A Less Conservative LMI Condition for Stability of Discrete-Time Systems With Slope-Restricted Nonlinearities

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***Abstract*—Many conditions have been found for the absolute stability of discrete-time Lur'e systems in the literature. It is advantageous to find convex searches via LMIs where possible. In this technical note, we construct two less conservative LMI conditions for discrete-time systems with slope-restricted nonlinearities. The first condition is derived via Lyapunov theory while the second is derived via the theory of integral quadratic constraints (IQCs) and noncausal Zames–Falb multipliers. Both conditions are related to the Jury–Lee criterion most appropriate for systems with such nonlinearities, and the second generalizes it. Numerical examples demonstrate a significant reduction in conservatism over competing approaches.**

***Index Terms*—Convex LMI, discrete-time, Lyapunov stability, multiplier theory.**

I. Introduction

A. Motivation

Most theories for absolute stability problems in the discrete-time setting were historically developed for the SISO case via the frequency-domain [7], [18]–[20], [42], [44]. It has become standard to state such conditions in terms of LMIs (linear matrix inequalities) where possible [6]. In particular these can easily be applied to multi-input multi-output problems that might not otherwise be tractable.

In this technical note we provide new LMI stability conditions for a discrete-time Lur'e system where there is a slope restriction and (possibly different) sector bound. This is appropriate for many practical control systems: in particular MIMO loops with actuator saturation where the slope restriction and sector bound are the same. If the control structure is otherwise linear then the results are immediately applicable. We are also motivated by optimizing anti-windup [1] and input-constrained model predictive control where the nonlinearity of the controller can be shown to satisfy such a slope restriction [15]. The implementation of such controllers is inherently in the discrete domain; hence, there is a clear motivation for understanding the discrete-time domain problem. The case where the slope restriction and sector bound are not the same has received recent interest for continuous systems [43].

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B. The Problem

This note gives LMI conditions that guarantee the absolute stability of the feedback interconnection between a discrete-time linear time-invariant (LTI) system and a memoryless, time-invariant, sector-bounded and slope-restricted nonlinearity. We provide the mathematical definition of the absolute stability problem in this section.

We are concerned with the strictly proper discrete-time LTI system

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 \in \mathbb{R}^{n_x}, \qquad y_k = Cx_k \tag{1}$$

where A is Schur with $x_k \in \mathbb{R}^{n_x}$ and with u_k, $y_k \in \mathbb{R}^p$; thus $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times p}$, and $C \in \mathbb{R}^{p \times n_x}$. The transfer function of this linear block is $G(z) = C(zI - A)^{-1}B$. Throughout the paper, we will assume that (1) is a minimal representation. It is in negative feedback with a memoryless, time-invariant (static) nonlinearity $\phi : \mathbb{R}^p \to \mathbb{R}^p$ with the relation

$$u_k = -\phi(y_k). \tag{2}$$

We assume that $\phi(0) = 0$, that ϕ is sector-bounded in the sense that there exists some $K \in \mathbb{R}^{p \times p}$, $K > 0$ such that

$$\phi(y)^T \left[K^{-1}\phi(y) - y\right] \le 0 \tag{3}$$

for all $y \in \mathbb{R}^p$ and that ϕ is slope-restricted in the sense that there exists some $S \in \mathbb{R}^{p \times p}$, $S > 0$ such that

$$[\phi(y) - \phi(\hat{y})]^T \left[S^{-1}\left[\phi(y) - \phi(\hat{y})\right] - [y - \hat{y}]\right] \le 0 \tag{4}$$

for all $y, \hat{y} \in \mathbb{R}^p$.

In this work we provide two conditions: Theorems 2.1 and 2.2. Theorem 2.1 uses Lyapunov theory and provides conditions for global asymptotic stability: the system (1), (2) is said to be stable if $\lim_{k\to\infty} x_k = 0$ for any initial $x_0 \in \mathbb{R}^{n_x}$. Theorem 2.2 uses multiplier and IQC theory and provides conditions for input-output stability; this in turn is sufficient for global asymptotic stability. Moreover, we add "absolute" to state that stability is ensured for all the nonlinearities within the specified class.

We use the standard notation Re (M) to denote the real value of M for $M \in \mathbb{C}$ and He (M) to denote $M + M^*$ for $M \in \mathbb{C}^{r \times r}$. If $*$ appears in the upper half triangle of a real-valued symmetric matrix it represents the transpose of the corresponding lower term.

C. Historical Background

Much of the development of absolute stability has been concerned with so-called diagonal nonlinearities where there exist $\phi_i : \mathbb{R} \to \mathbb{R}$ such that $[\phi(y)]_i = \phi_i(y_i)$ for $i = 1, \ldots, p$. We say such nonlinearities ϕ_i are sector-bounded on the interval $[0, k_i]$ and slope-restricted on the interval $[s_{li}, s_{ui}]$ if $\phi_i(0) = 0$ and $0 \le \{\phi_i(y)\}/\{y\} \le k_i$ for all $y \in \mathbb{R}$ and $s_{li} \le \{\phi_i(y) - \phi_i(\hat{y})\}/\{y - \hat{y}\} \le s_{ui}$ for all $y, \hat{y} \in \mathbb{R}$ and $y \ne \hat{y}$. With some abuse of terminology we also say ϕ is sector-bounded on $[0, K]$ and slope-restricted on $[S_L, S_U]$ where $K = \mathrm{diag}(k_1, \ldots, k_p)$, $S_L = \mathrm{diag}(s_{l1}, \ldots, s_{lp})$, and $S_U = \mathrm{diag}(s_{u1}, \ldots, s_{up})$. The problem is to derive conditions involving only the transfer function $G(z)$ and the sector and slope matrices K, S_L, and S_U such that the closed-loop system is absolutely stable. Furthermore, the more precise the description of the nonlinearity is, the less stringent the stability condition is likely to be. In our technical development below we are concerned with the class of nonlinearities satisfying $\phi(0) = 0$, sector-bounded on $[0, K]$ and slope-restricted on $[0, S]$ (the zero condition and slope restriction imply all such nonlinearities are also sector-bounded on $[0, S]$). The slope-restriction is equivalent to condition (4) with S diagonal.

Three Jury–Lee criteria, which are sometimes considered to be the discrete-time counterparts of the Popov criterion [25], were initially derived in the frequency domain [18]–[20] using an embryonic version of the passivity theorem. These are discussed in more detail in Section I-D. After the introduction of the passivity theorem, the multiplier approach was developed [7] to provide less restrictive conditions on $G(z)$. Willems and Brockett (1968) [44] provide the most general class of multipliers for SISO nonlinearities that are slope-restricted on $[0, s]$; these are the discrete-time counterparts of the Zames–Falb multipliers [46]. Their generalization to repeated MIMO nonlinearities is presented in [27], and a weaker generalization of the discrete-time multiplier for the SISO case can be found in Narendra and Cho (1968) [29], which is more conservative than [44] and [27]. The aforementioned literatures, however, only characterise the multipliers rather than actually provide a standard optimization method to solve a given Lur'e problem. Therefore, the multiplier-based stability criteria are often restricted to the SISO case as the complexity may become prohibitive in the multivariable case.

Meanwhile the Lyapunov theorem gives an alternative approach to provide stability conditions for Lur'e systems. In the discrete-time Lyapunov theorem, different stability criteria may be derived via different forms of Lyapunov function (e.g.: [11], [13], [21], [33], [36]) and/or, different properties of the nonlinearities introduced in the difference Lyapunov equation (e.g.: [5], [21], [33], [36]). Nevertheless, finding a Lyapunov function for a given system may not be straightforward. In some cases, the Lyapunov method leads to a convex search over an LMI [6], which can then be easily extended to the multivariable case with diagonal nonlinearities (see [5], [11], [12], [31]–[33]). In Ahmad *et al.* (2012) [5], an improved LMI-based criterion is derived via the Lyapunov method and the S-procedure, which is also applicable to non-diagonal nonlinearities. However, there are also cases where the method does not reduce to a convex LMI search (see [13]).

D. The Jury–Lee Criteria

1) JL criterion 1 [20]: For diagonal ϕ sector-bounded on $[0, K]$ and slope-restricted on $[-S, \infty)$ with diagonal positive definite $K, S \in \mathbb{R}^{p\times p}$, the system (1), (2) is absolutely stable if there exists $M = \mathrm{diag}(m_1, \ldots, m_p) \geq 0$ such that the frequency condition below is satisfied for all $|z| = 1$:

$$\mathrm{He}\left[K^{-1}+(I+(1-z^{-1})M)G(z)-\frac{1}{2}SM\left|(z-1)G(z)\right|^2\right]>0. \quad (5)$$

2) JL criterion 2 [18], [19]: For diagonal ϕ sector-bounded on $[0, K]$ and slope-restricted on $[-S, S]$ with diagonal positive definite $K, S \in \mathbb{R}^{p\times p}$, the system (1), (2) is absolutely stable if there exists $M = \mathrm{diag}(m_1, \ldots, m_p)$ such that the frequency condition below is satisfied for all $|z| = 1$:

$$\mathrm{He}\left[K^{-1}+(I+(z-1)M)G(z)-\frac{1}{2}S|M|\left|(z-1)G(z)\right|^2\right]>0. \quad (6)$$

3) JL criterion 3 [20]: For diagonal ϕ sector-bounded on $[0, K]$ and slope-restricted on $[0, S]$ with diagonal positive definite $K, S \in \mathbb{R}^{p\times p}$, the system (1), (2) is absolutely stable if there exists $M = \mathrm{diag}(m_1, \ldots, m_p)$ such that the frequency condition below is satisfied for all $|z| = 1$:

$$\mathrm{He}\left[K^{-1}+\frac{1}{2}S^{-1}|M||z-1|^2+(I+MQ(z))\,G(z)\right]>0, \quad (7)$$

where $q_i(z) = 1 - z^{-1}$ for $m_i \geq 0$, and $q_i(z) = z - 1$ for $m_i < 0$ and where $Q(z) = \mathrm{diag}(q_1(z), \ldots, q_p(z))$.

JL criterion 2 (6) above has received most attention and has been extended to an LMI-based condition via the discrete-time KYP Lemma [35] and the Lyapunov method [5], [12], [34], [40].

If the nonlinearity is slope-restricted on $[0, S]$, JL criterion 3 (7) may give the least conservative condition as the nonlinearities are described more precisely [23], [30]. It is demonstrated in [23] that JL criterion 3 (7) is less conservative than the Tsypkin criterion [42] and the other two Jury–Lee criteria (5), (6) (via the root-locus method in the SISO case). A simple SISO example where $G(z) = 0.1z/(z^2 - 1.8z + 0.81)$ shows that the JL criterion 3 (7) gives stability for $S \in (0, 12.25)$, which is significantly better than the recent results from [5] and [11] (which give $S \in (0, 5.88)$ and $S \in (0, 0.79)$, respectively). However, to the best of the authors' knowledge, no convex LMI search has been proposed for it in the literature (see, e.g., [25]), and its application is usually limited to the SISO case [20], [23], [30].

E. Contribution

In this technical note, two LMI conditions for the stated problem are derived. The second (Theorem 2.2) is equivalent to a frequency condition thatgeneralizess JL criterion 3 (7); the first (Theorem 2.1) provides sufficient conditions for the same frequency condition's satisfaction. The results are stated for diagonal MIMO nonlinearities with the application to SISO nonlinearities being a special case. We present a corresponding result for a class of MIMO nonlinearities we term "unstructured" (with some abuse of terminology) and discuss further possible generalizations.

Some numerical examples for SISO and MIMO cases are provided to compare the performance of the result with other existing LMI-based criteria in the literature including the recent contributions [3], [5], [11].

II. Main Results

A. Lyapunov Approach for Diagonal Nonlinearities

Theorem 2.1: Consider the system (1), (2) with A Schur and static diagonal nonlinearity ϕ satisfying $\phi(0) = 0$, sector-bounded on $[0, K]$ and slope-restricted on $[0, S]$ with $K > 0$ and $S > 0$ both diagonal. The system is absolutely stable if there exist non-negative, diagonal R_1, R_2 and R_3 in $\mathbb{R}^{p\times p}$ and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ with $P_{11} \in \mathbb{R}^{n_x\times n_x}$ and $P_{22} \in \mathbb{R}^{p\times p}$ such that the following LMI is satisfied:

$$P > 0 \text{ and } M_P + M_1 + M_2 + N_2 + M_3 < 0 \quad (8)$$

with

$$M_P = \begin{bmatrix} A^T P_{11} A - P_{11} & * & * \\ -B^T P_{11} A - P_{12}^T & B^T P_{11} B - P_{22} & * \\ P_{12}^T A & -P_{12}^T B & P_{22} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & * & * \\ 0 & -S^{-1} R_1 & * \\ R_1 C(A-I) & -R_1 CB + S^{-1} R_1 & -S^{-1} R_1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & * & * \\ -R_2C(A-I) & X & * \\ 0 & S^{-1}R_2 & -S^{-1}R_2 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} A^TC^TSR_2CA - C^TSR_2C & * & * \\ -B^TC^TR_2SCA & B^TC^TSR_2CB & * \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ R_3CA & -R_3CB & -2K^{-1}R_3 \end{bmatrix} \quad (9)$$

where $X = R_2CB + B^TC^TR_2 - S^{-1}R_2$.

Proof: See Appendix. ■

Note we may set any one of R_1, R_2 or R_3 to the identity without loss of generality. The proof is based on Lyapunov theory. The corresponding Lur'e-Lyapunov function is given by

$$V(\hat{x}_k) = \hat{x}_k^T P \hat{x}_k + 2\int_0^{y_k} \phi(\sigma)^T R_1 d\sigma + 2\int_0^{y_k} \tilde{\phi}(\sigma)^T R_2 d\sigma \quad (10)$$

where $\tilde{\phi}(\sigma) = S\sigma - \phi(\sigma)$, $P > 0$, and $\hat{x}_k = [x_k^T, \phi(y_k)^T]^T$. Similarly structured Lyapunov functions have been proposed in the literature [14], [31], [33], [45]; the resulting LMI in this case is different.

B. Frequency Interpretation

Invoking the KYP Lemma [37], the conditions of Theorem 2.1 imply the frequency condition

$$L_b(z)^*(M_1 + M_2 + N_2 + M_3)L_b(z) < 0 \quad \forall |z| = 1 \quad (11)$$

where L_b is defined as follows:

$$A_b = \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix}, \; B_b = \begin{bmatrix} 0 \\ I \end{bmatrix}, \; L_b(z) = \begin{bmatrix} (zI - A_b)^{-1}B_b \\ I \end{bmatrix} \quad (12)$$

since A_b and B_b define the evolution of the augmented state $\hat{x}$. Further, we have the identity $L_b(z)^* N_2 L_b(z) = 0$ for all $|z| = 1$. After some algebra, we find (11) can be written

$$\mathrm{He}\Big[R_3K^{-1} + \frac{1}{2}|z-1|^2 S^{-1}(R_1 + R_2) + (R_3 + (1-z^{-1})R_1 + (1-z)R_2)G(z)\Big] > 0 \quad \forall |z| = 1. \quad (13)$$

This frequency-domain condition is closely related to JL criterion 3 as (7) can be obtained by fixing $R_3 = I$ and setting $R_1R_2 = 0$. However, while (13) is necessary for the existence of a Lyapunov function, it is not sufficient as we have no guarantee that $P > 0$. An incorrect inference is made in [2] and elsewhere.

C. Multiplier Approach for Diagonal Nonlinearities

Motivated by the results of the previous section, we may conjecture that frequency condition (13) is sufficient for stability. This turns out to be true and can be demonstrated via multiplier and IQC theory. Specifically the frequency condition (13) can be reformulated as

$$\mathrm{He}\left[R_3(K^{-1} + G(z)) + M(z)(S^{-1} + G(z))\right] > 0 \quad \forall |z| = 1 \quad (14)$$

with

$$M(z) = R_1(1 - z^{-1}) + R_2(1 - z). \quad (15)$$

The diagonal entries of $M(z)$ are Zames–Falb multipliers (in the sense of [28]).

Theorem 2.2: Consider the system (1), (2) with A Schur and static diagonal nonlinearity ϕ satisfying $\phi(0) = 0$, sector-bounded on $[0, K]$ and slope-restricted on $[0, S]$ with $K > 0$ and $S > 0$ both diagonal. The system is absolutely stable if there exist non-negative, diagonal R_1, R_2 and R_3 in $\mathbb{R}^{p\times p}$ such that (13) is satisfied. Equivalently there exists symmetric $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ with $P_{11} \in \mathbb{R}^{n_x \times n_x}$ and $P_{22} \in \mathbb{R}^{p\times p}$ such that the following LMI is satisfied:

$$M_P + M_1 + M_2 + M_3 < 0 \quad (16)$$

with M_P, M_1, M_2 and M_3 given by (9).

Proof: See Appendix. ■

The similarity between our results and the continuous-time domain results in [32] could lead to the misinterpretation that both are equivalent. Their Zames–Falb interpretation shows that they are in fact very different. On one hand, results in [32] are equivalent to the use of rational first order continuous-time Zames—Falb multipliers [8]. On the other hand, our results provide equivalence with left and right-shift discrete-time Zames–Falb multipliers. Hence our characterization of the multiplier is the simplest case of the discrete-time version developed in [9] and references therein.

D. Unstructured Nonlinearities and Further Generalizations

In both our overview of the historical development and our statement of Theorems 2.1 and 2.2 we assume ϕ to be diagonal. For some applications this is over-restrictive, and a generalization may be desirable (see [14] and [4] for examples). In particular, we are interested in the case where ϕ represents the quadratic program that must be solved on-line for either input-constrained model predictive control [16] and for optimizing antiwindup [1]. Both theorems have their direct counterpart for this case. The generalization of Theorem 2.1 requires the use of multivariable convex analysis for Lemma 5.2.

Theorem 2.3: Consider the system (1), (2) with A Schur and nonlinearity ϕ satisfying $\phi(0) = 0$ and slope-restricted on $[0, S]$ with $S > 0$. Suppose in addition that ϕ is the gradient of some convex potential function.

a) *Lyapunov approach*: The system is absolutely stable if there exist non-negative scalars r_1, r_2 and r_3 and positive definite $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ such that the LMI (8) is satisfied with M_1, M_2, N_2, M_P and M_S given by (9) and $R_1 = r_1 I$, $R_2 = r_2 I$, $R_3 = r_3 I$. This is sufficient for frequency condition (13) to hold.

b) *IQC approach*: The system is absolutely stable if there exist non-negative scalars r_1, r_2 and r_3 such that the frequency condition (13) holds with $R_1 = r_1 I$, $R_2 = r_2 I$ and $R_3 = r_3 I$. Equivalently there exist non-negative scalars r_1, r_2 and r_3 and a symmetric $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ such that the LMI (16) is satisfied with M_1, M_2, M_P and M_S given by (9) and $R_1 = r_1 I$, $R_2 = r_2 I$, $R_3 = r_3 I$.

Proof: The proof of a) is similar to that of Theorem 2.1 while the proof of part b) is similar to that of Theorem 2.2. ■

We will refer to nonlinearities that satisfy the conditions of Theorem 2.3 as "unstructured nonlinearities" by way of contrast with "diagonal nonlinearities." Nevertheless, it should be understood that both the slope-restriction and the condition on the convex potential function imply considerable structure.

It is now straightforward to derive corresponding results and statements for block diagonal nonlinearities.

TABLE I
MAXIMUM (κ) FOR EXAMPLES 1 AND 2

Criterion	Example 1	Example 2
Circle Criterion [41]	0.7933	0.549
Tsypkin Criterion [21]	3.8000	0.848
Park & Kim [33]	3.8000	0.848
Haddad & Bernstein [12]	2.9054	1.865
Ahmad et al. [5]	5.8818	2.696
Gonzaga et al. [11]	0.7933	0.623
Theorem 2.1	12.43	3.4155
Theorem 2.2	12.99	3.5557
Ahmad et al. (2013) [3]	12.436	N/A
Nyquist Criterion	36.10	3.850

III. APPLICATIONS

In the following numerical examples, we consider only the case $K = S$. All results are obtained in Matlab using YALMIP [26] and SeDuMi [39].

To compare the performance of Theorems 2.1 and 2.2 with the existing LMI-based criteria, a discrete-time Lur'e system with $\phi(0) = 0$ and ϕ slope-restricted on $[0, \kappa I]$ is used, and the maximum value of $\kappa > 0$ for each criterion is computed. In this subsection, both a SISO example and a diagonal MIMO example are given. The maximum κ for different stability criteria are compared in Table I. Note that some of these criteria are developed for a subclass of nonlinearities, i.e., saturation; for these classes κ means the maximum slope of this subclass. The Nyquist value means the supremum of the values of κ such that this linear feedback system is stable for all gains within $[0, \kappa]$. Results show that the new approach gives less conservative results compared to the existing ones.

Example 1

$$G_1(z) = \frac{0.1z}{z^2 - 1.8z + 0.81}$$

Example 2

$$G_2(z) = \begin{bmatrix} \frac{0.2}{z-0.98} & \frac{-0.2}{z-0.92} \\ \frac{0.3}{z-0.97} & \frac{0.1}{z-0.91} \end{bmatrix}$$

If the nonlinearity ϕ is represented by a quadratic program (QP) as follows: $\phi(y) = \arg\min_v (1/2)v^T Hv - v^T y$ s.t. $Lv \leq b$ with $b \geq 0$ with fixed Hessian matrix $H > 0$ and fixed L and b, the KKT conditions can be used to show that the nonlinearity satisfies the conditions of Theorem 2.3 with $S^{-1} = H$ and $K = S$ [15]. Such conditions arise naturally in optimizing antiwindup [1] and input-constrained MPC [16].

Example 3: A stable, strictly proper 2-input–2-output plant $G_3(z) = 0.1z/(z^2 - 1.8z + 0.81)\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ is in negative feedback with a controller that is expressed by a QP with a positive definite Hessian matrix $H_3 = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$. A positive constant gain κ is applied on the plant's output, and the maximum stable gain for the new criterion is compared with the existing results in Table II.

IV. CONCLUSION

The main contributions of this note are Theorems 2.1 and 2.2, the construction of convex LMI searches to test a generalized form of JL criterion 3 (7) for slope-restricted nonlinearities. Theorem 2.2 is derived via IQC and multiplier theory. This suggests that our method may be superseded by a more comprehensive search over Zames–Falb multipliers, and the more recent results reported in [3] are promising in this respect for SISO plants. Further investigation is needed to produce tractable non-conservative searches. Nevertheless the numerical examples of this note demonstrate a significant reduction in conservatism over prior competing approaches in the literature.

The results are also shown to be applicable to the unstructured MIMO case which has applications to stability analysis of some optimizing-based controllers. From the numerical examples, it is demonstrated that Theorem 2.3 is less conservative than the existing discrete-time LMI-based criteria.

TABLE II
MAXIMUM (κ) FOR EXAMPLE 3

Criterion	**Maximum gain (κ)**
Generalized Circle Criterion [16]	0.4518
Primbs (2001) [36]	1.1018
Ahmad et al. [5]	3.1714
Theorem 2.3 (a)	6.5213
Theorem 2.3 (b)	6.7953

APPENDIX

A. Proof of Theorem 2.1

The following lemmas are instrumental for the construction of the Lyapunov function (10).

Lemma 5.1: Let $\tilde{\phi}(\sigma) = S\sigma - \phi(\sigma)$ where $\phi(0) = 0$ and ϕ is slope-restricted on $[0, S - \epsilon I]$, where $(S - \epsilon I) \in \mathbb{R}^{p\times p}$ is diagonal and positive definite. Then $\psi = \tilde{\phi}^{-1}$ exists, and the following inequality holds:

$$\left[\psi(\hat{x}) - \psi(x) - S^{-1}(\hat{x} - x)\right]^T [\hat{x} - x] \geq 0. \tag{17}$$

Proof: Proof is straightforward using monotonicity conditions on $\tilde{\phi}$. Similar arguments have been used in [10] ■

Lemma 5.2: Suppose either: the nonlinearity $\tilde{\phi}$ is diagonal and as in Lemma 5.1, and R is a diagonal non-negative matrix, or the nonlinearity $\tilde{\phi}$ is unstructured and as in Lemma 5.1, and $R = I$. Then, the following two equalities hold:

$$\int_{y_k}^{y_{k+1}} \tilde{\phi}(\sigma)^T R d\sigma + \int_{\tilde{\phi}(y_k)}^{\tilde{\phi}(y_{k+1})} [\psi(\sigma) - \psi(\tilde{\phi}(y_k))]^T R d\sigma = \tilde{\phi}(y_{k+1})^T R(y_{k+1} - y_k) \tag{18}$$

$$\int_{y_k}^{y_{k+1}} \tilde{\phi}(\sigma)^T R d\sigma - \int_{\tilde{\phi}(y_k)}^{\tilde{\phi}(y_{k+1})} [\psi(\tilde{\phi}(y_{k+1})) - \psi(\sigma)]^T R d\sigma = \tilde{\phi}(y_k)^T R(y_{k+1} - y_k). \tag{19}$$

Proof: The result follows from the Legendre transformation properties of convex and invertible functions (see Section 26 in [38]). ■

Lemma 5.3: Consider the diagonal nonlinearity $\tilde{\phi}$ as in Lemma 5.1 and let R be a diagonal non-negative matrix. The nonlinearity $\tilde{\phi}$ is then bounded as follows:

$$\text{(a) } 2\int_{y_k}^{y_{k+1}} \tilde{\phi}(\sigma)^T R d\sigma$$
$$\leq \left[y_{k+1}^T S R y_{k+1} - y_k^T S R y_k\right] - 2\phi(y_k)^T R[y_{k+1} - y_k]$$
$$- \left[\phi(y_{k+1}) - \phi(y_k)\right]^T S^{-1} R \left[\phi(y_{k+1}) - \phi(y_k)\right].$$

$$\text{(b) } 2\int_{y_k}^{y_{k+1}} \left[S\sigma - \tilde{\phi}(\sigma)\right]^T R d\sigma$$
$$\leq 2\phi(y_{k+1})^T R[y_{k+1} - y_k]$$
$$- \left[\phi(y_{k+1}) - \phi(y_k)\right]^T S^{-1} R[\phi(y_{k+1}) - \phi(y_k)].$$

Proof: The proof for Lemma 5.3(a) consists in bounding the integrals that appear in (18). Let us parameterize this integral using $\sigma = \tilde{\phi}(y_k) + \lambda(\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k))$ with $0 \leq \lambda \leq 1$, i.e.,

$$\int_{\tilde{\phi}(y_k)}^{\tilde{\phi}(y_{k+1})} \left[\psi(\sigma) - \psi\left(\tilde{\phi}(y_k)\right)\right]^T R d\sigma$$
$$= \int_0^1 \left[\psi(\sigma) - \psi\left(\tilde{\phi}(y_k)\right)\right]^T R \left[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)\right] d\lambda.$$

From Lemma 5.1 it follows that

$$\left[\psi(\hat{x}) - \psi(x)\right]^T R[\hat{x} - x] \geq [\hat{x} - x]^T S^{-1} R[\hat{x} - x]$$

thus this integral can be bounded as follows:

$$\int_0^1 \left[\psi(\sigma) - \psi\left(\tilde{\phi}(y_k)\right)\right]^T R \left[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)\right] d\lambda$$
$$\geq \int_0^1 \lambda \left[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)\right]^T S^{-1} R \left[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)\right] d\lambda$$
$$= \frac{1}{2}\left[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)\right]^T S^{-1} R[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)].$$

Finally, the result is obtained by substituting $\tilde{\phi}(\sigma) = S\sigma - \phi(\sigma)$ and some algebraic manipulations.

The proof for Lemma 5.3(b) is similar to the proof of Lemma 5.3(a) with $\sigma = \tilde{\phi}(y_{k+1}) - \lambda[\tilde{\phi}(y_{k+1}) - \tilde{\phi}(y_k)]$ and using identity (19). ■

Proof of Theorem 2.1: Initially, we consider ϕ with slope-restriction $[0, S - \epsilon I]$ for some $\epsilon > 0$.

In order to prove the absolute stability of the system using the Lyapunov stability theory, we need to show that the function (10) satisfies all the properties of Lyapunov theorem [22]. Let V_k denote $V(\hat{x}_k)$. It is straightforward to show that V is a valid candidate as Lyapunov function since $P > 0$, $R_1 > 0$, $R_2 > 0$, and $\tilde{\phi}$ is in the sector $[\epsilon I, S]$. We need to prove that $\Delta V < 0$ for all $x_k \neq 0$. Subtracting V_k from V_{k+1}, gives

$$\Delta V = \hat{x}_{k+1}^T P \hat{x}_{k+1} - \hat{x}_k^T P \hat{x}_k$$
$$+ 2\int_{y_k}^{y_{k+1}} \left[S\sigma - \tilde{\phi}(\sigma)\right]^T R_1 d\sigma + 2\int_{y_k}^{y_{k+1}} \tilde{\phi}(\sigma)^T R_2 d\sigma. \quad (20)$$

Define $\xi_k = [x_k^T, \phi(y_k)^T, \phi(y_{k+1})^T]^T$. After some algebra and using Lemma 5.3 to bound these integrals, it follows that $\Delta V \leq \xi_k^T(M_1 + M_2 + N_2 + M_P)\xi_k$, but we also have the relation $\xi_k^T M_3 \xi_k \geq 0$ from the sector bound of ϕ along the trajectories of the system. Then, applying the S-procedure [6], the absolute stability of the system (1), (2) is obtained if

$$M_1 + M_2 + N_2 + M_3 + M_P < 0 \quad (21)$$

since this implies that $\Delta V < 0$ if $x_k \neq 0$ along the trajectories of the system. Moreover, when $x_k = 0$ then $\xi_k = 0$ since $\phi(y_k) = \phi(Cx_k) = 0$ and $\phi(C(Ax_k + B\phi(Cx_k)) = 0$; hence, $\Delta V = 0$. In summary, the system (1), (2) is absolutely stable if LMI (21) is satisfied for some positive definite matrices P, R_1, R_2, and R_3.

The last step of the proof is to show that the result is also valid when $\epsilon = 0$. Given some positive definite matrices P, R_1, R_2, and R_3, the left hand side of (8) can be seen as a matrix that depends on S, i.e., $Q(S) > 0$. If $S > 0$, then $Q(S)$ depends continuously on S. Using a classical result in matrix perturbation theory (see [17], Appendix D), the continuity of the eigenvalues of Q with respect to S implies that $Q(S + \delta I) > 0$ for sufficiently small $\delta > 0$. Choose any such δ and apply the above result with $S_\delta = S + \delta I$ and ϕ with slope restriction $[0, S_\delta - \epsilon I]$ for some $\epsilon > 0$. As the unique condition on ϵ is to be positive, take $\epsilon = \delta$. The result follows straightforwardly. ■

B. *Proof of Theorem 2.2*

First let us state the IQC Theorem for discrete-time systems [24] for an appropriately structured IQC multiplier.

Theorem 5.1: Consider the system (1), (2) and let Π be a bounded LTI self-adjoint operator on ℓ_2 where $\Pi_{11} \geq 0$ and $\Pi_{22} \geq 0$. Suppose that

1) For any $\tau \in [0, 1]$, the system (1), (2) is well posed with $\tau\phi$.
3) 2) ϕ satisfies the IQC defined by Π.

$$\begin{bmatrix} -G(z) \\ I \end{bmatrix}^* \Pi(z) \begin{bmatrix} -G(z) \\ I \end{bmatrix} < 0 \quad \forall |z| = 1. \quad (22)$$

Then the system is absolutely stable.

Proof of theorem 2.2: Let us assume that ϕ is slope restricted $S[0, S - \epsilon I]$ and define $M(z)$ given by (15). Following the definition and analysis of [44], the diagonal entries of $M(z)$ are Zames–Falb multipliers. Then, following the notation of [28], the sector bound and slope restriction imply $\phi \in \mathrm{IQC}(\Pi)$ with

$$\Pi(z) = \begin{bmatrix} 0 & M(z)^* + R_3 \\ M(z) + R_3 & -S^{-1}\left(M(z) + M(z)^*\right) - 2R_3K^{-1} \end{bmatrix}. \quad (23)$$

Hence, [28] the system is stable provided

$$\begin{bmatrix} -G(z) \\ I \end{bmatrix}^* \Pi(z) \begin{bmatrix} -G(z) \\ I \end{bmatrix} < 0 \quad \text{for all } |z| = 1. \quad (24)$$

Using the identities $\mathrm{Re}(z) = \mathrm{Re}(z^{-1}) = 1/2\mathrm{Re}(z + z^{-1})$ for all $|z| = 1$, we may write this as (14) and equivalently (13). Finally, for any $\tau \in [0, 1]$ the feedback interconnection $\tau\phi$ is well posed since it is Lipschitz continuous [22]. As a result, conditions in Theorem 5.1 are fulfilled; hence, the system (1), (2) is absolutely stable.

Similarly to previous proof, using the continuity properties of $G(z)$ and $\Pi(z)$, if (24) holds for some $S > 0$, then there exists $\delta > 0$ such that (24) also holds for $S_\delta = S + \delta$, then the result is obtained by taking $\epsilon = \delta$.

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