-order continuous-time systems. We show that it is false in general for second-order discrete-time systems by construction of counterexamples with stable periodic solutions. We discuss a class of second-order discrete-time systems for which it is true provided the nonlinearity is odd, but false in general. This has strong implications for the analysis of saturated systems.

1 Introduction

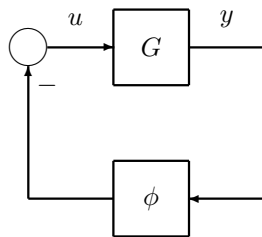


Fig. 1. Lur'e problem

Absolute stability of Lur'e¹ systems (see Fig. 1) has attracted much attention in the literature; Aizerman and Gantmacher (1964) give a classical perspective of the Lur'e problem, and Altshuller (2013) and Carrasco *et al.* (2015a) give recent overviews. Given a class of nonlinearities Φ , the Lur'e problem consists of finding conditions on the LTI system G which ensure that the negative feedback system between G and ϕ is globally asymptotically stable for all $\phi \in \Phi$. The class Φ is typically described by sector conditions, but several other classes are given by Megretski and Rantzer (1997). Results have

been mostly focused on continuous-time systems, with less attention to the discrete-time counterpart.

The Kalman conjecture (Kalman, 1957) has played an important role in the development of absolute stability theory and is stated in several textbooks (e.g. Vidyasagar, 1993; Brogliato *et al.*, 2006; Haddad and Chellaboina, 2008). Continuous-time counterexamples were first proposed by Fitts (1966) and have been the subject of interesting discussion in the literature (see Barabanov, 1988; Leonov *et al.*, 2010; Bragin *et al.*, 2011; Leonov and Kuznetsov, 2011, 2013, and references therein). They are useful for evaluating the performance of stability tests including searches for Zames-Falb multipliers (Safonov and Wyetzner, 1987; Chen and Wen, 1996; Carrasco *et al.*, 2012; Chang *et al.*, 2012; Carrasco *et al.*, 2014).

Both modern digital control implementation and robust stability results for optimizing controllers (Heath and Wills, 2007) require a complete study in the discrete-time domain. Recently, new stability conditions for discrete-time Lur'e systems have been proposed in the literature (Ahmad *et al.*, 2013b; Gonzaga *et al.*, 2012; Ahmad *et al.*, 2015, 2013a; Wang *et al.*, 2014; Park *et al.*, 2015), while earlier results include those of Tsypkin (1962); Kapila and Haddad (1996); Park and Kim (1998); Haddad and Bernstein (1994). The need for good benchmarks is exemplified by the numerical values used to illustrate the results of Gonzaga *et al.* (2012) and Park *et al.* (2015); the numerical values given by Gonzaga *et al.* (2012) are no better than the circle criterion,

* DOI:10.1016/j.automat.2015.07.005

©Creative Commons Attribution Non-Commercial No Derivatives License

Email addresses: william.heath@manchester.ac.uk (W. P. Heath), joaquin.carrascogomez@manchester.ac.uk (J. Carrasco), manuel.delasen@ehu.es (M. de la Sen).

¹ Also written as Lurie or Lurye.

while the numerical values given by Park *et al.* (2015) are claimed better than the Nyquist value. This motivates the study of the discrete-time Kalman conjecture, henceforth DTKC.

In this technical communiqué we present counterexamples with saturation functions that demonstrate that the Kalman conjecture is false in general for second-order discrete-time systems. To the best of our knowledge, these are the first explicit counterexamples to the discrete-time Kalman conjecture, even though the existence of high order counterexamples has been widely assumed. In particular it should be possible to generate fourth-order counterexamples by sampling their continuous-time counterparts. Periodic solutions to Lur'e systems where the LTI system is second-order discrete-time but open-loop unstable have been discussed elsewhere in the literature (e.g. Hu and Lin, 2001; Yang *et al.*, 2013); since the LTI system is open-loop unstable these cannot be considered counterexamples to the Kalman conjecture.

We present two counterexamples in Section 2 where the saturation function is non-odd. We discuss some implications of these counterexamples for systems under constant disturbance or set point demand in Section 3, where we also present a further counterexample where the saturation function is odd (previously reported by Carrasco *et al.*, 2015b).

2 Counterexamples

A matrix is said to be Schur if all its eigenvalues have absolute value strictly lower than 1. The notation $G \sim [A, B, C, D]$ means that the set of matrices $[A, B, C, D]$ is a state-space representation of the LTI system G .

Definition 1 (Nyquist value) *The Nyquist value for a stable transfer function $G(z)$ is*

$$k_N = \sup_k \{k > 0 : (1 + \tau k G(z))^{-1} \text{ is stable } \forall \tau \in [0, 1]\}.$$

The discrete Lur'e system is represented in Fig. 1. The nonlinearity ϕ is memoryless, so there exists an $N : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi(v)(i) = N(v(i))$. The state-space representation of the closed-loop system is given by

$$x(i+1) = Ax(i) - BN(Cx(i)). \quad (1)$$

The origin of the system $x(i+1) = f(x(i))$ is said to be global asymptotically stable, henceforth GAS, if $f(0) = 0$ and for any $\epsilon > 0$ and $x(0)$ there exists n such that $|x(i)| < \epsilon$ for all $i > n$. With some abuse of notation, the system $x(i+1) = f(x(i))$ is said to be GAS if its origin is GAS. We will use the notation $x(n) = f^n(x(0))$. The system $x(i+1) = f(x(i))$ is said to have a periodic solution if there exist x such that $f^n(x) = x$ for some

$n > 0$. The least integer n is called the period of the solution. It is standard to show that the origin is not GAS by finding a periodic solution with period larger than 1 (see Hu and Lin, 2001; Yang *et al.*, 2013).

Let us define the static nonlinearity N_L as

$$N_L(x) = \begin{cases} -L & \text{if } x < -L, \\ x & \text{if } -L \leq x \leq L, \\ L & \text{if } x > L, \end{cases} \quad (2)$$

The nonlinearity N_L is odd if and only if $L = 1$. We use $N_1(x)$ and $\text{sat}(x)$ interchangeably.

In this paper we consider a slightly modified version of the DTKC, where the nonlinearity is not required to be smooth:

Conjecture 1 (DTKC) *Let ϕ be a memoryless slope-restricted nonlinearity such that there exists a continuously differentiable $N : \mathbb{R} \rightarrow \mathbb{R}$ and $S > 0$ such that $\phi(y)(i) = N(y(i))$ and*

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq S, \quad (3)$$

for all $x_1, x_2 \in \mathbb{R}$. Then, the negative feedback interconnection of the discrete-time LTI system $G \sim [A, B, C, 0]$ and ϕ (as in Fig 1) is GAS if $A - BCk$ is Schur for all $k \in [0, S]$.

Remark 1 Carrasco *et al.* (2015b) show how to construct a counterexample with smooth nonlinearity given a counterexample to Conjecture 1.

Definition 2 *Let \mathcal{G} be the class of open-loop systems with transfer function*

$$G(z) = \frac{-a_{22}z + a_{11}a_{22} - a_{12}a_{21}}{z(z - a_{11})}. \quad (4)$$

Theorem 1 *The class of systems \mathcal{G} does not satisfy the Kalman conjecture.*

Proof: It suffices to construct a counterexample. Consider, as an example, the case

$$x(i+1) = \begin{bmatrix} 0.5x_1(i) + 1.6x_2(i) \\ N_L(-1.2x_1(i) - 2x_2(i)) \end{bmatrix} \quad (5)$$

with corresponding transfer function $G(z) = \frac{2z+0.92}{z^2-0.5z}$, so that the Nyquist value of G is $k_N = 25/23$.

Consider the system $x(i+1) = f_1(x(i))$ given by (5) with $L = 2$. Note that N_L satisfies (3) with $S = 1 < k_N$ for all L . Let us define

$$H_1 = \left\{ \begin{bmatrix} -880 \\ 367 \\ 1 \end{bmatrix}, \begin{bmatrix} 736 \\ 1835 \\ 322 \\ 367 \end{bmatrix}, \begin{bmatrix} 2944 \\ 1835 \\ -2 \end{bmatrix} \right\}.$$

It is straightforward to show that for any $x \in H_1$, then $f_1^3(x) = x$. Hence it is not GAS since there exists a periodic solution with period 3. ■

Following LaSalle (1976) and considering the system $x(i+1) = f(x(i))$, a closed set H is said to be an attractor if there exists a neighbourhood U of H such that $f^n(x) \rightarrow H$ for all $x \in U$. The expression $f^n(x) \rightarrow H$ means that the distance between $f^n(x)$ and H approaches 0 as $n \rightarrow \infty$. H is said to be stable if given a neighbourhood V of H there exists a neighbourhood W of H such that $f^n(x) \in V$ for all $x \in W$ and $n > 0$. H is asymptotically stable if it is an attractor and stable (see LaSalle, 1976, for further details).

Theorem 2 Consider the system $x(i+1) = f_1(x(i))$ given by (5) with $L = 2$. Then, the set H_1 is asymptotically stable.

Proof: Consider the neighbourhood of H_1 given by

$$U = \left\{ \begin{array}{l} \left[\begin{array}{c} -\frac{880}{367} + \varepsilon_1 \\ 1 + \varepsilon_2 \end{array} \right], \left[\begin{array}{c} \frac{736}{1835} + \varepsilon_3 \\ \frac{322}{367} + \varepsilon_4 \end{array} \right], \left[\begin{array}{c} \frac{2944}{1835} + \varepsilon_5 \\ -2 + \varepsilon_6 \end{array} \right], \\ \left. \begin{array}{l} 6\varepsilon_1 + 10\varepsilon_2 > -225/367, \\ 3\varepsilon_1 + 5\varepsilon_2 < 2640/367, \\ 45\varepsilon_1 + 52\varepsilon_2 < 2166/367, \\ 167\varepsilon_1 + 240\varepsilon_2 < 7500/367, \\ 167\varepsilon_1 + 240\varepsilon_2 > -28880/1101, \\ \text{such that } 3\varepsilon_3 + 5\varepsilon_4 > -1083/1835, \\ 5\varepsilon_3 + 16\varepsilon_4 > -750/367, \\ 5\varepsilon_3 + 16\varepsilon_4 < 2888/1101, \\ 3\varepsilon_5 + 5\varepsilon_6 < 9861/3670, \\ 5\varepsilon_5 + 16\varepsilon_6 < 1444/1101, \\ 5\varepsilon_5 + 16\varepsilon_6 > -375/367, \end{array} \right\}$$

and analyse the solutions $f_1^n(x)$ for all $x \in U$. Firstly, if $x = (-\frac{880}{367} + \varepsilon_1, 1 + \varepsilon_2) \in U$, it follows that

$$\begin{aligned} f_1^1(x) &= \begin{bmatrix} \frac{736}{1835} \\ \frac{322}{367} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{8}{5} \\ -\frac{6}{5} & -2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \\ f_1^2(x) &= \begin{bmatrix} \frac{2944}{1835} \\ -2 \end{bmatrix} + \begin{bmatrix} -\frac{167}{100} & -\frac{12}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \\ f_1^3(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{167}{200} & -\frac{6}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \\ f_1^{3+3n}(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \left(\frac{167}{200}\right)^n \begin{bmatrix} -\frac{167}{200} & -\frac{6}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \end{aligned}$$

for $n = 1, 2, \dots$ Secondly, if $x = (\frac{736}{1835} + \varepsilon_3, \frac{322}{367} + \varepsilon_4) \in U$, it follows that

$$\begin{aligned} f_1^1(x) &= \begin{bmatrix} \frac{2944}{1835} \\ -2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{8}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \\ f_1^2(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} & -\frac{4}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \\ f_1^{2+3n}(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \left(\frac{167}{200}\right)^n \begin{bmatrix} -\frac{1}{4} & -\frac{4}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \end{aligned}$$

for $n = 1, 2, \dots$ Finally, if $x = (\frac{2944}{1835} + \varepsilon_5, -2 + \varepsilon_6) \in U$, it follows that

$$\begin{aligned} f_1^1(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{8}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \\ f_1^{1+3n}(x) &= \begin{bmatrix} -\frac{880}{367} \\ 1 \end{bmatrix} + \left(\frac{167}{200}\right)^n \begin{bmatrix} \frac{1}{2} & \frac{8}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \end{aligned}$$

for $n = 1, 2, \dots$. This development shows that $f_1^n(x) \rightarrow H_1$ for all $x \in U$, i.e. H_1 is an attractor. Moreover, using this machinery it is then straightforward to choose a neighbourhood V and find W such that if $x \in W$, then $f^n(x) \in V$ for all n ; hence H_1 is stable. Therefore H_1 is asymptotically stable. ■

Remark 2 It is possible to construct further interesting counterexamples from the class \mathcal{G} . Consider for example the system $x(i+1) = f_2(x(i))$ given by (5) with $L = \infty$ (i.e. a one-sided saturation). Let us define

$$H_2 = \left\{ \begin{bmatrix} \frac{80}{367} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3136}{1835} \\ -\frac{830}{367} \end{bmatrix}, \begin{bmatrix} -\frac{5072}{1835} \\ 1 \end{bmatrix} \right\}.$$

Similar to the previous counterexample, it is straightforward to show that for any $x \in H_2$, then $f_2^3(x) = x$. Hence it is not GAS since there exists a nontrivial periodic solution with period 3. It is straightforward to show the solution set is stable in a similar manner to Theorem 2.

3 Discussion on the oddness condition

The class \mathcal{G} is inspired by the analysis of Hu and Lin (2001). As part of their wider analysis of discrete-time planar systems under odd saturation, they consider the two-state system

$$x(i+1) = \begin{bmatrix} a_{11}x_1(i) + a_{12}x_2(i) \\ \text{sat}(a_{21}x_1(i) + a_{22}x_2(i)) \end{bmatrix}, \quad (6)$$

where $x(i) = (x_1(i), x_2(i))$. The system (6) can be written as the Lur'e system (1) with $G(z) \in \mathcal{G}$ (Definition 2).

While Theorem 1 shows that counterexamples to the Kalman conjecture can be found within the class of systems \mathcal{G} , Heath and Carrasco (2015) show that the class of systems \mathcal{G} satisfies the Kalman conjecture provided the nonlinearity is odd. This is demonstrated using a Lyapunov function for $|a_{11}| = 1$ and discrete-time Zames-Falb multipliers (Willems and Brockett, 1968) for $|a_{11}| < 1$. In some cases the requisite multiplier is only valid for odd nonlinearities.

In the literature discussing saturation systems and anti-windup synthesis it is standard to distinguish symmetric (odd) and asymmetric saturation (e.g. Zaccarian and Teel, 2011), but in the analysis it is common to assume the nonlinearity is odd (e.g. Hu and Lin, 2001; Tarbouriech *et al.*, 2011). In the two counterexamples of Section 2 stability is guaranteed when the nonlinearity is odd but breaks down otherwise. This has implications for stability with constant disturbances or set-point tracking. Although we have presented our results for unforced systems, it is straightforward to extend them to input-output \mathcal{L}_2 stability with exogenous inputs. Such analysis requires renormalizing about steady-state values; if these are non-zero then there is no reason to assume the renormalized nonlinearity is odd even when it is an odd saturation function in the nominal case. Note that Hu *et al.* (2002) address a related but different problem in their consideration of systems with *persistent* disturbances; their analysis gives no information about the structure (or presence) of limit cycles under *constant* disturbances.

Similarly, although the distinction between Zames-Falb multipliers for odd and non-odd nonlinearities is well-known (in both the classical and recent literature), examples and implications are not widely discussed. The counterexamples of Section 2 provide benchmark examples where the distinction is significant. In this case, the maximum slope for odd slope-restricted nonlinearities (k_{odd}) must be strictly bigger than the maximum slope for non-odd slope-restricted nonlinearities ($k_{\text{non-odd}}$), i.e.

$$k_{\text{non-odd}} < k_{\text{odd}} = k_N.$$

To the best of authors' knowledge, such a counterexample has not yet been found in continuous-time, though some Zames-Falb multipliers searches can be setup for either case (Safonov and Wyetzner, 1987; Chen and Wen, 1996). Examples given by Chen and Wen (1996) provide almost the same value for each case. The existence of such a counterexample would provide a suitable benchmark to analyse the relative conservatism of such searches in each case.

We pose an open question: suppose a (general) Lur'e system is stable with odd nonlinearity. How robust is it to perturbances in ϕ which destroy the symmetry? In the case of saturation this question may be quantified. Suppose ϕ is given by the static nonlinearity N_L given in (2). What is the maximum L_M for which stability is

guaranteed for all $L \in [1, L_M]$? For our examples it is easy to check that periodic solutions exist when $L = 436/275$, but no periodic solution with period 3 exists when $L < 436/275$.

Notwithstanding the previous discussion, second-order counterexamples exist even when the nonlinearity is odd. The following was first presented by Carrasco *et al.* (2015b)², where the related Aizerman and Markus-Yamabe conjectures are also discussed.

Let $G(z) = \frac{z}{z^2 - 2az + a^2}$ where $0 < a < 1$, or equivalently

$$x(i+1) = \begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix} x(i) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(i), \quad (7)$$

$$y(i) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(i). \quad (8)$$

where $x(i) \in \mathbb{R}^2$. It is straightforward to check that $A - BCk$ is Schur if $k < k_N = (a+1)^2$. From (7) and (8) $x(i) = (y(i-1), y(i))$; thus let us define the feedback interconnection as follows

$$\Sigma : \begin{cases} y(i) = 2ay(i-1) - a^2y(i-2) + u(i-1), \\ u(i) = -S\text{sat}(y(i)). \end{cases} \quad (9)$$

for some $S > 0$ and initial conditions $y_0(-1)$ and $y_0(-2)$. The next result shows that there exists a periodic solution if $S > k_p$, and it leads to a counterexample of DTKC since $k_p < k_N$ for some $0 < a < 1$:

Theorem 3 (Carrasco *et al.* (2015b)) *The system Σ has a non-trivial periodic solution $y(i) = -y(i-2)$ if $(\sqrt{2}-1) < a < 1$ and $S > k_p = \frac{(a^2+1)^2}{a^2+2a-1}$.*

Corollary 1 (Carrasco *et al.* (2015b)) *The system Σ is a counterexample of the DTKC if $k_p < k_N$, i.e. $2a^3 + a^2 - 1 > 0$. Analytically,*

$$a > \frac{(53 + 6\sqrt{78})^{1/3} + (53 - 6\sqrt{78})^{1/3} - 1}{6} \simeq 0.657298.$$

Remark 3 *As an example, let us consider $a = 0.9$, then $k_N = 3.61$ and $k_p = \frac{32761}{16100}$. Then using $S = 2.1$, the system has a periodic output $\{y_1, y_2, -y_1, -y_2\}$ where*

$$y_1 = 2.1 \frac{19900}{32761} > 1 \text{ and } y_2 = 2.1 \frac{16100}{32761} > 1.$$

In the phase plane, the periodic solution is described by

$$H_3 = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}, \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix}, \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} \right\}.$$

² Counterexamples are available in Maplecloud, accessible from the second author's personal page: <http://goo.gl/NX6OXS>

For any $x \in H_3$, then $f_3^4(x) = x$, where f_3 can be obtained by taking $u(i) = -\text{Sat}(y(i))$ in (7). Hence it is not GAS since there exists a periodic solution with period 4. Once again it is straightforward to show the periodic solution set is stable. This system is used as a benchmark example for stability tests by Ahmad et al. (2015, 2013a).

4 Conclusion

We have shown that, although DTKC is true for first-order systems, it is false in general for second-order systems. The construction of counterexamples using second-order systems is given. As a result, stability properties of discrete-time Lur'e systems with second-order and higher plants cannot be derived from the feedback interconnection between the linear system G and a linear gain k as in the continuous-time domain. All three counterexamples are hidden oscillations in the sense of Leonov and Kuznetsov (2013), and the periodic orbits are stable. Theorem 2 shows formally the stability of the periodic orbit for the first counterexample.

The counterexamples are remarkable in that the behaviour is straightforward to compute and verify, contrasting with the continuous-time case. The counterexamples of Section 2 show that it is not sufficient to consider odd nonlinearities to determine absolute stability, even with low order systems. We have discussed implications for the analysis of systems under constant disturbance or set-point demand.

5 Acknowledgement

The authors would like to thank the anonymous reviewers for their many helpful comments and suggestions.

References

- Ahmad, N. S., J. Carrasco and W. P. Heath (2013a). LMI searches for discrete-time Zames-Falb multipliers. In: *Proceedings of the IEEE Conference on Decision and Control*. pp. 5258–5263.
- Ahmad, N. S., J. Carrasco and W. P. Heath (2015). A less conservative LMI condition for stability of discrete-time systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control* **60**(6), 1692–1697.
- Ahmad, N. S., W. P. Heath and G. Li (2013b). LMI-based stability criteria for discrete-time Lur'e systems with monotonic, sector- and slope-restricted nonlinearities. *IEEE Transactions on Automatic Control* **58**(2), 459–465.
- Aizerman, M. A. and F. R. Gantmacher (1964). *Absolute Stability of Regular Systems*. Holden-Day.
- Altshuller, D. (2013). *Frequency Domain Criteria for Absolute Stability: A Delay-integral-quadratic Constraints Approach*. Springer.
- Barabanov, N. E. (1988). On the Kalman problem. *Siberian Mathematical Journal* **29**, 333–341.
- Bragin, V., V. Vagaitsev, N. Kuznetsov and G. Leonov (2011). Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits. *Journal of Computer and Systems Sciences International* **50**(4), 511–543.
- Brogliato, B., B. Maschke, R. Lozano and O. Ege-land (2006). *Dissipative Systems Analysis and Control: Theory and Applications*. Springer.
- Carrasco, J., M. C. Turner and W. P. Heath (2015a). Zames-Falb multipliers for absolute stability: from O'Shea's contribution to convex searches. In: *Proceedings of the European Control Conference, ECC15*.
- Carrasco, J., M. Maya-Gonzalez, A. Lanzon and W.P. Heath (2014). LMI searches for anticausal and non-causal rational Zames-Falb multipliers. *Systems and Control Letters* **70**, 17–22.
- Carrasco, J., W. P. Heath and M. de la Sen (2015b). Second-order counterexample to the discrete-time Kalman conjecture. In: *Proceedings of the European Control Conference, ECC15*.
- Carrasco, J., W.P. Heath, G. Li and A. Lanzon (2012). Comments on "on the existence of stable, causal multipliers for systems with slope-restricted nonlinearities". *IEEE Transactions on Automatic Control* **57**(9), 2422–2428.
- Chang, M., R. Mancera M. and Safonov (2012). Computation of Zames-Falb multipliers revisited. *IEEE Transactions on Automatic Control* **57**(4), 1024–1029.
- Chen, X. and J. Wen (1996). Robustness analysis for linear time invariant systems with structured incrementally sector bounded feedback non-linearities. *Applied Mathematics and Computer Science* **6**, 623648.
- Fitts, R. (1966). Two counterexamples to Aizerman's conjecture. *IEEE Transactions on Automatic Control* **11**(3), 553–556.
- Gonzaga, C. A., M. Jungers and J. Daafouz (2012). Stability analysis of discrete-time Lur'e systems. *Automatica* **48**(9), 2277–2283.
- Haddad, W. M. and D. S. Bernstein (1994). Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle and Popov theorems and their application to robust stability. part ii: Discrete time theory. *International Journal of Robust and Non-linear Control* **4**(2), 249–265.
- Haddad, W. M. and V. Chellaboina (2008). *Nonlinear Dynamical Systems and Control*. Princeton University Press.
- Heath, W. P. and J. Carrasco (2015). Global asymptotic stability for a class of discrete-time systems. In: *Proceedings of the European Control Conference, ECC15*.
- Heath, W.P. and A.G. Wills (2007). Zames-Falb multipliers for quadratic programming. *IEEE Transactions on Automatic Control* **52**(10), 1948–1951.
- Hu, T. and Z. Lin (2001). A complete stability analysis of planar discrete-time linear systems under saturation. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **48**(6), 710–725.

- Hu, T., Z. Lin and B. M. Chen (2002). Analysis and design for discrete-time linear systems subject to actuator saturation. *Systems and Control Letters* **45**(2), 97–112.
- Kalman, R. E. (1957). Physical and mathematical mechanisms of instability in nonlinear automatic control systems. *Trans. ASME* **79**(3), 553–566.
- Kapila, V. and W. M. Haddad (1996). A multivariable extension of the Tsypkin criterion using a Lyapunov-function approach. *IEEE Transactions on Automatic Control* **41**(1), 149–152.
- LaSalle, J. P. (1976). *The Stability of Dynamical Systems*. SIAM.
- Leonov, G. A. and N. V. Kuznetsov (2013). Hidden attractors in dynamical systems. from hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits. *International Journal of Bifurcation and Chaos* **23**(1), 1–69.
- Leonov, G., N. Kuznetsov and V. Bragin (2010). On problems of Aizerman and Kalman. *Vestnik St. Petersburg University: Mathematics* **43**, 148–162.
- Leonov, G.A. and N.V. Kuznetsov (2011). Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems. *Doklady Mathematics* **84**(1), 475–481.
- Megretski, A. and A. Rantzer (1997). System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control* **42**(6), 819–830.
- Park, B. Y., P. Park and N. K. Kwon (2015). An improved stability criterion for discrete-time Lur’e systems with sector- and slope-restrictions. *Automatica* **51**(1), 255 – 258.
- Park, P. and S. W. Kim (1998). A revisited Tsypkin criterion for discrete-time nonlinear Lur’e systems with monotonic sector-restrictions. *Automatica* **34**(11), 1417–1420.
- Safonov, M. G. and G. Wyetzner (1987). Computer-aided stability analysis renders Popov criterion obsolete. *Transactions on Automatic Control* **32**(12), 1128–1131.
- Tarbouriech, S., G. Garia, J. M. Gomes da Silva Jr. and I. Qieinnec (2011). *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer.
- Tsypkin, Y. Z. (1962). On the stability in the large of nonlinear sampled-data systems. *Doklady Akademii Nauk SSSR* **145**, 52–55.
- Vidyasagar, M. (1993). *Nonlinear Systems Analysis*. Prentice-Hall International Editions, reprinted SIAM 2002.
- Wang, S., W. P. Heath and J. Carrasco (2014). A complete and convex search for discrete-time noncausal FIR Zames-Falb multipliers. In: *Proceedings of the IEEE Conference on Decision and Control*.
- Willems, J. and R. Brockett (1968). Some new rearrangement inequalities having application in stability analysis. *IEEE Transactions on Automatic Control*, **13**(5), 539 – 549.
- Yang, T., A. A. Stoorvogel and A. Saberi (2013). Dynamic behavior of the discrete-time double integrator with saturated locally stabilizing linear state feedback laws. *International Journal of Robust and Nonlinear Control* **23**(17), 1899–1931.
- Zaccarian, L. and A. R. Teel (2011). *Modern Anti-windup Synthesis*. Princeton.