A Modern Conjecture on Absolute Stability

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Outline

Results and insights

In collaboration with...

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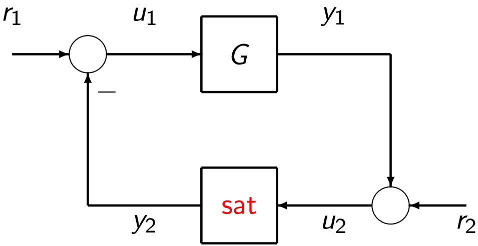
Classical motivation



Every actuator is affected by saturation, saturation is a standard nonlinearity in every control system.

Core history of Control Engineering

The golden decade (1962-1972)

- Input-Output stability
- Small Gain Theorem
- Passivity Theorem
- Circle Criterion
- Popov Criterion
- Kalman-Yakubovich-Popov lemma
- Dissipativity Theory

Stand on the shoulders of giants

Kalman, Yakubovich, Zames, Popov, Falb, (JC&JL) Willems, Brocket, Desoer, Vidyasagar,...

Recent motivation

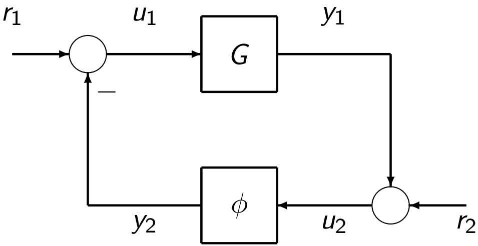
Zames-Falb multipliers for quadratic programming
WP Heath, AG Wills - … of the 44th IEEE Conference on Decision …, 2005 - ieeexplore.ieee.org
In constrained linear model predictive control a quadratic program must be solved on-line at each control step. If zero is feasible the resultant static nonlinearity is sector bound. We show that the nonlinearity is also monotone nondecreasing and slope restricted; furthermore it …
Cited by 60 Related articles

Zames-falb multipliers for quadratic programming
WP Heath, AG Wills - IEEE Transactions on Automatic Control, 2007 - cheric.org
In constrained linear model predictive control, a quadratic program must be solved on-line at each control step, and this constitutes. a nonlinearity. If zero is a feasible point for this quadratic program then the resultant nonlinearity is sector bounded. We show that if the …

Analysis and design of **optimization algorithms** via integral quadratic constraints
L **Lessard**, B Recht, A Packard - SIAM Journal on **Optimization**, 2016 - SIAM
This paper develops a new framework to analyze and design iterative optimization algorithms built on the notion of integral quadratic constraints (IQCs) from robust control theory. IQCs provide sufficient conditions for the stability of complicated interconnected …
Cited by 270 Related articles All 10 versions

Stability analysis using quadratic constraints for systems with neural network controllers
H **Yin**, P **Seiler**, M Arcak - arXiv preprint arXiv:2006.07579, 2020 - arxiv.org
A method is presented to analyze the stability of feedback systems with neural network controllers. Two stability theorems are given to prove asymptotic stability and to compute an ellipsoidal inner-approximation to the region of attraction. The first theorem addresses linear …
Cited by 7 Related articles All 3 versions

Lurye system



G is L.T.I.
ϕ is a nonlinear/uncertain system.

Lurye in memoriam of Dmitry A Altshuller (1961-2017)

Stability definitions

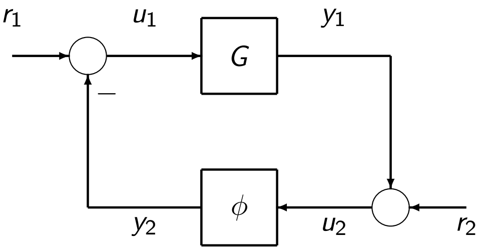
Lyapunon stability

For a system $\dot{x}(t) = f(x(t), t)$, with $x_0 = x(0)$.
We say that the system is globally asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all x_0.

Input-output stability

Our system is a causal operator, i.e. $y = Su$.
Stability is defined in terms of the properties of the signal, i.e. S is stable if y is energy-bounded for any energy bounded u

Input-Output stability of a feedback system

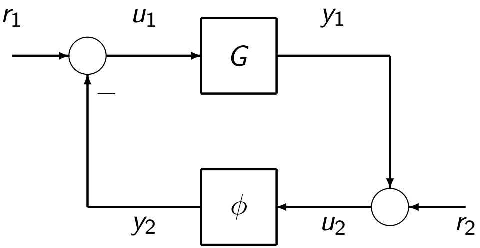


$$y_1 = Gu_1 \quad y_2 = \phi u_2$$
$$u_1 = r_1 - y_2 \quad u_2 = r_2 + y_1$$

Definition

The feedback is input-output stable if for any energy bounded pair of inputs (r_1, r_2), the pair of outputs (y_1, y_2) is also energy bounded.

Absolute stability problem



$$y_1 = Gu_1 \quad y_2 = \phi u_2, \quad \phi \in \Phi$$
$$u_1 = r_1 - y_2 \quad u_2 = r_2 + y_1$$

Problem

Find conditions on G such that the feedback interconnection between G and any $\phi \in \Phi$ is stable.

Solutions in three-step procedure

Step 1

Characterise the nonlinear class appropriately by means of a class of LTI multipliers

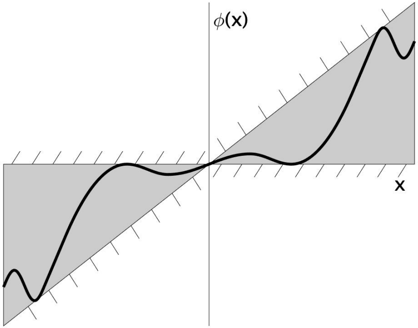
Step 2

Produce a frequency domain condition by using a stability result subject to the existence of one admissible multiplier

Step 3

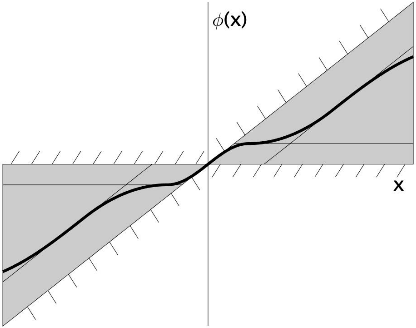
Given a system G, develop a procedure to search for the admissible multiplier

Sector-restricted nonlinearities



$$0 \leq \phi(x)/x \leq k$$

Slope-restricted nonlinearities



$$0 \leq \frac{\phi(x_1)-\phi(x_2)}{x_1-x_2} \leq k$$

Lurye problem

Lurye problem

For a given G, find the supremum k such that the feedback interconnection between G and any sector-restricted (or slope-resctricted) in the sector $[0, k]$ is stable.

Step 4

Find the minimum value of k such that Step 3 cannot be fulfilled.

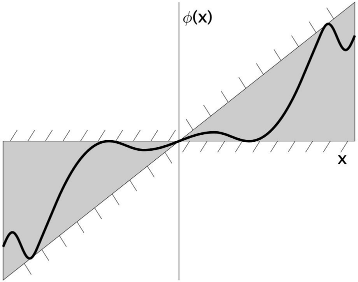
Nyquist gain

The Nyquist gain k_N of a stable LTI system G is the supremum of the set of gains k such that the feedback interconnection between G and the linear gain τk is stable for all $\tau \in [0, 1]$

Conjectures in the 50's

Aizerman Conjecture (1949)

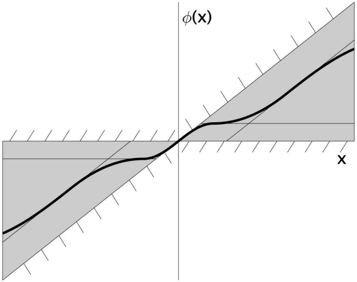
The negative feedback interconnection between an LTI stable system G and any nonlinearity in the sector $[0, k]$ is stable if and only if the feedback interconnection between G and the linear gain τk is stable for all $\tau \in [0, 1]$.



Conjectures in the 50's

Kalman Conjecture (1957)

The negative feedback interconnection between an LTI stable system G and any nonlinearity in the slope $[0, k]$ is stable if and only if the feedback interconnection between G and the linear gain τk is stable for all $\tau \in [0, 1]$.



Circle Criterion

Circle Criterion

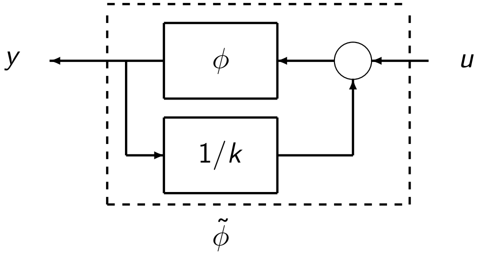
The negative feedback interconnection between an LTI stable system G and a nonlinearity in the sector $[0, k]$ is stable if there exists $\epsilon > 0$ such that

$$\mathrm{Re}(1 + kG(j\omega)) > \epsilon \qquad \forall \omega \in \mathbb{R}.$$

Circle Criterion

Step 1. Characterization

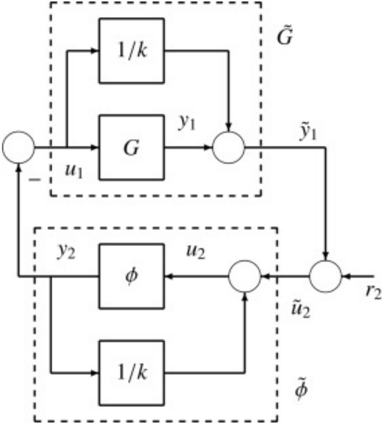
$$0 \leq x\phi(x) \leq k - \epsilon \implies 0 \leq x\tilde{\phi}(x)$$



Circle Criterion

Step 2 (and 3). Use the passivity theorem

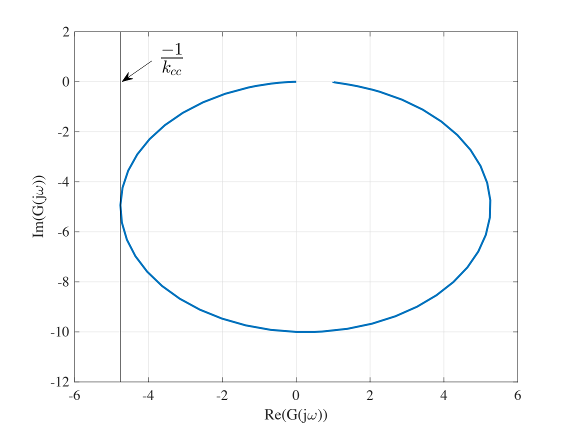
There exists $\epsilon > 0$ such that $\text{Re}(1 + kG(j\omega)) > \epsilon \quad \forall \omega \in \mathbb{R}$.



Circle Criterion

Step 4

The Circle Criterion hold for any $k < k_{cc}$



Popov Criterion

Popov Theorem

The negative feedback interconnection between a strictly proper LTI stable system G and a nonlinearity in the sector $[0, k]$ is stable if there exists $q \in \mathbb{R}$ and $\epsilon > 0$ such that

$$\mathrm{Re}((1 + jq\omega)(1/k + G(j\omega)) \geq \epsilon \qquad \forall \omega \in \mathbb{R}.$$

Popov multiplier

Step 1. Characterization

If ϕ is sector restricted, the composition operator $\tilde{\phi}(1+jq\omega)^{-1}$ is passive for all q.

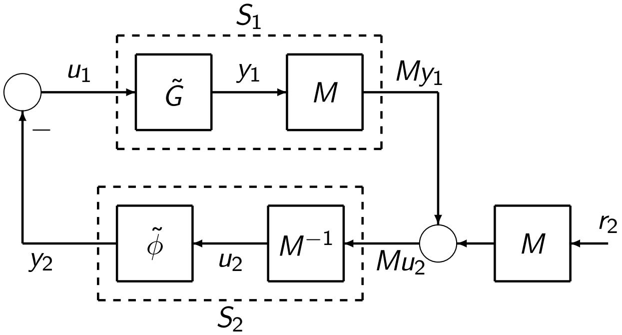
Multiplier as a mathematical operator

The object $(1+jq\omega)$ is referred to as the Popov multiplier.

Popov Criterion

Step 2. Use the passivity theorem

There exist $\epsilon > 0$ and q such that
$\text{Re}((1 + jq\omega)(1 + kG(j\omega)) > \epsilon \quad \forall\omega \in \mathbb{R}.$



Popov Criterion

Popov plot

The Popov plot of a system G is the plot of $\mathrm{Re}(G(j\omega))$ versus $\omega\mathrm{Im}(G(j\omega))$ for all $\omega \geq 0$.

Step 3. Graphical interpretation

The negative feedback interconnection between a strictly proper LTI stable system G and a nonlinearity in the sector $[0, k]$ is stable if the Popov plot of G lies to the right of a line passing through the point $(-1/k + \epsilon, 0)$ with arbitrary slope.

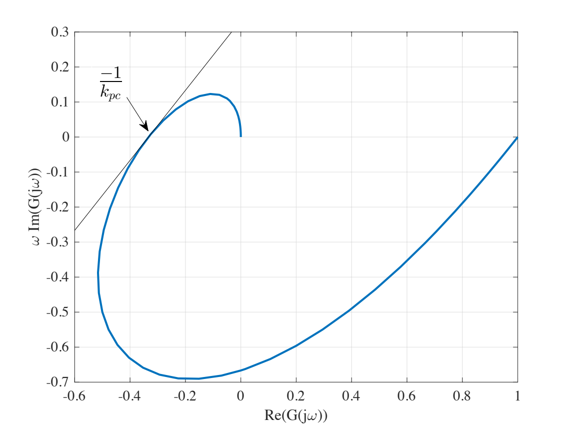
Where is the multiplier?

The slope of the line is related to q.

Popov Criterion

Step 4

The Popov Criterion hold for any $0 \leq k < k_{pc}$



Off-Axis Circle Criterion

Step 1 Development of the class of RL/RC multipliers preserving the positivity of ϕ (Brockett and J. L. Willems, 1965);

Step 2 Frequency domain condition on G via Passivity Theorem subject to the existence of one RL/RC multiplier;

Step 3 Selection of an RL/RC multiplier for a given G by restricting attention to multipliers with quasi-constant phase (Cho and Narendra, 1968);

Step 4 Procedure to find the solution of the Lurye Problem.

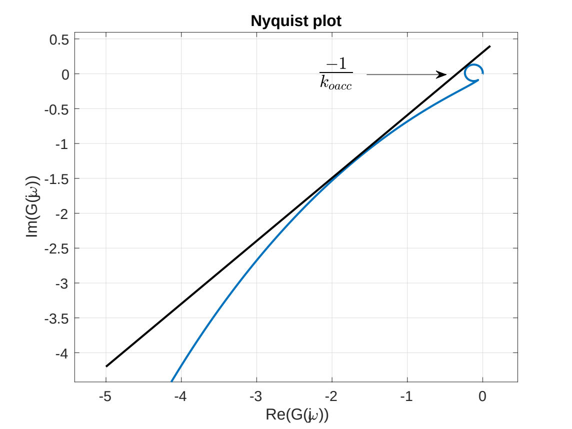
Further reading

Absolute Stability, Carrasco and Heath, Wiley EEE Encyclopedia.

Off-Axis Circle Criterion

Step 4

The Off-Axis Circle Criterion holds for any $k < k_{oacc}$



Discrete-time result

Circle Criterion

Direct translation replacing the $j\omega$ axis by the unit circle $e^{j\omega}$.

Popov Criterion

It is not possible to extend the Popov Criterion to the discrete-time domain.

Off-Axis Circle Criterion

It is not possible to extend the Popov Criterion to the discrete-time domain.

KYP Lemma

KYP Lemma, Rantzer 1996

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ for all $\omega \in \mathbb{R}$ and (A, B) controllable, the following two statements are equivalent:

1. $$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \leq 0, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$
2. There exists $P = P^T \in \mathbb{R}^{n\times n}$ such that
$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \leq 0.$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.

IQC: One theorem to rule them up

IQC Theorem

The positive feedback interconnection between a stable LTI system $G \in \mathrm{RH}_\infty^{l\times m}$ and a bounded operator $\Delta : \mathcal{L}_2^l[0,\infty) \to \mathcal{L}_2^m[0,\infty)$ is stable if the feedback between G and $\tau\Delta$ is well-posed for all $\tau \in [0,1]$ and there exists a measurable Hermitian-valued function $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ such that:

1. for any $u \in \mathcal{L}_2[0,\infty)$, the integral quadratic constraint holds
$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \tau\widehat{\Delta u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \tau\widehat{\Delta u}(j\omega) \end{bmatrix} d\omega \geq 0$$
for all $\tau \in [0,1]$;
2. there exists $\epsilon > 0$ such that
$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall\omega \in \mathbb{R}.$$

Tailored for absolute stability

Step 1 We characterise Δ by means of a class of suitable objects $\Pi(j\omega)$, which can be seen as generalised multipliers, such that the Integral Quadratic Constraint (IQC)

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \tau\widehat{\Delta u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \tau\widehat{\Delta u}(j\omega) \end{bmatrix} d\omega \geq 0$$

is satisfied;

Step 2 the IQC Theorem ensures the stability of the Lurye system if the frequency condition Item 2

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R}.$$

Step 3 with the use of a finite parametrisation of the multiplier and the KYP Lemma to search over finite number of parameters

List of IQCs

Small gain systems (with gain γ)

$$\Pi(j\omega) = \begin{bmatrix} 1 & 0 \\ 0 & -1/\gamma^2 \end{bmatrix}$$

Passivity

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Sector-restricted nonlinearities

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 \\ 1 & -2/k \end{bmatrix}$$

Sector-restricted nonlinearities

$$\Pi(j\omega) = \begin{bmatrix} 0 & (1-jq\omega) \\ (1+jq\omega) & -2/k \end{bmatrix}$$

Recommended reading: Veenman, Scherer, Köroğlu, EJC, 2016

Step 1 - Widest suitable LTI class for slope-restricted nonlinearities

O'Shea (1967)

The class of O'Shea-Zames-Falb multipliers $\mathcal{M}$ is defined by the convolution operators M whose impulse response is of the form

$$m(t) = \delta(t) - \sum_{i=1}^{\infty} h_i \delta(t - t_i) - h(t),$$

where δ is the Dirac delta function, $t_i \neq 0$ and $h_i > 0$ for all i and $h(t) > 0$ for all t, and where

$$\sum_{i=1}^{\infty} h_i + \int_{-\infty}^{\infty} h(t)dt \leq 1.$$

Step 2 - Stability result

Zames-Falb theorem (1968)

The negative feedback interconnection between a proper LTI stable system G and a slope-restricted nonlinearity in $[0, k]$ is stable if there exist $M \in \mathcal{M}$ and $\epsilon > 0$ such that

$$\mathrm{Re}\{M(j\omega)(1 + kG(j\omega))\} \geq \epsilon \qquad \forall \omega \in \mathbb{R}. \tag{1}$$

Step 3 - Searches

Safonov (1987)

$$m(t) = 1 - \sum_i h_i \delta(t - t_i)$$

Non-convex, requires time and/or frequency sweep, can work very well

Chen and Wen (1995)

$$m(t) = 1 - \sum_i h_i \frac{t^i e^{-t}}{i!} \sigma(t) - \sum_i \bar{h}_i \frac{t^i e^t}{i!} \sigma(-t)$$

LMI, ill-conditioned for high order

Turner et al. (2009), Carrasco et al. (2014)

Conservative bound the $\mathcal{L}_1$ norm - LMI. Causal only (Turner) or anti-causal only (Carrasco)

Comparisons in:

- Carrasco, Turner, and Heath, European Journal of Control 2016

Discrete-time O'Shea-Zames-Falb multipliers

O'Shea & Younis (1967)

The class of O'Shea-Zames-Falb multipliers $\mathcal{M}$ is defined by the convolution operators M whose impulse response $m(k)$ satisfies $\sum_k |m(k)| \leq 2m(0)$ where $m(k) < 0$ for all $k \neq 0$

Step 2 - Stability Theorem (J. C. Willems & Brockett, 1968)

The negative feedback interconnection between a proper LTI stable system G and a slope-restricted nonlinearity in $[0, k]$ is stable if there exist $M \in \mathcal{M}$ and $\epsilon > 0$ such that

$$\mathrm{Re}\{M(e^{j\omega})(1 + kG(e^{j\omega}))\} \geq \epsilon \quad \forall \omega \in [0, \pi].$$

Step 3 - Searches (Carrasco et al, IEEE TAC, 2020)

The search in Discrete-Time is simple in comparison with continuous-time counterparts. The canonical basis $1, z^{-1}, z^{-2}, ..., z^{-n}$ is used for the parametrisation.

Only one horse in town

TABLE I
EXAMPLES

Ex.	Plant
1 [36]	$G_1(z) = \frac{0.1z}{z^2-1.8z+0.81}$
2 [36]	$G_2(z) = \frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$
3 [36]	$G_3(z) = -\frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$
4 [36]	$G_4(z) = \frac{z^4-1.5z^3+0.5z^2-0.5z+0.5}{4.4z^5-8.957z^4+9.893z^3-5.671z^2+2.207z-0.5}$
5 [36]	$G_5(z) = \frac{-0.5z+0.1}{z^3-0.9z^2+0.79z+0.089}$
6 [40]	$G_6(z) = \frac{2z+0.92}{z^2-0.5z}$
7 (new)	$G_7(z) = \frac{1.341z^4-1.221z^3+0.6285z^2-0.5618z+0.1993}{z^5-0.935z^4+0.7697z^3-1.118z^2+0.6917z-0.1352}$

Criterion	Odd ϕ?	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7
Circle Criterion [56]	N	0.7934	0.1984	0.1379	1.5312	1.0273	0.6510	0.1069
Tsypkin Criterion [57]	N	3.8000	0.2427	0.1379	1.6911	1.0273	0.6510	0.1069
Ahmad et. al. (2015), Thm 1 [36]	N	12.4309	0.7261	0.3027	2.5904	2.4475	0.9067	0.1695
Park et al. (2019)[37]	N	12.9960	0.7397	0.3054	2.5904	2.4475	0.9108	0.1695
Causal DT Zames-Falb (Prop. III.2.)	Y	12.4355	0.7687	0.2341	3.3606	2.3328	0.9222	0.1966
Anticausal DT Zames-Falb (Prop. III.6.)	Y	1.4994	0.4816	0.3058	3.2365	2.4474	1.0869	0.2365
FIR Zames-Falb ($n_f = 1$, $n_b = 1$)	N	12.9960	0.7397	0.3054	2.5904	2.4475	0.9108	0.1695
FIR Zames-Falb ($n_f = 2$, $n_b = 2$)	N	12.9959	0.7397	0.3054	2.5904	2.4475	0.9115	0.1695
FIR Zames-Falb ($n_f = 3$, $n_b = 3$)	N	12.9960	0.7397	0.3054	3.2254	2.4475	0.9115	0.4347
FIR Zames-Falb ($n_f = 100$, $n_b = 100$)	N	12.9766	0.7984	0.3100	3.8227	2.4475	0.9115	0.4921
FIR Zames-Falb ($n_f = n_b = n^*$)	N	13.0283 (7)	0.8027 (15)	0.3120 (14)	3.8240 (5)	2.4475 (1)	0.9115 (2)	0.4922 (25)
FIR Zames-Falb ($n_f = 1$, $n_b = 1$)	Y	12.9959	0.7782	0.3076	3.1350	2.4475	1.0870	0.2366
FIR Zames-Falb ($n_f = 2$, $n_b = 2$)	Y	12.9959	1.1056	0.3104	3.8240	2.4475	1.0870	0.2940
FIR Zames-Falb ($n_f = 3$, $n_b = 3$)	Y	13.4822	1.1056	0.3121	3.8240	2.4475	1.0870	0.4759
FIR Zames-Falb ($n_f = 100$, $n_b = 100$)	Y	13.5101	1.1056	0.3121	3.8240	2.4475	1.0870	0.5278
FIR Zames-Falb ($n_f = n_b = n^*$)	Y	13.5113 (17)	1.1056 (2)	0.3121 (3)	3.8240 (2)	2.4475 (1)	1.0870 (1)	0.5280 (19)
Nyquist Value	N/A	36.1000	2.7455	0.3126	7.9070	2.4475	1.0870	1.1766

Limitations of O'Shea-Zames-Falb multipliers in DT

Dual problem (Jönsson, Megretski)

For a given G, minimise k such that it is not possible to find a suitable O'Shea-Zames-Falb multiplier for $1 + kG$.

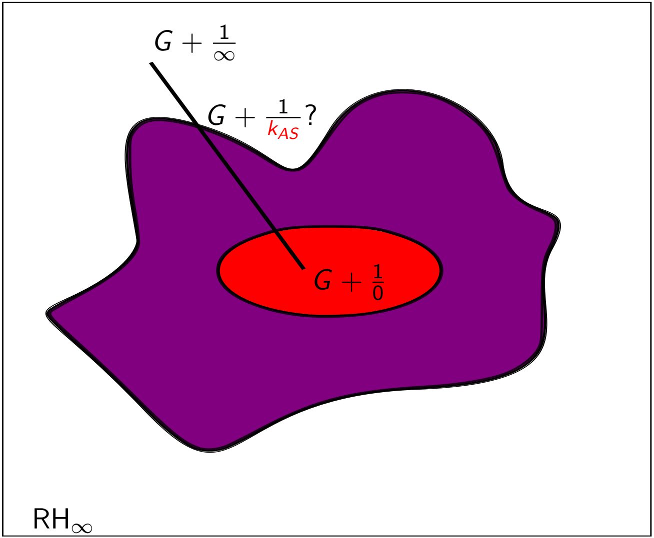
Original motivation

As we use a subset of multipliers, we want to know the efficiency of the parametrisation

My conjecture (Carrasco, Turner, Heath, EJC, 2016)

The feedback interconnection between G and any nonlinearity in the slope in the sector $[0, k]$ if and only if there is a suitable O'Shea-Zames-Falb for the plant $1 + kG$

Graphical representation of the Lurye problem



Jönsson's DT counterpart

Theorem (Zhang, Carrasco, Heath, IEEE TAC 2022)

Let $G \in \mathrm{RH}_\infty$ and let $\beta > 1$. Given $\lambda_1, \ldots, \lambda_{\beta-1} \geq 0$ such that $\sum_{r=1}^{\beta-1} \lambda_r > 0$; if

$$\sum_{r=1}^{\beta-1} \mathrm{Re}\left\{\lambda_r G(e^{j\omega_r})\right\} \leq \min_{l \in \mathbb{Z}} \left[\sum_{r=1}^{\beta-1} \mathrm{Re}\left\{\lambda_r G(e^{j\omega_r}) e^{-j\omega_r l}\right\}\right],$$

where $\omega_r = \frac{r}{\beta}\pi$ for $r = 1, \ldots, \beta - 1$, then there is no O'Shea–Zames–Falb multiplier M such that

$$\mathrm{Re}\left\{M(e^{j\omega})G(e^{j\omega})\right\} > 0, \quad \forall \omega \in [0, \pi].$$

Dual result for DT O'Shea–Zames–Falb multiplier

Numerical Implementation

Let $G \in \mathrm{RH}_\infty$ and let $\beta > 1$. For $l = 0, 1, \cdots, 2\beta - 1$, let us define

$$\mathrm{u} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{\beta-1} \end{bmatrix}, \quad \mathrm{v}_l = \begin{bmatrix} \mathrm{Re}\{(1 - e^{-j\omega_1 l})G(e^{j\omega_1})\} \\ \mathrm{Re}\{(1 - e^{-j\omega_2 l})G(e^{j\omega_2})\} \\ \vdots \\ \mathrm{Re}\{(1 - e^{-j\omega_{\beta-1} l})G(e^{j\omega_{\beta-1}})\} \end{bmatrix},$$

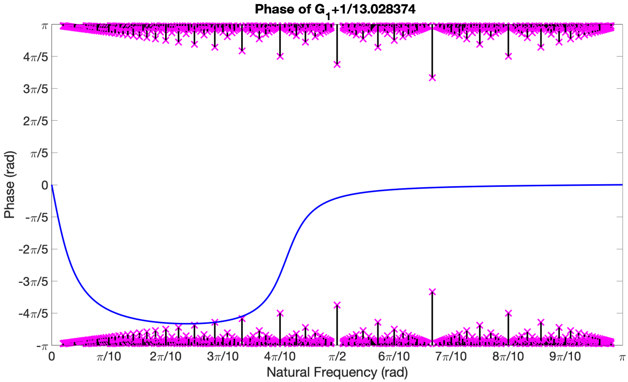
Assume there exists $\mathrm{u} \succeq 0$ such that $|\mathrm{u}| > 0$ and

$$\mathrm{u}^\top \mathrm{v}_l \leq 0 \text{ for all } l = 0, 1, \cdots, 2\beta - 1.$$

Then there is no O'Shea–Zames–Falb multiplier M such that

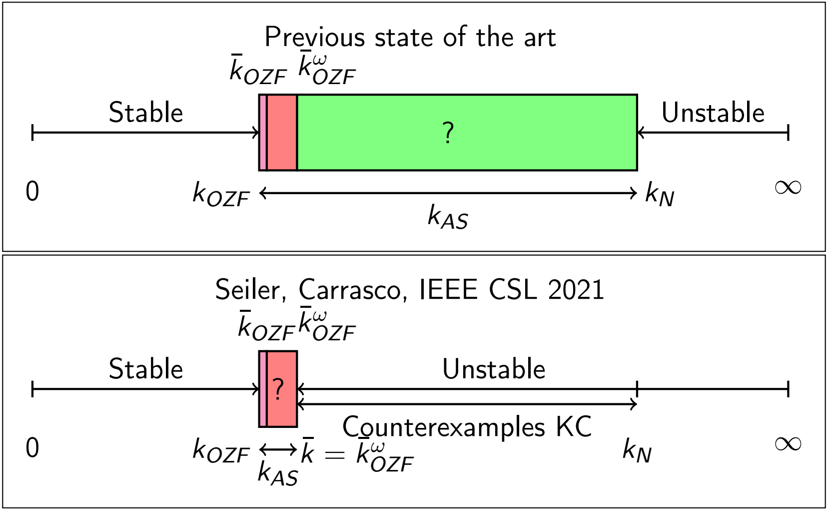
$$\mathrm{Re}\left\{M(e^{j\omega})G(e^{j\omega})\right\} > 0, \quad \forall \omega \in [0, \pi].$$

One-frequency condition



- Every crossing between the Bode plot and the black lines indicates the existence of a periodic solution.
- For a given system, we find the critical $\bar{k}_{OZF}$ such that the phase of $G + 1/k$ reaches the magenta crosses (Stern-Brocot tree).

Construction of Counterexamples of the Kalman Conjecture



Current bottleneck on the necessity

Step 1 - Done

The class of LTV OZF multipliers proposed by Willems and Brockett (1968) is a perfect characterization of the nonlinearity.

Step 2 - Open

It is possible to write any LTV multiplier as a conic combination of a basis. Is it possible to apply the lossless S-procedure with infinite terms?

Step 3 - Done (Also in Kharitenko and Scherer, accepted in Automatica)

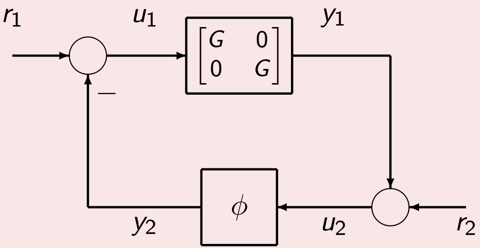
The class of LTV OZF multipliers is "phase-equivalent" to LTI OZF multipliers.

(Su, Seiler, Carrasco, Khong, in press, Automatica)

Kharitenko and Scherer, under review

Arbitrary dimension

By using duality in the linear programming problem, the conjecture holds for some dimension, e.g. $N = 2$



Questions