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LMI searches for anticausal and noncausal rational Zames–Falb multipliers ☆

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ABSTRACT

Given a linear time-invariant plant, the search for a suitable multiplier over the class of Zames–Falb multipliers is a challenging problem which has been studied for several decades. Recently, a new linear matrix inequality search has been proposed over rational and causal Zames–Falb multipliers. This letter analyzes the conservatism of the restriction to causality on the multipliers and presents a complementary search for rational and anticausal multipliers. The addition of a Popov multiplier to the anticausal Zames–Falb multiplier is implemented by analogy with the causal search. As a result, a search over a noncausal subset of Zames–Falb multipliers is obtained. A comparison between all the search methods proposed in the literature is given.

1. Introduction

The use of noncausal multipliers in absolute stability was widely studied in the sixties, with particular attention to the class of slope-restricted nonlinearities. O'Shea [1,2] was the first to propose a class of noncausal multipliers; see also [3]. Zames and Falb [4] propose a general framework for the use of noncausal multipliers in passivity theory and provide a formal proof for the results given in [2], since the validity of the results given by O'Shea in [2] was limited by "the a priori assumption that the solutions are bounded" [4]. Nowadays these multipliers are referred to as Zames–Falb multipliers.

Definition 1.1. The class of Zames–Falb multipliers is given by the operators $M : \mathcal{L}_2(-\infty,\infty) \rightarrow \mathcal{L}_2(-\infty,\infty)$ [1] whose transfer function is within the following set

$$\mathcal{M} = \left\{ M(s) = 1 - H(s) : H(s) = \mathfrak{L}(h(t)), \int_{-\infty}^{\infty} |h(t)| dt < 1 \right\}, \tag{1}$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ and $H(s)$ means the bilateral Laplace transform of $h(t)$, i.e. $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$.

By use of a loop-transformation [5], the stability of a system $G \in \mathbf{RH}_\infty$ in feedback interconnection with any slope restricted $S[0, k]$ and odd nonlinearity can be guaranteed if there exists a Zames–Falb multiplier M such that $M(G + 1/k)$ is strictly positive, i.e.

$$\text{Re}\left\{ M(j\omega)\left(G(j\omega) + \frac{1}{k} \right) \right\} > 0 \quad \forall \omega \in \mathbb{R}. \tag{2}$$

But given G, it is not straightforward to find such an M. The difficulty arises from the characterization of the Zames–Falb multipliers: their definition includes a bound on the time-domain response (1). The problem to be addressed is: given a system $G \in \mathbf{RH}_\infty$ and a constant $k > 0$, under what conditions is the existence of a Zames–Falb multiplier $M \in \mathcal{M}$ ensured?

To date, different partial solutions have been given. In [6–8], a linear program is proposed to find a suitable irrational multiplier whose impulse response is parameterized with delta functions. This method requires the computation of the Nyquist plot over an infinitely dense frequency sweep, which while not computationally attractive gives results that are very competitive. In general, the positiveness of the solution cannot be checked in a linear matrix inequality (LMI) framework; hence it may provide a false positive solution. In [9], a rational parametrization of a transfer function is proposed in such a way that its $\mathcal{L}_1$-norm can be bounded. A search over the set of parameters under the condition in (1) must be carried out. As an advantage, it can be optimized

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[1] Proper definitions of the signal spaces are given in Section 2.

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by using LMIs; moreover, the localization of the poles may be selected using the IQC toolbox [10]. Recently, it has been shown that this search is asymptotically complete [11].

These methods avoid any conservatism in the characterization of the multiplier when the nonlinearity is slope restricted, since the parametrization is chosen in order to compute analytically the integral in (1). However, the rational parametrization will introduce some conservatism when the nonlinearity is odd, since the integral can only be bounded using a triangular inequality. This conservatism may be avoided by using very high order multipliers, but the problem becomes numerically ill-conditioned. The result depends on the skill of the user, since a prior parameter selection must be done and a posterior check could be required. In general, the results using exponential functions are less competitive than using delta functions, but delta functions may lead to unreliable results.

Park [12] develops an LMI search for a particular Lyapunov function. The result is interpreted in the frequency domain, and the resulting multipliers are equivalent to first order Zames–Falb multipliers [13]. Hence, one can think of Park's LMIs as a search over the first order Zames–Falb multipliers. Recently, an LMI search over rational and causal Zames–Falb multipliers has been proposed in [14] (see also [15]), by using the multiobjective synthesis technique presented in [16]. The restriction on the causality of the multiplier has partially been overcome by adding a Popov multiplier [17,18], which can include a pole at $+\infty$, with the resulting multiplier noncausal.

The LMI methods are independent of the skills of the user and are easily reproducible. The existence of a suitable multiplier can be guaranteed by checking the feasibility of a set of LMIs, but two main drawbacks to the method proposed in [14] can be stated:

- The search has an inherent conservativeness. For the check if a transfer function is a Zames–Falb multiplier, the integral in (1) is not computed, but bounded via an LMI. As commented in [16], this upper bound "can be fairly conservative".
- The multiplier is restricted to be causal and the same order of the plant.

The authors [14] justify the last restriction stating that other classes of multipliers, as used in the Circle and Popov criteria (see [19,20]), and Park's method [12], are within this characterization.

However, Park's method uses the following class of multipliers:

$$M_p(s) = 1 + \frac{bs}{a^2 - s^2} \tag{3}$$

where $a, b \in \mathbb{R}$. Hence causality is not required in Park's method. The numerical results in [14,15] are competitive with Park's method [12] for some examples and worse than Park's method for other examples discussed, hinting that the causal restriction may be significant. The extension proposed in [17,18] adds a Popov multiplier to the Zames–Falb multiplier. The Popov multiplier can be interpreted as an anti-causal component in the Zames–Falb multiplier [13]. This extension gives an improvement for some examples, but fails to reach the Nyquist value for Example 1 in [14,18]. Since the Kalman conjecture is guaranteed for third order systems [21], this example shows some conservatism.

In this letter we analyze the limitations on the phase of the multiplier when it is restricted to be either causal or anticausal. We give specific bounds on the phase for first order multipliers. The analysis motivates the development of anticausal counterparts of the searches proposed in [14,15,17,18]. The key novelty is the possibility of using $P < 0$. Since the previous literature in multiobjective techniques [16] has been focused on control synthesis, the condition $P > 0$ has been always required. This condition can be relaxed in the multiplier synthesis. Technical details are omitted in some places where their direct counterparts are found in [14,17]. We show that the main source of conservatism in [14,15,17,18] is the limitation of the multiplier to be causal. When causal, anticausal and noncausal searches are used together, it is clear that the loose bound on the $\mathcal{L}_1$-norm is not the main source of conservatism. Finally, we are able to answer the question raised in [18] on the performance of the algorithms. Limitations on the phase of causal and anticausal multipliers are the key idea to understanding *a priori* the performance of the corresponding search algorithms. Thus we have been able to develop an example (Example 9), where the algorithms presented in this paper improve *all* search methods proposed in the literature. In particular, our result for this example is 16 times better than [14] and 5 times better than [18]. Preliminary results for anticaulsal multipliers were presented in [22].

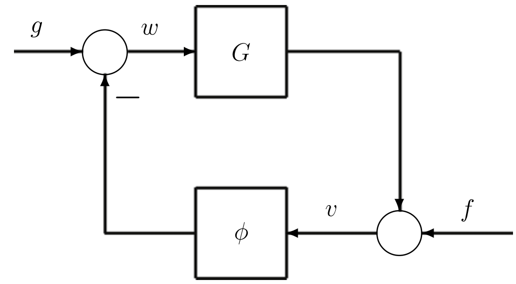

Fig. 1. Lur'e problem.

2. Notation and preliminary results

Let $\mathcal{L}_2^m[0,\infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}^m$ with the inner product defined as $\langle f, g\rangle = \int_0^\infty f(t)^\top g(t)dt$, for $f,\ g \in \mathcal{L}_2^m[0,\infty)$. The symbol $^\top$ means transpose. Similarly, $\mathcal{L}_2^m(-\infty,\infty)$ can be defined and its inner product is given by $\langle f, g\rangle = \int_{-\infty}^\infty f(t)^\top g(t)dt$.

A truncation of the function f at T is given by $f_T(t) = f(t),\ \forall t \le T$ and $f_T(t) = 0,\ \forall t > T$. The function f belongs to the extended space $\mathcal{L}_{2e}^m[0,\infty)$ if $f_T \in \mathcal{L}_2^m[0,\infty)$ for all $T > 0$. In addition, $\mathcal{L}_1^m(-\infty,\infty)$ is the space of all absolute integrable functions, and given a function $h : \mathbb{R} \to \mathbb{R}$ such that $h \in \mathcal{L}_1$; then its $\mathcal{L}_1$-norm is given by $\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|dt$.

A nonlinearity $\phi : \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ is said to be memoryless if there exists $N : \mathbb{R} \to \mathbb{R}$ such that $(\phi v)(t) = N(v(t))$ for all $t \in \mathbb{R}$. Henceforward we assume that $N(0) = 0$. A memoryless nonlinearity ϕ is said to be bounded if there exists C such that $|N(x)| < C|x|$ for all $x \in \mathbb{R}$. A memoryless nonlinearity ϕ is slope restricted in the interval $S[0,k]$, henceforward ϕ_k, if

$$0 \le \frac{N(x_1) - N(x_2)}{x_1 - x_2} \le k \tag{4}$$

for all $x_1 \ne x_2$. The nonlinearity ϕ is said to be odd if $N(x) = -N(-x)$ for all $x \in \mathbb{R}$.

This paper focuses on the stability of the feedback interconnection of a stable LTI system G and a slope-restricted nonlinearity ϕ_k, represented in Fig. 1 and given by

$$\begin{cases} v = f + Gw, \\ w = g - \phi_k v. \end{cases} \tag{5}$$

It is well-posed if the map $(v, w) \mapsto (g, f)$ has a causal inverse on $\mathcal{L}_{2e}^2[0,\infty)$. Since G is a stable LTI system, the exogenous input g can be taken as the zero signal without loss of generality. This interconnection is stable if for any $f \in \mathcal{L}_2[0,\infty)$, both $w \in \mathcal{L}_2[0,\infty)$ and $v \in \mathcal{L}_2[0,\infty)$. In addition, $G(s)$ means the transfer function of the LTI system G.

The standard notation $\mathbf{L}_\infty$ is used for the space of all transfer functions bounded on the imaginary axis and at infinity. $\mathbf{RL}_\infty$ is used for the space of all proper real rational transfer functions bounded on the imaginary axis, $\mathbf{RH}_\infty$ is used for the space of all (proper real rational) transfer functions such that all their poles have strictly negative real parts, and $\mathbf{RH}_\infty^-$ is used for the space of all proper real rational transfer functions such that all their poles have strictly positive real parts. With some reasonable abuse of the notation, given a rational strictly proper transfer function $H(s)$ bounded at the imaginary axis, $\|H\|_1$ means the $\mathcal{L}_1$-norm of impulse response of $H(s)$.

Let $\bar{M}$ denote a linear time-invariant operator mapping a time domain input signal to a time domain output signal and M denote the corresponding transfer function, for some particular region of convergence of the bilateral Laplace integral, mapping the bilateral Laplace transform of the time-domain input signal to the bilateral Laplace transform of the time domain output signal. To avoid ambiguity in impulse responses that correspond to transfer functions when the bilateral Laplace transform is used (see [23]), we insist on a causal $\bar{M}$ when $M \in \mathbf{RH}_\infty$ with the RHP contained in the region of convergence and an anticausal $\bar{M}$ when $M \in \mathbf{RH}_\infty^-$ with the LHP contained in the region of convergence. Since any $M \in \mathbf{RL}_\infty$ with a region of convergence that includes the imaginary axis can be split into the sum of two functions, one in $\mathbf{RH}_\infty$ and one in $\mathbf{RH}_\infty^-$, the corresponding $\bar{M}$ is noncausal corresponding to the sum of a causal part and an anticausal part. Henceforward and with some abuse of notation, we will use the same notation for the operator and its transfer function.

The following theorem provides the absolute stability of system (5) subject to the search of an appropriate Zames–Falb multiplier.

Theorem 2.1 (*[4,5]*). *Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$, and ϕ_k a slope restricted $S[0, k]$ and odd nonlinearity. Suppose that there exists $M \in \mathcal{M}$ (Definition 1.1) such that*

$$\mathrm{Re}\,\{M(j\omega)(1 + kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \tag{6}$$

Then the feedback interconnection (5) is $\mathcal{L}_2$-stable. ■

This theorem characterizes the class of Zames–Falb multipliers. The search presented in this letter focuses on rational multipliers, i.e. $M \in \mathbf{RL}_\infty$.

In this letter the symbol $M^\sim$ means the $\mathcal{L}_2$-adjoint of M. This operator satisfies $\langle y, Mx\rangle = \langle M^\sim y, x\rangle$ for all $u \in \mathcal{L}_2(-\infty, \infty)$ and $y \in \mathcal{L}_2(-\infty, \infty)$. As a result, $M^\sim$ is anticausal if and only if M is causal [5]. In particular, the $\mathcal{L}_2$-adjoint of a rational transfer function $M(s)$ is given by $M^\top(-s)$. In the time domain, the impulse response is reflected with respect to $t = 0$, i.e. given a linear operator M with an impulse response $m(t)$ the impulse response of $M^\sim$ is $m^\top(-t)$. As a result, $M^\sim$ is an anticausal Zames–Falb multiplier if and only if M is a causal Zames–Falb multiplier.

The following lemma identifies when a transfer function is a Zames–Falb multiplier.

Lemma 2.2 (*[15]*). *Let $M \in \mathbf{RL}_\infty$ be a rational transfer function with $M(s) = M(\infty) + \hat{M}(s)$, where $\hat{M}(s)$ denotes its associated strictly proper transfer function. Then, $M(s)/M(\infty)$ is a Zames–Falb multiplier if and only if $\|\hat{M}\|_1 < M(\infty)$.* ■

The Nyquist value is defined and the Kalman conjecture is stated as follows.

Definition 2.3. Given a stable linear plant $G \in \mathbf{RH}_\infty$, the Nyquist value, k_N is the supremum of the values $\bar{k}$ such that $kG(s)$ satisfies the Nyquist criterion for all $k \in [0, \bar{k}]$, i.e.

$$k_N = \sup\{\bar{k} \in \mathbb{R}^+ : \inf_{\omega\in\mathbb{R}}\{|1 + kG(j\omega)|\} > 0\ \forall k \in [0, \bar{k}]\}. \tag{7}$$

This value is used as a benchmark in other papers discussing Zames–Falb multiplier searches [6,9], and it is straightforward to show that Theorem 2.1 cannot be satisfied for $k \geq k_N$. As a result, given $G \in \mathbf{RH}_\infty$, the search for Zames–Falb multipliers must only be carried out for $0 < k < k_N$.

Conjecture 2.4 (*Kalman Conjecture*). *The feedback interconnection of a strictly proper plant G and ϕ_k is stable for any $k < k_N$.* ■

Remark 2.5. This conjecture has an important role in the development of absolute stability and is true for $n \leq 3$ [21], where n is the order of $G(s)$, but is false in general.

Lemma 2.6 (*[21,13]*). *Given a strictly proper plant G with order 3 or less, and $k < k_N$, there exists a first order Zames–Falb multiplier M such that*

$$\mathrm{Re}\,\{M(j\omega)(1 + kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}. \tag{8}$$

3. Discussion on causal multipliers

In this section we show that causality can be a significant source of conservatism. Let us consider Example 1 in [14], which considers the plant

$$G(s) = \frac{s^2 - 0.2s - 0.1}{s^3 + 2s^2 + s + 1} \tag{9}$$

where a factor -1 has been applied to take into account negative feedback. A linear search shows that $k_N \in (4.5894, 4.5895)$; thus the search of suitable Zames–Falb multipliers satisfying Theorem 2.1 can be restricted to $k \in [0, 4.5894]$. Then there exists a first order Zames–Falb multiplier M such that $\mathrm{Re}\{M(1+4.5894G)\} > 0$ for all $\omega \in \mathbb{R}$ (see Lemma 2.6).

After a simple trial and error procedure, a suitable noncausal Zames–Falb multiplier given by inspection is

$$M_{nc}(s) = \frac{s - 0.0012}{s - 1.09}. \tag{10}$$

We can check the following properties:

- It is a Zames–Falb multiplier since
$$\|\hat{M}_{nc}\|_1 = \left\|\frac{1.0888}{s - 1.09}\right\|_1 = 0.9989 < 1 = M_{nc1}(\infty), \tag{11}$$
where Lemma 2.2 has been used.
- $\mathrm{Re}\{M_{nc}(j\omega)(1 + 4.5894G(j\omega))\} > 0$ for all $\omega \in \mathbb{R}$.

Thus, Theorem 2.1 ensures the absolute stability of G and $\phi \in S[0, 4.5894]$. This property has also been checked via LMI using the KYP lemma [24]. Some questions can be immediately asked: why is this multiplier anticausal? Is it possible to find a causal first order Zames–Falb multiplier satisfying the above condition?

There exists a trade-off between phase and $\mathcal{L}_1$-norm [25], which is exacerbated when the multiplier is limited to be either causal or anticausal. Analytic results can be shown if we restrict our attention to first order Zames–Falb multipliers.

Lemma 3.1. *If M_c is a causal first order Zames–Falb multiplier, then $\angle M_c(j\omega) > -\arcsin(1/3)$ for all $\omega \in \mathbb{R}$. Moreover, given $\epsilon > 0$, there exists a causal Zames–Falb multiplier such that its phase is $90° - \epsilon$ at some frequency.* ■

Lemma 3.2. *If M_{ac} is an anticausal first order Zames–Falb multiplier, then $\angle M_{ac}(j\omega) < \arcsin(1/3)$ for all $\omega \in \mathbb{R}$. Moreover, given $\epsilon > 0$, there exists an anticausal Zames–Falb multiplier such that its phase is $-90° + \epsilon$ at some frequency.* ■

The above two lemmas can be proved by using Lemma 2.2 on a first order transfer function. Summarizing, they state that the phase of a causal first order Zames–Falb multiplier must be within $(-\arcsin(1/3), 90^\circ)$ and the phase of an anticausal first order Zames–Falb multiplier must be within $(-90^\circ, \arcsin(1/3))$.

We can now investigate the dependence of the phase of $1 + kG(j\omega)$ with respect to k. For $0 \le k \le 1.2431$, the phase will be with the interval $(-90^\circ, 90^\circ)$ and the circle criterion can be applied. For $1.2431 < k \le 4.5894$, the phase lead defect increases from 0 at $k = 1.2431$ up to 75° at $k = 4.5894$. Therefore, for a causal first order Zames–Falb multiplier, a theoretical limitation can be set when $(1 + kG)$ has a phase lead larger than $90^\circ + \arcsin(1/3)$, approximately, 109.47°. This limit is crossed at $k = 1.805$.

Now we can answer the two questions given at the beginning of the section. Since the system $(1 + 4.5894G)$ has a phase lead larger than 109.47°, there exists no causal first order Zames–Falb multiplier M_c satisfying (6). If we are restricted to first order Zames–Falb multipliers then anticausal Zames–Falb multipliers must be used.

If we consider causal third order Zames–Falb multipliers by using the search in [14,15], we find that the maximum slope is 2.2428, improving the value of the causal first order Zames–Falb multiplier. Thus, one could postulate that the theoretical limitations given by restricting the search over the set of causal first order Zames–Falb multipliers may be avoided by using higher order or irrational causal Zames–Falb multipliers. Table 1 shows the result obtained with different methods for searches for Zames–Falb multipliers proposed in the literature. The other methods [6,9] consider noncausal multipliers, so they have been modified to search over causal multipliers only. Although they have been optimized with more powerful tools than the above inspection method, these causal multipliers remain conservative. In conclusion and for this example, all searches proposed in the literature are conservative if the search is restricted to causal multipliers. Even the relaxation of causal Zames–Falb multipliers plus Popov multipliers [18] is not able to reach the Nyquist value.

4. LMI search for anticausal multipliers

This section presents a modification to the causal method of [14] able to search for an anticausal Zames–Falb multiplier. This modification must be considered as a complementary method. It is known that any rational Zames–Falb multiplier $M(s)$ has a canonical factorization [4], i.e. $M = M_- M_+$ where $M_-^\sim \in \mathbf{RH}_\infty$, $(M_-^\sim)^{-1} \in \mathbf{RH}_\infty$, $M_+ \in \mathbf{RH}_\infty$, and $M_+^{-1} \in \mathbf{RH}_\infty$. Loosely speaking, in [14] M_- is taken as the identity, whereas we propose a equivalent synthesis taking M_+ as the identity.

Some proofs in this section are omitted since they can be developed with the same mathematical machinery as in [14,17] and setting $P < 0$.

4.1. Anticausal search

The method proposed in [14] is based on the multiobjective synthesis developed in [16]. In our complementary method, we substitute the condition $P < 0$ (for $P > 0$ in [14]). Note that P nonsingular ensures that the change of variable is feasible. A prior lemma is needed to bound the $\mathcal{L}_1$-norm of an anticausal transfer function.

Lemma 4.1. *Given a strictly proper transfer function $H \in \mathbf{RH}_\infty^-$ given by $H(s) = C(sI - A)^{-1}B$, where $-A$ is Hurwitz, assume that there exist $Y < 0$, $\mu > 0$, $\xi > 0$, and $\lambda > 0$ such that*

$$\begin{bmatrix} A^\top Y + YA - \lambda Y & YB \\ \star & -\mu \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} -\lambda Y & 0 & C^\top \\ \star & (\xi - \mu) & 0 \\ \star & \star & \xi \end{bmatrix} > 0. \tag{13}$$

Then $\|H\|_1 < \xi$.

Proof. The result is straightforward since $\|H\|_1$ is the same as $\|H^\sim\|_1$, where $H^\sim(s)$ is given by

$$H^\sim(s) = (C(sI - (-A))^{-1}(-B)). \tag{14}$$

Taking $W = -Y$ in (12) and (13), there exist $W > 0$, $\mu > 0$, and $\lambda > 0$ such that

$$\begin{bmatrix} (-A^\top)W + W(-A) + \lambda W & WB \\ \star & -\mu \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} \lambda W & 0 & C^\top \\ \star & (\xi - \mu)I & 0 \\ \star & \star & \xi \end{bmatrix} > 0. \tag{16}$$

Then applying results in [16], it is obtained that $\|-H^\sim\|_1 = \|H\|_1 < \xi$. ■

Using this lemma, the result for anticausal multipliers can be stated as follows.

Proposition 4.2. *Let $G \in \mathbf{RH}_\infty$ be represented by the state space matrices A_g, B_g, C_g, and D_g. Let ϕ be a slope restricted $S[0, k]$ and odd nonlinearity. Assume that there exist positive definite symmetric matrices $S_{11} > 0$, $P_{11} > 0$, unstructured matrices $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$, and positive constant $\mu > 0$ and $\lambda > 0$, such that the LMIs (17)–(19) are satisfied (Eqs. (17)–(19) are given in Box I). Then the feedback interconnection (5) is $\mathcal{L}_2$-stable.*

4.2. Addition of a Popov multiplier

As shown in [26], the Popov multiplier is a limiting case of a Zames–Falb multiplier

$$1 + qs = \lim_{\epsilon \to 0} \frac{1 + qs}{1 + \epsilon s}. \tag{20}$$

A detailed analysis of this limit has been carried out in [13]. Therefore, since the method proposed originally in [14] is restricted to causal multipliers and its anticausal counterpart has been developed in the previous section, the addition of a Popov multiplier as proposed in [17,18] to these causal or anticausal searches improves the parametrization of the Zames–Falb multiplier. The result in [18] will generate noncausal Zames–Falb multipliers with a limited anticausal part whereas the following result will generate Zames–Falb multipliers with a limited causal part.

Table 1
Maximum slope for different classes of multipliers.

Multiplier	Maximum slope k
Causal high order [9]	1.624
Causal irrational [6]	1.775
Causal order 1, Lemma 3.1	1.8 (approx)
Causal order 3, Turner method [14]	2.2428
Causal order 3 plus Popov multiplier [18]	3.5026
Noncausal order 1, Eq. (10)	4.5894
Nyquist value	4.5894

$$\begin{bmatrix} S_{11}A_g + A_g^\top S_{11} & S_{11}A_g + A_g^\top P_{11} + kC_g^\top \mathbf{B}_u^\top + \mathbf{A}_u^\top & S_{11}B_g - kC_g^\top + \mathbf{C}_u^\top \\ \star & P_{11}A_g + A_g^\top P_{11} + \mathbf{B}_u kC_g + kC_g^\top \mathbf{B}_u^\top & P_{11}B_g + \mathbf{B}_u(1+kD_g) - kC_g^\top \\ \star & \star & -(I+kD_g) - (I+kD_g)^\top \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} -\mathbf{A}_u - \mathbf{A}_u^\top - \lambda(P_{11} - S_{11}) & \mathbf{B}_u \\ \star & -\mu \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \mathbf{C}_u^\top \\ \star & (1-\mu)I & 0 \\ \star & \star & 1 \end{bmatrix} > 0. \tag{19}$$

Box I.

$$\begin{bmatrix} S_{11}A_g + A_g^\top S_{11} & S_{11}A_g + A_g^\top P_{11} + kC_g^\top \mathbf{B}_u^\top + \mathbf{A}_u^\top & S_{11}B_g - kC_g^\top - k(\gamma + \nu A_g^\top)C_g^\top + \mathbf{C}_u^\top \\ \star & P_{11}A_g + A_g^\top P_{11} + \mathbf{B}_u kC_g + kC_g^\top \mathbf{B}_u^\top & P_{11}B_g - kC_g^\top - k(\gamma + \nu A_g^\top)C_g^\top + \mathbf{B}_u \\ \star & \star & -2 - 2\gamma - \nu kC_gB_g - \nu kB_g^\top C_g^\top \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} -\mathbf{A}_u - \mathbf{A}_u^\top - \lambda(P_{11} - S_{11}) & \mathbf{B}_u \\ \star & -\mu \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \mathbf{C}_u^\top \\ \star & (1-\mu)I & 0 \\ \star & \star & 1 \end{bmatrix} > 0. \tag{23}$$

Box II.

Proposition 4.3. *Let $G \in \mathbf{RH}_2$ be represented by the state space matrices A_g, B_g, and C_g. Let ϕ be a slope restricted $S[0,k]$ and odd nonlinearity. Assume that there exist positive definite symmetric matrices $S_{11} > 0$, $P_{11} > 0$, unstructured matrices $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$, and positive constants $\gamma > 0$, $\mu > 0$ and $\lambda > 0$, and a real constant ν, such that the LMIs (21)–(23) are satisfied (Eqs. (21)–(23) are given in Box II). Then the feedback interconnection (5) is $\mathcal{L}_2$-stable.*

Remark 4.4. As commented in [15], a search over λ is required for obtaining competitive results. In the causal Zames–Falb search [14,15] as well as in the anticausal search presented in the previous section, the maximum slope k appears to have a quasi-convex dependence with respect to λ. However, the addition of the Popov multiplier, in [18] and in this section, changes this behavior, and several local maxima can appear.

Remark 4.5. The Zames–Falb multiplier can be reconstructed from $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$ as in [15], whereas the Popov multiplier is given by $1 + \nu/\gamma s$. Hence

$$M_{\text{P+ZF}}(s) = M_{\text{ZF}}(s) + \gamma\left(1 + \frac{\nu}{\gamma}s\right) \tag{24}$$

is the multiplier obtained from the search.

It is straightforward to obtain the counterpart version of the $\mathcal{L}_2$-gain bound result in [17], when the Zames–Falb multiplier is an anticausal multiplier.

5. Numerical examples

Table 2 shows nine examples. Examples 1-6 are discussed in [14,15] (Example 1 has been used in Section 3), while Examples 7 and 8 are given in [27,8], respectively. Example 9 is new. Results are obtained using the MATLAB LMI Toolbox. For Examples 1, 2 and 9, results of the anticausal methods are obtained using $1/k + G(j\omega)$ rather than $1 + kG(j\omega)$ as the numerical results sometimes differ. In the use of IQC-β, four poles have been placed at 1 and another four poles at -1.

Table 2
Examples.

Ex.	$G(s)$
1 [14]	$G_1(s) = \frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$
2 [14]	$G_2(s) = -G_1(s)$
3 [14]	$G_3(s) = \frac{s^2}{s^4+0.2s^3+6s^2+0.1s+1}$
4 [14]	$G_4(s) = -G_3(s)$
5 [14]	$G_5(s) = \frac{s^2}{s^4+0.0003s^3+10s^2+0.0021s+9}$
6 [14]	$G_6(s) = -G_5(s)$
7 [27]	$G_7(s) = \frac{s^2}{s^3+2s^2+2s+1}$
8 [8]	$G_8(s) = 9.432\frac{(s^2+15.6s+147.8)(s^2+2.356s+56.21)(s^2-0.332s+26.15)}{(s^2+2.588s+90.9)(s^2+11.79s+113.7)(s^2+14.84s+84.05)(s+8.83)}$
9 (new)	$G_9(s) = \frac{s^2}{s^4+5.001s^3+7.005s^2+5.006s+6}$

Table 3 gives the results for the plants in Table 2. As expected, results for the anticausal method improve the maximum slope for the plants where Park's method is better than the causal method [14] (Examples 1, 3, and 6). Park's multipliers are competitive for slightly damped plants, since they carry no conservativeness in the bound of the $\mathcal{L}_1$-norm. Nevertheless, the addition of the Popov multiplier in [17,18] and its implementation for the anticausal method provides a reliable and competitive method if they are combined.

Example 9 has been designed to show under what circumstances the methods proposed in this letter are expected to provide better results than alternative methods in the literature. Loosely speaking, anticausal multipliers are expected to be more appropriate than causal multipliers for achieving negative values of the phase. In addition, temporal searches such as the delta and exponential methods are conservative for multipliers within the full set of Zames–Falb multipliers, due to the use of a triangular inequality for bounding the $\mathcal{L}_1$-norm as commented earlier (see Eq. (10) in [9]); hence in plants with slightly damped poles it can be a drawback. Therefore, the proposed example has two resonant poles at $(-0.0005 \pm i)$, two zeros at 0 to ensure the Nyquist value is infinity, and two other poles at -2 and -3 so the order is more than 3 poles.

The numerical results show that the difference between causal and anticausal multipliers is larger at low order than at high

Table 3
Sector/slope bounds obtainable using various stability criteria.

Criteria	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8	Ex. 9
Circle	1.2431	0.7640	0.3263	0.3081	0.00040	0.00039	8.1235	4.3159	0.0482
Park's method [12]	**4.5894**	**1.0894**	0.7883	0.7083	0.00183	0.00183	**10,000+**	62.5691	26.0097
Causal method [14]	2.2428	1.0894	0.7049	0.8526	0.00181	0.00095	17.605	**87.3854**	5.2643
Anticausal method	**4.5894**	1.0745	0.9846	0.6135	0.00095	0.00182	**10,000+**	21.6190	38.5982
Causal + Popov method [17,18]	3.2897	**1.0894**	0.7760	1.0792	**0.00333**	0.00318	17.724	**87.3854**	13.7834
Anticausal + Popov method	**4.5894**	1.0745	1.4513	0.7222	0.00319	**0.00333**	**10,000+**	22.4304	**91.0858**
Delta method [6–8]	**4.5894**	**1.0894**	**1.6122**	**1.2652**	Unreliable	Unreliable	95.406	83.1430	80.2735
Exponential method via IQC-β [9,10]	4.5885	1.0893	1.1700	0.9541	0.00067	0.00068	**10,000+**	9.1375	48.7639
Nyquist value	4.5894	1.0894	∞	3.5000	∞	1.7142	∞	87.3854	∞

order, but several other factors, such as the amount of phase required, are important. From numerical results, causal multipliers are more appropriate when the Nyquist plot of the plant reaches the minimum value of its real part in the third quadrant (Examples 4 and 6), whereas anticausal multipliers are more appropriate when this minimum is reached in the second quadrant (Examples 1, 3, 5, and 7). If the Nyquist plot has similar real parts in the second and third quadrants, then the results are similar for causal and anticausal multipliers (Examples 2 and 8). This empirical rule agrees with the analysis of Section 3.

6. Conclusion

This letter has analyzed the consequences of restricting the set of Zames–Falb multipliers to causal multipliers. For first order Zames–Falb multipliers, theoretical results have shown that causal Zames–Falb multipliers have a strong constraint on their phase lag and anticausal Zames–Falb multipliers have a corresponding constraint on their phase lead. An example given in the literature has been used to show that a noncausal multiplier obtained by inspection beats all the convex searches if they are restricted to causal Zames–Falb multipliers.

Using the method developed in [14], a search of anticausal multipliers has been proposed, which is a complementary solution to the search of causal multipliers. The new search has been tested and it improves the results given by Turner's method [14] in the examples where this method is not competitive. A similar extension to that of [17] is proposed to avoid the anticausal limitation. The anticausal search developed in this paper confirms that a major source of conservatism for some examples in [14] is the restriction to causal multipliers. The combination of causal and anticausal methods with the addition of the Popov multiplier generates results at least competitive with the best in the literature. However, the delta method can provide better results in some cases due to its advantages measuring the $\mathcal{L}_1$-norm of the multiplier. Finding an efficient search over the entire class of Zames–Falb multipliers remains an open problem.

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