INTRODUCTION OF HUYGHENS ABSORBING BOUNDARY CONDITIONS INTO LOD METHOD FOR FDTD

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The finite difference time domain (FDTD) method is based on Yee algorithm which employs a very simple way to discretize maxwell equations. Any structure of interest is decomposed into a cubic unit cells called voxels and the size of the area that can be simulated is limited by computer resources.

LOD method is an alternative method for the application of the FDTD method and it is by design implicit in nature. Implicit methods were introduced to overcome the time step limitation inherent in the conventional explicit methods. This implies that a larger time step can be used for the computational domain when compared to the normal explicit FDTD method. This results in the speed up of the overall simulation time, highly desirable when electromagnetic fields are to be determined for a large computational space or whenever objects having very fine details are to be modeled.

However during simulation as the wave propagates outward, it will eventually come to the edge of the allowable space, which is dictated by how the arrays have been dimensioned in the program. If nothing were done to address this, unpredictable reflections would be generated that would go back inward. Thus, there would be no way to determine which is the real wave and which is the reflected noise.

This is the reason that Absorbing Boundary Conditions (ABCs) have been an issue for as long as FDTD has been used. In using Absorbing boundary conditions the objective is to achieve an ideal ABC which absorbs all the outgoing waves and produces no reflection, along with catering for all incident angles of the waves propagating towards it. The idea is to simulate the open space in such a way that the waves appear to propagate infinitely. Huygens Absorbing boundary conditions (HABC) are proactive in both in its design and implementation. They incorporate the idea of a hypothetical “Huygens surface” separating the two connecting field regions such that any field propagating towards the HABC can be canceled by generating a counter field that is equal in magnitude and opposite in direction to its original counterpart.
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Chapter 1

Finite Difference Time Domain Method

The finite-difference time-domain (FDTD) method is one of the full-wave techniques used to solve problems in electromagnetics. It is easy to understand and easy to implement in software. As it is a time-domain method, solutions can cover a wide range of frequencies with a single simulation run.

The FDTD method belongs in the general class of grid-based differential time-domain numerical modeling methods. The time-dependent Maxwell’s equations in partial differential form are discretized using central-difference approximations to the space and time partial derivatives. The FDTD algorithm as first proposed by Kane Yee in 1966 employs second-order central differences. The FDTD method has been widely used in engineering design problems throughout the scientific community. However, its performance yield and usability still hinge on the inherent limits imposed by the numerical dispersion errors and the Courant-Frederick-Levy (CFL) constraints.

At the numerical dispersion error front, higher-order methods were developed to tackle it. For the CFL constraint, implicit methods like Alternating direction implicit ADI-FDTD method, Crank Nicolson CN-FDTD and more recently locally-one-dimensional LOD-FDTD method has been proposed.

This section describes issues related to FDTD, including the numerical formulation, constraints on the parameters, boundary conditions and the derivation of FDTD equations.
1.1 Numerical formulation of Explicit FDTD

The time-dependent Maxwell’s curl equations are given as Faraday’s law,
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},
\]
and Ampere’s law,
\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},
\]
\[
\mathbf{D} = \varepsilon \mathbf{E}
\]
\[
\mathbf{B} = \mu \mathbf{H}
\]
expanding (1.1) and (1.4) in a vectorial manner,
\[
\nabla \times \mathbf{E} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \mathbf{i}_x + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \mathbf{i}_y + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \mathbf{i}_z
\]
\[
= -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial (H_x \mathbf{i}_x + H_y \mathbf{i}_y + H_z \mathbf{i}_z)}{\partial t}.
\]
similarly, expanding (1.2) and (1.3) yields
\[
\nabla \times \mathbf{H} = \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{i}_x + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{i}_y + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{i}_z
\]
\[
= \varepsilon \frac{\partial (E_x \mathbf{i}_x + E_y \mathbf{i}_y + E_z \mathbf{i}_z)}{\partial t}
\]
where \( \mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z \) are the unit vectors in \( x, y, \) and \( z \) directions. Faraday’s law is expressed in a scalar manner as,
\[
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t}
\]
\[
(1.7)
\]
\[
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}
\]
\[
(1.8)
\]
\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t}. \] (1.9)

Similarly, Ampere’s law when written in scalar form becomes,
\[ \frac{\partial D_x}{\partial t} = \frac{\partial H_z}{\partial z} - \frac{\partial H_y}{\partial x} \] (1.10)
\[ \frac{\partial D_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \] (1.11)
\[ \frac{\partial D_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}. \] (1.12)

Writing (1.6) in scalar manner,
\[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t} \] (1.13)
\[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t} \] (1.14)
\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \epsilon \frac{\partial E_z}{\partial t} \] (1.15)

Discretizing the above equations by taking the central difference approximations for the temporal and spatial derivatives yields,
\[ \frac{H_z^{n+\frac{1}{2}}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2}) - H_z^{n-\frac{1}{2}}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta z} - \frac{H_y^{n+\frac{1}{2}}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2}) - H_y^{n-\frac{1}{2}}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta z} \]
\[ = \epsilon \frac{E_x^{n}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2}) - E_x^{n-1}(i, j \pm \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta t} \] (1.16)

Discretizing (1.14) yields,
\[ \frac{H_x^{n+\frac{1}{2}}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2}) - H_x^{n-\frac{1}{2}}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta x} - \frac{H_y^{n+\frac{1}{2}}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2}) - H_y^{n-\frac{1}{2}}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta x} \]
\[ = \epsilon \frac{E_y^{n}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2}) - E_y^{n-1}(i, j \mp \frac{1}{2}, k, \pm \frac{1}{2})}{\Delta t} \] (1.17)
Discretizing \((1.15)\) gives,
\[
\frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2},j,k) - H_y^{n-\frac{1}{2}}(i-\frac{1}{2},j,k)}{\Delta x} - \frac{H_y^{n-\frac{1}{2}}(i,j+\frac{1}{2},k) - H_y^{n-\frac{1}{2}}(i,j-\frac{1}{2},k)}{\Delta y} - \frac{H_y^{n-\frac{1}{2}}(i,j,k+\frac{1}{2}) - H_y^{n-\frac{1}{2}}(i,j,k-\frac{1}{2})}{\Delta z} = \frac{\epsilon^n(i,j,k)E_x^n(i,j,k) - \epsilon^{n-1}(i,j,k)E_x^{n-1}(i,j,k)}{\Delta t}
\]  
\(1.18\)

Similarly \((1.7)\) is discretized as,
\[
\frac{E_z^n(i,j,k) - E_z^n(i,j-1,k)}{\Delta y} - \frac{E_y^n(i,j-\frac{1}{2},k+\frac{1}{2}) - E_y^n(i,j-\frac{1}{2},k-\frac{1}{2})}{\Delta z} - \mu^n(i,j,k)H_x^{n+\frac{1}{2}}(i,j-\frac{1}{2},k) - \mu^{n-\frac{1}{2}}(i,j-\frac{1}{2},k)H_x^{n-\frac{1}{2}}(i,j-\frac{1}{2},k) = \frac{\mu^n(i,j,k)H_y^n(i,j,k) - \mu^{n-1}(i,j,k)H_y^{n-1}(i,j,k)}{\Delta t}
\]  
\(1.19\)

Discretizing \((1.8)\),
\[
\frac{E_x^n(i,\frac{1}{2},j,k+\frac{1}{2}) - E_x^n(i,\frac{1}{2},j,k-\frac{1}{2})}{\Delta z} - \frac{E_z^n(i-j,\frac{1}{2},k) - E_z^n(i-j,\frac{1}{2},k)}{\Delta x} - \frac{\mu^n(i-j,\frac{1}{2},k)H_y^{n+\frac{1}{2}}(i-j,\frac{1}{2},k) - \mu^{n-\frac{1}{2}}(i-j,\frac{1}{2},k)H_y^{n-\frac{1}{2}}(i-j,\frac{1}{2},k)}{\Delta t}
\]  
\(1.20\)

Discretizing \((1.9)\) yields,
\[
\frac{E_y^n(i,j+\frac{1}{2},k+\frac{1}{2}) - E_y^n(i,j+\frac{1}{2},k+\frac{1}{2})}{\Delta x} - \frac{E_z^n(i+j,\frac{1}{2},k) - E_z^n(i+j,\frac{1}{2},k)}{\Delta y} - \frac{\mu^n(i+j,\frac{1}{2},k)H_x^{n+\frac{1}{2}}(i+j,\frac{1}{2},k) - \mu^{n-\frac{1}{2}}(i+j,\frac{1}{2},k)H_x^{n-\frac{1}{2}}(i+j,\frac{1}{2},k)}{\Delta t}
\]  
\(1.21\)

Here when a medium is homogeneous, the material dependent permittivity \(\epsilon^n(i,j,k)\) which when an electromagnetic wave passes through it depends on the frequency of the wave, and permeability \(\mu^n(i,j,k)\) satisfy the equations \((1.3)\) and \((1.4)\). When the origins of \(E_x(0,0,0), E_y(0,0,0), E_z(0,0,0), H_x(0,0,0), H_y(0,0,0)\) and \(H_z(0,0,0)\) are placed in Cartesian coordinates, six coupled explicit finite difference equations are defined as follows. From \((1.19)\) :
\[
\frac{E_z^n(i,j,k) - E_z^n(i,j-1,k)}{\Delta y} - \frac{E_y^n(i,j,k) - E_y^n(i,j,k-1)}{\Delta z} - \frac{\mu^n(i,j,k)H_x^n(i,j,k) - \mu^{n-1}(i,j,k)H_x^{n-1}(i,j,k)}{\Delta t} = \frac{\mu^{n-1}(i,j,k)H_x^{n-1}(i,j,k)}{\mu^n(i,j,k)}
\]
\(1.22\)
Similarly, (1.16) becomes:

\[
\frac{E_x^n(i,j,k) - E_x^n(i,j,k-1)}{\Delta z} - \frac{E_x^n(i,j,k) - E_x^n(i-1,j,k)}{\Delta x} = \frac{H_y^n(i,j,k) - H_y^{n-1}(i,j,k)}{\Delta t}
\]

\[
H_y^n(i,j,k) = \frac{\mu^{n-1}(i,j,k)H_y^{n-1}(i,j,k)}{\mu^n(i,j,k)}
\]

\[
\frac{\Delta t}{\mu^n(i,j,k)} \left[ \frac{E_x^n(i,j,k) - E_x^n(i,j,k-1)}{\Delta z} - \frac{E_x^n(i,j,k) - E_x^n(i-1,j,k)}{\Delta x} \right]
\]

\[
[i_{\min} + 1 \leq i \leq i_{\max} - 1, j_{\min} + 1 \leq j \leq j_{\max}, k_{\min} + 1 \leq k \leq k_{\max}].
\]

Similarly, (1.20) becomes:

\[
\frac{E_y^n(i,j,k) - E_y^n(i,j,k-1)}{\Delta z} - \frac{E_y^n(i,j,k) - E_y^n(i-1,j,k)}{\Delta x} = \frac{H_x^n(i,j,k) - H_x^{n-1}(i,j,k)}{\Delta t}
\]

\[
H_x^n(i,j,k) = \frac{\mu^{n-1}(i,j,k)H_x^{n-1}(i,j,k)}{\mu^n(i,j,k)}
\]

\[
\frac{\Delta t}{\mu^n(i,j,k)} \left[ \frac{E_y^n(i,j,k) - E_y^n(i,j,k-1)}{\Delta z} - \frac{E_y^n(i,j,k) - E_y^n(i-1,j,k)}{\Delta x} \right]
\]

\[
[i_{\min} + 1 \leq i \leq i_{\max} - 1, j_{\min} + 1 \leq j \leq j_{\max}, k_{\min} + 1 \leq k \leq k_{\max}].
\]

Similarly, (1.16) is simplified to:

\[
\frac{H_x^n(i,j+1,k) - H_x^n(i,j,k)}{\Delta y} - \frac{H_y^n(i,j,k+1) - H_y^n(i,j,k)}{\Delta z} = \frac{\varepsilon^{n+1}(i,j,k)E_x^{n+1}(i,j,k) - \varepsilon^n(i,j,k)E_x^n(i,j,k)}{\Delta t}
\]

\[
\varepsilon^{n+1}(i,j,k)E_x^{n+1}(i,j,k) = \frac{\Delta t}{\varepsilon^{n+1}(i,j,k)} \left[ H_x^n(i,j+1,k) - H_x^n(i,j,k) \right]
\]

\[
+ \frac{\Delta t}{\varepsilon^{n+1}(i,j,k)} \left[ H_y^n(i,j+1,k) - H_y^n(i,j,k) \right]
\]

\[
[i_{\min} + 1 \leq i \leq i_{\max}, j_{\min} \leq j \leq j_{\max}, k_{\min} \leq k \leq k_{\max}].
\]
Similarly, (1.17) is re-written as:

\[
\frac{H^n_z(i,j,k+1) - H^n_x(i,j,k)}{\Delta z} - \frac{H^n_z(i+1,j,k) - H^n_x(i,j,k)}{\Delta x} = \epsilon^{n+1}(i,j,k)E^n_y(i,j,k) - \epsilon^n(i,j,k)E^n_y(i,j,k)
\]

\[
\therefore E^n_y(i,j,k) = \frac{\epsilon^n(i,j,k)E^n_y(i,j,k)}{\epsilon^{n+1}(i,j,k)} + \frac{\Delta t}{\epsilon^{n+1}(i,j,k)} \left[ \frac{H^n_x(i,j,k+1) - H^n_x(i,j,k)}{\Delta z} - \frac{H^n_x(i+1,j,k) - H^n_x(i,j,k)}{\Delta x} \right]
\]

\[
[i_{\min} \leq i \leq i_{\max}, j_{\min} \leq j \leq j_{\max}, k_{\min} \leq k \leq k_{\max}]
\]

and finally (1.18) is also simplified as

\[
\frac{H^n_y(i,j,k+1) - H^n_y(i,j,k)}{\Delta y} - \frac{H^n_x(i,j,k+1) - H^n_x(i,j,k)}{\Delta x} = \epsilon^{n+1}(i,j,k)E^n_z(i,j,k) - \epsilon^n(i,j,k)E^n_z(i,j,k)
\]

\[
\therefore E^n_z(i,j,k) = \frac{\epsilon^n(i,j,k)E^n_z(i,j,k)}{\epsilon^{n+1}(i,j,k)} + \frac{\Delta t}{\epsilon^{n+1}(i,j,k)} \left[ \frac{H^n_y(i,j,k+1) - H^n_y(i,j,k)}{\Delta y} - \frac{H^n_x(i,j,k+1) - H^n_x(i,j,k)}{\Delta x} \right]
\]

\[
[i_{\min} \leq i \leq i_{\max}, j_{\min} \leq j \leq j_{\max}, k_{\min} + 1 \leq k \leq k_{\max}]
\]

Here, \(E^n_x(i,j,k), E^n_y(i,j,k), E^n_z(i,j,k), E^n_x(i,j,k+1), E^n_y(i,j,k+1), E^n_z(i,j,k+1), E^n_x(i,j,k+1), E^n_y(i,j,k+1), E^n_z(i,j,k+1), E^n_x(i,j,k), E^n_y(i,j,k), E^n_z(i,j,k)\) are calculated by the boundary condition.
Chapter 2

Huygens Absorbing Boundary Conditions

2.1 Absorbing Boundary Conditions (ABCs)

In Finite Difference Time Domain (FDTD) method, the size of the area that can be simulated is limited by computer resources. For instance, in three dimensional simulation values of all the six components of both E and H fields are mapped onto three dimensional arrays. During simulation as the wave propagates outward, it will eventually come to the edge of the allowable space, which is dictated by how the arrays have been dimensioned in the program. If nothing were done to address this, unpredictable reflections would be generated that would go back inward. Thus, there would be no way to determine which is the real wave and which is the reflected noise. This is the reason that Absorbing Boundary Conditions (ABCs) have been an issue for as long as FDTD has been used.

In using Absorbing boundary conditions the objective is to achieve an ideal ABC which absorbs all the outgoing waves and produces no reflection, along with catering for all incident angles of the waves propagating towards it. The idea is to simulate the open space in such a way that the waves appear to propagate infinitely. In one dimension, when operating at the Courant limit of one (when the Courant number is unity, the distance the wave travels in one temporal step \( \Delta t \) is equal to one spatial step \( \Delta x \), i.e., \( c\Delta t = \Delta x \)), an exact ABC can be realized. Unfortunately in higher dimensions, or even in one dimension when not operating at the Courant limit, ABCs are only approximate. The better the approximation
the better the ABC and less energy it will reflect back into the interior of the grid. There are basically two prevalent approaches in ABC formulation.

1. Analytical Approaches: Uses analytical techniques to get estimates of the field values at a particular point of interest for example by using different operators like Higdon operator or space-time extrapolation operator or as in the case of Mur’s and Liao’s ABC’s.

2. Material based Approaches are the conventional approaches towards boundary condition design and employ enclosing an object of interest with a material medium having special absorbing properties, so that the waves travelling through the medium are attenuated such as the Perfectly Matched Layer (PML).

### 2.2 Huygens Absorbing boundary conditions

Huygens Absorbing boundary conditions (HABC) are proactive in both in its design and implementation. They incorporate the idea of a hypothetical “Huygens surface” separating the two connecting field regions such that any field propagating towards the HABC can be canceled by generating a counter field that is equal in magnitude and opposite in direction to its original counterpart. This principle is derived from the equivalence theorem which states that “the field produced within a given part of space by sources located outside this part can be reproduced by impressing the electric and magnetic current densities upon the surface separating the two parts”.

The surface where the equivalent currents are set is called a Huygens surface and the fields generated have magnitude equal to the actual physical fields and direction opposite to them. Moreover, since an operator is used to obtain an estimate of the equivalent currents on the Huygens surface, the cancellation is not complete which results in a certain amount of reflection from the proposed ABCs. In [2] they are regarded as the generalization of two already published Absorbing boundary conditions namely the re-Radiating boundary condition (rRABC) and the Multiple Absorbing surfaces (MAS). In re-Radiating boundary condition (rRABC) [3] the idea of field teleportation was initially presented i.e. creating
an identical but opposite copy of the incident field which is then used for the attenuation of that incident field. Whereas Multiple absorbing Surfaces [6], present an idea of a connecting surface in the total field/scattered field scenario in which by imposing “connecting conditions”, the surface is able to absorb incident fields and allows only the scattered fields to propagate through the surface. Multiple surfaces are used to prevent the amplification of residual waves upon reflection form the PEC (Perfect Electric Conductor) wall.

Huygens Absorbing boundary Condition incorporates both these ideas and presents an idea of a hypothetical “Huygens surface” separating the total field scattered field regions such that any field propagating towards the HABC can be cancelled by generating a counter field that is equal in magnitude and opposite in direction to its original counterpart. In fact, it is in essence the extension of the same concept as indicated in [2] where it is regarded as an alternative implementation of ABCs based on the use of Higdon operator ABCs [5]. Here fields are generated by enforcing equivalent surface currents on the Huygens surface given by the relation,

$$\vec{J}_S = \vec{n} \times \vec{H}_i$$  \hfill (2.1)

$$\vec{K}_S = -\vec{n} \times \vec{E}_i$$  \hfill (2.2)

here, $\vec{n}$ is the unit vector normal to the Huygens surface and its orientation is opposite to that of the field originating from the source. Vectors $\vec{J}_S$ and $\vec{K}_S$ are the equivalent surface current impressed on the Huygens surface giving rise to fields equal in magnitude and opposite in direction of the physical fields.

### 2.3 Implementation Issues

In theory, the sum of the physical field with the impressed field equals zero and no additional ABC is required on the outer boundary of the domain because no field reaches this boundary. In actual implementation however, the original fields in the absence of the hypothetical Huygens surface cannot be known at the exact location of surface[2], giving rise to analytical nature of Huygens ABC. The incident field is then approximated using operators like the space-time extrapolation operator used in [2].
Figure 2.1: Basic principle of Huygens Absorbing boundary conditions as in [2]
2.4 Mathematical Formulation

Consider \( U_i \) as the incident field approaching the hypothetical Huygens surface placed at the coordinates \((x_c, t)\). Since the field is discontinuous at point \( x_c \) it cannot be known exactly at this point. The estimate of the the incident field \( U_{est}(x_c, t) \) is given by,

\[
U_{est}(x_c, t) = \sum_{k=1}^{N} a_k U_a(x_c - \delta x_k, t - \delta t_k) \tag{2.3}
\]

By retaining only one instant and one location, (2.3) reduces to,

\[
U_{est}(x_c, t) = PU_a(x_c, t) \tag{2.4}
\]

where \( P = K(-\delta x)Z(-\delta t) \) is known as the space-time extrapolation operator used by Higdon as an ABC. Using the operator, the estimated field is given by,

\[
U_{est}(x_c, t) = U_a(x_c - \delta x, t - \delta t) \tag{2.5}
\]

which is the estimate of the incident field at \((x_c, t)\) and equals the field that was present at location \((x_c - \delta x)\) and at time \((t - \delta t)\). Consider the three incidence scenarios as shown in figure 2.2.

In case A, \( U_i \) is the physical field from the source which is incident on the Huygens surface located at \( x_c \). The total field passing through the Huygens surface is the difference of the actual incident field \( U_i(x_c, t) \) and the approximation of this incident field \( U_{est}(x_c, t) \) determined analytically on the Huygens surface and radiated to oppose it. The transmitted field \( U_t(x_c, t) \) is given by,

\[
U_t(x_{c+}, t) = U_i(x_c, t) - U_{est}(x_c, t) \tag{2.6}
\]

as in this case the incident field \( U_i(x_c, t) \) is the actual field \( U_a \), the field passing through the Huygens surface is given by using (2.6) and (2.5)

\[
U_t(x_{c+}, t) = U_i(x_c, t) - U_{i+}(x_c - \delta x, t - \delta t) \tag{2.7}
\]
since the space and time steps are taken to be small for better approximation of field at point \( x_c \),

\[
U_{i+}(x_c - \delta x, t - \delta t) = U_{i+}(x_c, t) - \frac{\partial U_{i+}(x_c, t)}{\partial x} \delta x - \frac{\partial U_{i+}(x_c, t)}{\partial t} \delta t
\]  

(2.8)

replacing (2.8) in (2.7) yields,

\[
U_{i+}(x_c, t) = \frac{\partial U_{i+}(x_c, t)}{\partial x} \delta x - \frac{\partial U_{i+}(x_c, t)}{\partial t} \delta t
\]  

(2.9)

the space and time derivatives of the incident field are related as [2],

\[
\frac{\partial U_{i+}(x_c, t)}{\partial x} = -\frac{1}{c} \frac{\partial U_{i+}(x_c, t)}{\partial t}
\]  

(2.10)

replacing (2.10) in (2.9) results in,

\[
U_{i+}(x_c, t) = -\frac{1}{c} \frac{\partial U_{i+}(x_c, t)}{\partial t} \delta x - \frac{\partial U_{i+}(x_c, t)}{\partial t} \delta t
\]  

(2.11)
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\[ U_{i+}(x_c, t) = (\delta t - \frac{\delta x}{c}) \frac{\partial U_{i+}(x_c, t)}{\partial t} \]  \hspace{1cm} (2.12)

Here equation 2.12 shows that field transmitted on the other side of the Huygens surface is the time derivative of the incident field and the performance of the ABC depends on space and time steps of the space-time extrapolating operator with the ABC behaving ideally when both \( \delta x \) and \( \delta t \) tend to be zero. For the case B where the wave is originating on the right hand side of the Huygens surface, the transmitted field \( U_t(x_c, t) \) is given by,

\[ U_{t-}(x_c-, t) = U_{i-}(x_c, t) + U_{est}(x_c, t) \]  \hspace{1cm} (2.13)

The transmitted field \( U_t(x_c, t) \) passing through the Huygens surface is no longer the residue field as in the case of equation 2.6. Since the source field is on the right of the Huygens surface, sign of the normal is opposite to its physical orientation and the Huygens surface will behave as an anti-Huygens surface generating a wave opposite to the that of the approaching source field from the left. The estimated field is given by,

\[ U_{est}(x_c, t) = U_{t-}(x_c- \delta x, t- \delta t) \]  \hspace{1cm} (2.14)

\[ U_{t-}(x_c-, t) = U_{i-}(x_c, t) + U_{t-}(x_c- \delta x, t- \delta t) \]  \hspace{1cm} (2.15)

the change in sign of the transmitted fields is due to the anti-Huygens surface.

Using,

\[ U_{t-}(x_c- \delta x, t- \delta t) = U_{t-}(x_c, t) - \frac{\partial U_{t-}(x_c, t)}{\partial x} \delta x - \frac{\partial U_{t-}(x_c, t)}{\partial t} \delta t \]  \hspace{1cm} (2.16)

Equation 2.15 becomes,

\[ U_{i-}(x_c, t) = \frac{\partial U_{t-}(x_c-, t)}{\partial x} \delta x - \frac{\partial U_{t-}(x_c, t)}{\partial t} \delta t \]  \hspace{1cm} (2.17)

Replacing derivative on space with derivative on time, equation 2.17 becomes

\[ \frac{\partial U_{t-}(x_c-, t)}{\partial t} = \frac{1}{(\delta t + \frac{\delta x}{c})} \frac{\partial U_{i-}(x_c, t)}{\partial t} \]  \hspace{1cm} (2.18)
\[ U_{l-}(x_{c-}, t) = \frac{1}{(\delta t + \delta x/c)} \int U_{l-}(x_{c}, t)\delta t \] (2.19)

Equation 2.19 shows that field transmitted on the other side of the Huygens surface is the integral of the incident field and the performance of the ABC depends on the space and time steps of the space-time extrapolating operator with the ABC behaving ideally when both $\delta x$ and $\delta t$ tend to be zero. In the case when the space is ended with a PEC condition, then the actual field $U_a$ is the superposition of both transmitted and reflected fields as given by equations 2.12 and 2.19 and the overall reflection is given by,

\[ r = \frac{\delta x - c\delta t}{\delta x + c\delta t} \] (2.20)

The reflection given by the equation 2.20 is not the one which is observed in the experiments when using the Higdon operator based approximations. In practice the reflection is zero for the magic time step.

This concludes the theoretical background for the Huygens Absorbing Boundary Conditions. In the remaining chapter the derivation of the HABC equations for three Dimensions will be carried out.

### 2.5 Derivation of HABC Equations

Here figure 2.3 represents the staggering of both E and H fields in a cubic FDTD voxel that has been employed in the generation of 3-D HABC equations. It can be observed that, $E_x$ at $(i + \frac{1}{2}, j, k - \frac{1}{2})$ is written as $E_{x(i,j,k)}$, $E_y$ at $(i, j + \frac{1}{2}, k - \frac{1}{2})$ is written as $E_{y(i,j,k)}$ and $E_z$ at $(i, j, k)$ is written as $E_{z(i,j,k)}$. Similarly, $H_x$ at $(i, j + \frac{1}{2}, k)$ is written as $H_{x(i,j,k)}$, $H_y$ at $(i + \frac{1}{2}, j, k)$ is written as $H_{y(i,j,k)}$ and $H_z$ at $(i + \frac{1}{2}, j + \frac{1}{2}, k - \frac{1}{2})$ is written as $H_{z(i,j,k)}$.

Furthermore $\Delta x_i$ represents the unit spatial increment on the x-axis between $E_{z(i,j,k)}$ and $E_{z(i+1,j,k)}$, $\Delta y_j$ represents the unit spatial increment on the y-axis between $E_{z(i,j,k)}$ and $E_{z(i,j+1,k)}$ and $\Delta z_k$ represents the unit spatial increment on the z-axis between $E_{z(i,j,k)}$ and $E_{z(i,j,k+1)}$.

From 2.3 and 2.4 it can be shown that following points are calculated by the boundary conditions in each plane to keep outgoing E and H fields from being reflected back into the problem space. Normally, in calculating the E field, we need to know the surrounding H values; this is a fundamental assumption of the
Figure 2.3: A Basic cubic voxel representing all the fields in 3-D
FDTD method. At the Edge of the problem space we will not have the value to one side. However, we have an advantage because we know there are no sources outside the problem space. Therefore, the fields at the edge must be propagating outward. We will use these two facts to estimate the value at the end by using the value next to it. Here $E_{x}^{n}(i,j_{\text{min}},k)$, $E_{x}^{n}(i,j_{\text{max}},k)$, $E_{x}^{n}(i,j_{\text{kmn}},k)$, $E_{y}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{y}^{n}(i,j_{\text{kmn}},k)$, $E_{y}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$, $E_{z}^{n}(i,j_{\text{kmn}},k)$ will be calculated using the boundary conditions. Writing the standard FDTD equations for the staggering scheme as shown in figure 2.3 yields,

$$\frac{E_{x}^{n}(i,j+1,k) - E_{x}^{n}(i,j,k)}{\Delta y(j)} - \frac{E_{x}^{n}(i,j,k+1) - E_{x}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)} = \mu \left( H_{x}^{n+1}(i,j,k) - H_{x}^{n}(i,j,k) \right)$$  \hspace{1cm} (2.21)

$$\therefore H_{x}^{n+1}(i,j,k) = H_{x}^{n}(i,j,k)$$

$$+ \frac{\Delta t}{\mu} \left[ E_{y}^{n}(i,j,k+1) - E_{y}^{n}(i,j,k) \right] - \frac{E_{y}^{n}(i,j,k+1) - E_{y}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)}$$

$$\left[ i_{\text{min}} + 1 \leq i \leq i_{\text{max}} - 1, j_{\text{min}} \leq j \leq j_{\text{max}} - 1, k_{\text{min}} \leq k \leq k_{\text{max}} - 1 \right]$$

$$- \frac{E_{z}^{n}(i+1,j,k) - E_{z}^{n}(i,j,k)}{\Delta x(i)} + \frac{E_{z}^{n}(i,j,k+1) - E_{z}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)} = \mu \left( H_{y}^{n+1}(i,j,k) - H_{y}^{n}(i,j,k) \right)$$  \hspace{1cm} (2.22)

$$\therefore H_{y}^{n+1}(i,j,k) = H_{y}^{n}(i,j,k)$$

$$+ \frac{\Delta t}{\mu} \left[ E_{z}^{n}(i+1,j,k) - E_{z}^{n}(i,j,k) \right] - \frac{E_{z}^{n}(i,j+1,k) - E_{z}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)}$$

$$\left[ i_{\text{min}} \leq i \leq i_{\text{max}} - 1, j_{\text{min}} + 1 \leq j \leq j_{\text{max}} - 1, k_{\text{min}} \leq k \leq k_{\text{max}} - 1 \right]$$

$$- \frac{E_{x}^{n}(i,j+1,k) - E_{x}^{n}(i,j,k)}{\Delta y(j)} + \frac{E_{x}^{n}(i+1,j,k) - E_{x}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)} = \mu \left( H_{z}^{n+1}(i,j,k) - H_{z}^{n}(i,j,k) \right)$$  \hspace{1cm} (2.23)

$$\therefore H_{z}^{n+1}(i,j,k) = H_{z}^{n}(i,j,k)$$

$$+ \frac{\Delta t}{\mu} \left[ E_{z}^{n}(i,j+1,k) - E_{z}^{n}(i,j,k) \right] - \frac{E_{y}^{n}(i+1,j,k) - E_{y}^{n}(i,j,k)}{\Delta z(k-1) + \Delta z(k)}$$
\[ i_{\text{min}} \leq i \leq i_{\text{max}} - 1, \quad j_{\text{min}} \leq j \leq j_{\text{max}} - 1, \quad k_{\text{min}} + 1 \leq k \leq k_{\text{max}} - 1 \]

\[
\begin{align*}
\frac{H_z^{n+1}(i,j,k) - H_z^n(i,j-1,k)}{\Delta y_{(j-1)} + \Delta y_{(j)}} & - \frac{H_z^{n+1}(i,j,k) - H_z^{n+1}(i,j,k-1)}{\Delta z(k)} = (2.24) \\
\epsilon(i,j,k) \left( E_x^{n+1}(i,j,k) - E_x^n(i,j,k) \right) & = \Delta t \\
\therefore E_x^{n+1}(i,j,k) & = E_x^n(i,j,k) \\
+ \frac{\Delta t}{\epsilon(i,j,k)} \left[ \frac{H_z^{n+1}(i,j,k) - H_z^{n+1}(i,j-1,k)}{\Delta z(k)} - \frac{H_y^{n+1}(i,j,k) - H_y^{n+1}(i,j,k-1)}{\Delta z(k)} \right] \] \\
\[ i_{\text{min}} + 1 \leq i \leq i_{\text{max}} - 1, j_{\text{min}} + 1 \leq j \leq j_{\text{max}} - 1, k_{\text{min}} + 1 \leq k \leq k_{\text{max}} - 1 \]

\[
\begin{align*}
\frac{H_x^{n+1}(i,j,k) - H_x^{n+1}(i,j,k-1)}{\Delta z(k)} & - \frac{H_x^{n+1}(i,j,k) - H_x^{n+1}(i-1,j,k)}{\Delta x_{(i-1)} + \Delta x_{(i)}} = (2.25) \\
\epsilon(i,j,k) \left( E_y^{n+1}(i,j,k) - E_y^n(i,j,k) \right) & = \Delta t \\
\therefore E_y^{n+1}(i,j,k) & = E_y^n(i,j,k) \\
+ \frac{\Delta t}{\epsilon(i,j,k)} \left[ \frac{H_x^{n+1}(i,j,k) - H_x^{n+1}(i,j,k-1)}{\Delta z(k)} - \frac{H_z^{n+1}(i,j,k) - H_z^{n+1}(i-1,j,k)}{\Delta z(k)} \right] \] \\
\[ i_{\text{min}} + 1 \leq i \leq i_{\text{max}} - 1, j_{\text{min}} + 1 \leq j \leq j_{\text{max}} - 1, k_{\text{min}} + 1 \leq k \leq k_{\text{max}} - 1 \]

\[
\begin{align*}
\frac{H_y^{n+1}(i,j,k) - H_y^{n+1}(i-1,j,k)}{\Delta x_{(i-1)} + \Delta x_{(i)}} & - \frac{H_y^{n+1}(i,j,k) - H_y^{n+1}(i,j-1,k)}{\Delta y_{(j-1)} + \Delta y_{(j)}} = (2.26) \\
\epsilon(i,j,k) \left( E_z^{n+1}(i,j,k) - E_z^n(i,j,k) \right) & = \Delta t \\
\therefore E_z^{n+1}(i,j,k) & = E_z^n(i,j,k) \\
+ \frac{\Delta t}{\epsilon(i,j,k)} \left[ \frac{H_y^{n+1}(i,j,k) - H_y^{n+1}(i-1,j,k)}{\Delta x_{(i-1)} + \Delta x_{(i)}} - \frac{H_z^{n+1}(i,j,k) - H_z^{n+1}(i,j-1,k)}{\Delta x_{(i-1)} + \Delta x_{(i)}} \right] \] \\
\[ i_{\text{min}} + 1 \leq i \leq i_{\text{max}} - 1, j_{\text{min}} + 1 \leq j \leq j_{\text{max}} - 1, k_{\text{min}} + 1 \leq k \leq k_{\text{max}} \]
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Using Higdon as the space-time approximation operator, updates for the \( X \) plane are given by [8]. For \( i=imin \) boundary,

\[
E^m_{y}(i,j,k) = E^{n-1}_{y}(i+1,j,k) + \frac{\Delta t - \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i+1,j,k)}}{\Delta t + \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{y}(i+1,j,k) - E^{n-1}_{y}(i,j,k))
\]

for \( i=imax \) boundary,

\[
E^m_{y}(i,j,k) = E^{n-1}_{y}(i-1,j,k) + \frac{\Delta t - \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i-1,j,k)}}{\Delta t + \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{y}(i-1,j,k) - E^{n-1}_{y}(i,j,k))
\]

Similarly writing updates for the \( Y \) plane from [8],

for \( j=jmin \) boundary,

\[
E^m_{x}(i,j,k) = E^{n-1}_{x}(i,j+1,k) + \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j+1,k)}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{x}(i,j+1,k) - E^{n-1}_{x}(i,j,k))
\]

for \( j=jmax \) boundary,

\[
E^m_{x}(i,j,k) = E^{n-1}_{x}(i,j-1,k) + \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j-1,k)}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{x}(i,j-1,k) - E^{n-1}_{x}(i,j,k))
\]

Again using [8] updates for the \( Z \) plane,

for \( k=kmin \) boundary,

\[
E^m_{z}(i,j,k) = E^{n-1}_{z}(i,j,k+1) + \frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k+1)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{z}(i,j,k+1) - E^{n-1}_{z}(i,j,k))
\]

for \( k=kmax \) boundary,

\[
E^m_{z}(i,j,k) = E^{n-1}_{z}(i,j,k-1) + \frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k-1)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j,k)}} (E^m_{z}(i,j,k-1) - E^{n-1}_{z}(i,j,k))
\]

Following are the HABC equations for the six faces of a FDTD cubic voxel.
for the $imin$ boundary, here HABC at

$$(i_o \rightarrow i_1, j_o \rightarrow j_1, k_o \rightarrow k_1) \quad (2.33)$$

HABC is placed at $i = i_o - \frac{1}{4}$

Now, estimating $\tilde{E}_y$,

$$\tilde{E}_{yest}^{n}(i,j+rac{1}{2},k-rac{1}{2}) = E_{y}^{n-1}(i+1,j+rac{1}{2},k-rac{1}{2}) \quad (2.34)$$

estimation exchange between

$$\tilde{E}_{yest}^{n}(i,j+rac{1}{2},k-rac{1}{2}) \rightarrow H_z^{n}(i_o - \frac{1}{2},j+rac{1}{2},k-rac{1}{2}) \quad (2.35)$$

results,

$$H_z^{n}(i_o - \frac{1}{2},j+rac{1}{2},k-rac{1}{2}) = H_z^{n}(i_o - \frac{1}{2},j+rac{1}{2},k-rac{1}{2}) + \frac{\Delta t}{\mu_0 \Delta x_i} \tilde{E}_{yest}^{n}(i_o,j+rac{1}{2},k-rac{1}{2}) \quad (2.36)$$

estimating $\tilde{H}_z$,

$$\tilde{H}_{zest}^{n}(i-rac{1}{2},j+rac{1}{2},k-rac{1}{2}) = H_z^{n-1}(i+rac{1}{2},j+rac{1}{2},k-rac{1}{2}) \quad (2.37)$$

estimation exchange between

$$\tilde{H}_{zest}^{n}(i-rac{1}{2},j+rac{1}{2},k-rac{1}{2}) \rightarrow E_{y}^{n}(i_o,j+rac{1}{2},k-rac{1}{2}) \quad (2.38)$$

results,

$$E_{y}^{n}(i_o,j+rac{1}{2},k-rac{1}{2}) = E_{y}^{n}(i_o,j+rac{1}{2},k-rac{1}{2}) + \frac{\Delta t}{\sqrt{\epsilon_0 \epsilon_i \epsilon_0 \epsilon_i \epsilon_i}} \tilde{H}_{zest}^{n}(i_o - \frac{1}{4},j+rac{1}{2},k-rac{1}{2}) \quad (2.39)$$
estimating $\tilde{E}_z$,

$$
\tilde{E}_{zest}^n(i,j,k) = E_{z}^{n-1}(i+1,j,k) \quad (2.40)
$$

estimation exchange between

$$
\tilde{E}_{zest}^n(i,j,k) \rightarrow H^n_y(i_o-\frac{1}{2},j,k) \quad (2.41)
$$

results,

$$
H^n_y(i_o-\frac{1}{2},j,k) = H^n_y(i_o-\frac{1}{2},j,k) - \frac{\Delta t}{\mu_0 \Delta x_i} \tilde{E}_{zest}^n(i_o,j,k) \quad (2.42)
$$

estimating $\tilde{H}_y$,

$$
\tilde{H}_{yest}^n(i-\frac{1}{2},j,k) = H^{-1}_y(i+\frac{1}{2},j,k) \quad (2.43)
$$

estimation exchange between

$$
\tilde{H}_{yest}^n(i-\frac{1}{2},j,k) \rightarrow E^n_z(i_o,j,k) \quad (2.44)
$$

results,

$$
E^n_z(i_o,j,k) = E^n_z(i_o,j,k) - \frac{\Delta t}{\epsilon_0 \epsilon(i_o,j,k)} \tilde{H}_{yest}^n(i_o-\frac{1}{2},j,k) \quad (2.45)
$$

for the $i_{max}$ boundary, here HABC at

$$
(i_o \rightarrow i_1, j_o \rightarrow j_1, k_o \rightarrow k_1) \quad (2.46)
$$

HABC is place at $i = i_1 + \frac{1}{4}$
estimating $\tilde{E}_y$,

$$\tilde{E}_{y_{est}}^{n}(i,j+\frac{1}{2},k-\frac{1}{2}) = E_{y}^{n-1}(i-1,j+\frac{1}{2},k-\frac{1}{2}) \tag{2.47}$$

$$+ \frac{\Delta t - \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i-1,j+\frac{1}{2},k-\frac{1}{2})}}{\Delta t + \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i+\frac{1}{2},k-\frac{1}{2})}} \left( E_{y}^{n}(i-1,j+\frac{1}{2},k-\frac{1}{2}) - \tilde{E}_{y_{est}}^{n-1}(i,j+\frac{1}{2},k-\frac{1}{2}) \right)$$

estimation exchange between

$$\tilde{E}_{y_{est}}^{n}(i,j+\frac{1}{2},k-\frac{1}{2}) \rightarrow H_{z}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) \tag{2.48}$$

results,

$$H_{z}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) = H_{z}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) - \frac{\Delta t}{\mu_0 \Delta x_i} \tilde{E}_{y_{est}}^{n}(i,\frac{1}{2},k-\frac{1}{2}) \tag{2.49}$$

estimating $\tilde{H}_z$,

$$\tilde{H}_{z_{est}}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) = H_{z}^{n-1}(i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) \tag{2.50}$$

$$+ \frac{\Delta t - \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2})}}{\Delta t + \Delta x_i \sqrt{\mu_0 \epsilon_0 \epsilon(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2})}} \left( H_{z}^{n}(i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) - \tilde{H}_{z_{est}}^{n-1}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) \right)$$

estimation exchange between

$$\tilde{H}_{z_{est}}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) \rightarrow E_{y}^{n}(i,j+\frac{1}{2},k-\frac{1}{2}) \tag{2.51}$$

results,

$$E_{y}^{n}(i,j+\frac{1}{2},k-\frac{1}{2}) = E_{y}^{n}(i,j+\frac{1}{2},k-\frac{1}{2}) - \frac{\Delta t}{\epsilon_0 \epsilon(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2})} \tilde{H}_{z_{est}}^{n}(i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}) \tag{2.52}$$

estimating $\tilde{E}_z$,

$$\tilde{E}_{z_{est}}^{n}(i,j,k) = E_{z}^{n-1}(i-1,j,k) \tag{2.53}$$
estimation exchange between

\[ \tilde{E}_{zest}^{n}(i,j,k) \rightarrow H_{y}^{n}(i+\frac{1}{2},j,k) \]  

(2.54)

results,

\[ H_{y}^{n}(i+\frac{1}{2},j,k) = H_{y}^{n}(i+\frac{1}{2},j,k) + \frac{\Delta t}{\mu_{0} \Delta x_{i}} \tilde{E}_{zest}^{n}(i,j,k) \]  

(2.55)

estimating \( H_{y} \),

\[ \frac{\Delta t - \Delta x_{i} \sqrt{\mu_{0} \varepsilon_{0}(i-\frac{1}{2},j,k)}}{\Delta t + \Delta x_{i} \sqrt{\mu_{0} \varepsilon_{0}(j-\frac{1}{2},j,k)}} \left( H_{y}^{n}(i-\frac{1}{2},j,k) - \tilde{H}_{yest}^{n}(i+\frac{1}{2},j,k) \right) + \Delta t - \Delta x_{i} \sqrt{\mu_{0} \varepsilon_{0}(i-\frac{1}{2},j,k)} \]  

(2.56)

estimation exchange between

\[ \tilde{H}_{yest}^{n}(i+\frac{1}{2},j,k) \rightarrow E_{x}^{n}(i,j,k) \]  

(2.57)

results,

\[ E_{x}^{n}(i,j,k) = E_{x}^{n}(i,j,k) + \frac{\Delta t}{\varepsilon_{0} \varepsilon_{i}(i,j,k)} \tilde{H}_{yest}^{n}(i+\frac{1}{2},j,k) \]  

(2.58)

for the \( j_{min} \) boundary, here HABC at

\[ (i_{o} \rightarrow i_{1}, j_{o} \rightarrow j_{1}, k_{o} \rightarrow k_{1}) \]  

(2.59)

HABC is placed at \( j = j_{o} - \frac{1}{4} \)

estimating \( \tilde{E}_{x} \),

\[ \frac{\Delta t - \Delta y_{j} \sqrt{\mu_{0} \varepsilon_{0}(i+\frac{1}{2},j_{o}+1,k-\frac{1}{2})}}{\Delta t + \Delta y_{j} \sqrt{\mu_{0} \varepsilon_{0}(i+\frac{1}{2},j_{o}+1,k-\frac{1}{2})}} \left( E_{x}^{n}(i+\frac{1}{2},j_{o}+1,k-\frac{1}{2}) - \tilde{E}_{xest}^{n-1}(i+\frac{1}{2},j_{o},k-\frac{1}{2}) \right) + \Delta t - \Delta y_{j} \sqrt{\mu_{0} \varepsilon_{0}(i+\frac{1}{2},j_{o}+1,k-\frac{1}{2})} \]  

(2.60)
estimation exchange between

\[ \tilde{E}_{x,est}^{\hat{n}}(i+\frac{1}{2},j_o,k-\frac{1}{2}) \rightarrow H_x^{n}(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) \] (2.61)

results,

\[ H_x^n(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) = H_x^n(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) - \frac{\Delta t}{\mu_0 \Delta y_j} \tilde{E}_{x,est}^{\hat{n}}(i+\frac{1}{2},j_o,k-\frac{1}{2}) \] (2.62)

estimating \( \tilde{H}_z \),

\[ \tilde{H}_{z,est}^{\hat{n}}(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) = H_{z}^{n-1}(i+\frac{1}{2},j_o+\frac{1}{2},k-\frac{1}{2}) \] (2.63)

estimation exchange between

\[ \tilde{H}_{z,est}^{\hat{n}}(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) \rightarrow E_x^n(i+\frac{1}{2},j_o,k-\frac{1}{2}) \] (2.64)

results,

\[ E_x^n(i+\frac{1}{2},j_o,k-\frac{1}{2}) = E_x^n(i+\frac{1}{2},j_o,k-\frac{1}{2}) - \frac{\Delta t}{\epsilon_0 \epsilon(i+\frac{1}{2},j_o,k-\frac{1}{2})} \tilde{H}_{z,est}^{\hat{n}}(i+\frac{1}{2},j_o-\frac{1}{2},k-\frac{1}{2}) \] (2.65)

estimating \( \tilde{E}_z \),

\[ \tilde{E}_{z,est}^{\hat{n}}(i,j_o,k) = E_z^{n-1}(i,j_o+1,k) \] (2.66)

estimation exchange between

\[ \tilde{E}_{z,est}^{\hat{n}}(i,j_o,k) \rightarrow H_x^n(i,j_o-\frac{1}{2},k) \] (2.67)

results,
\[ H^n_x(i_{jo} - \frac{1}{4}, k) = H^n_x(i_{jo} - \frac{1}{4}, k) + \frac{\Delta t}{\mu_0 \Delta y_j} \tilde{E}^n_{xest}(i_{jo}, k) \] (2.68)

estimating \( \tilde{H} x \),

\[ \tilde{H}^n_{xest}(i_{jo} - \frac{1}{2}, k) = H^{n-1}_x(i_{jo} + \frac{1}{2}, k) \] (2.69)

\[ + \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon (i_{jo} + \frac{1}{2}, k)}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon (i_{jo} - \frac{1}{2}, k)}} \left( H^n_x(i_{jo} + \frac{1}{2}, k) - \tilde{H}^{n-1}_{xest}(i_{jo} - \frac{1}{2}, k) \right) \]

estimation exchange between

\[ \tilde{H}^n_{xest}(i_{jo} - \frac{1}{2}, k) \rightarrow E^n_z(i_{jo}, k) \] (2.70)

results,

\[ E^n_z(i_{jo}, k) = E^n_z(i_{jo}, k) + \frac{\Delta t}{\epsilon_0 \epsilon(i_{ju}, k)} \frac{\Delta y_{jo} - \frac{1}{2} + \Delta y_{jo} + \frac{1}{2}}{2} \tilde{H}^n_{xest}(i_{jo} - \frac{1}{2}, k) \] (2.71)

for the \( jmax \) boundary, here HABC at

\[ (i_o \rightarrow i_1, j_o \rightarrow j_1, k_o \rightarrow k_1) \] (2.72)

HABC is place at \( j = j_1 + \frac{1}{4} \)

estimating \( \tilde{E} x \),

\[ \tilde{E}^n_{xest}(i + \frac{1}{2}, j_1 + \frac{1}{2}, k) = E^{n-1}_x(i + \frac{1}{2}, j_1 - \frac{1}{2}) \] (2.73)

\[ + \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon (i + \frac{1}{2}, j_1 - \frac{1}{2} - \frac{1}{2})}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon (i + \frac{1}{2}, j_1 - \frac{1}{2} + \frac{1}{2})}} \left( E^n_x(i + \frac{1}{2}, j_1 - \frac{1}{2}) - \tilde{E}^{n-1}_{xest}(i + \frac{1}{2}, j_1 + \frac{1}{2}) \right) \]

estimation exchange between

\[ \tilde{E}^n_{xest}(i + \frac{1}{2}, j_1 + \frac{1}{2}, k) \rightarrow H^n_z(i + \frac{1}{2}, j_1 + \frac{1}{2}, k - \frac{1}{2}) \] (2.74)

results,
\[ H_z^n(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2}) = H_z^n(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2}) + \frac{\Delta t}{\mu_0 \Delta y_j} \tilde{E}_{z,est}^n(i + \frac{1}{2} j_1, k - \frac{1}{2}) \] (2.75)

estimating \( \tilde{H}_z \),

\[ \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i + \frac{1}{2} j_1 - \frac{1}{2}, k - \frac{1}{2})}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2})}} (H_z^n(i + \frac{1}{2} j_1 - \frac{1}{2}, k - \frac{1}{2}) - \tilde{H}_{z,est}^{n-1}(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2})) \] (2.76)

estimation exchange between

\[ \tilde{H}_{z,est}^n(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2}) \rightarrow E_x^n(i + \frac{1}{2} j_1, k - \frac{1}{2}) \] (2.77)

results,

\[ E_x^n(i + \frac{1}{2} j_1, k - \frac{1}{2}) = E_x^n(i + \frac{1}{2} j_1, k - \frac{1}{2}) + \frac{\Delta t}{\epsilon_0 \epsilon(i + \frac{1}{2} j_1, k - \frac{1}{2})} \] (2.78)

\[ \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i + \frac{1}{2} j_1, k - \frac{1}{2})}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i + \frac{1}{2} j_1, k - \frac{1}{2})}} \tilde{H}_{z,est}^n(i + \frac{1}{2} j_1 + \frac{1}{2}, k - \frac{1}{2}) \]

estimating \( \tilde{E}_z \),

\[ \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i, j_1 - 1, k)}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i, j_1, k)}} (E_z^n(i, j_1 - 1, k) - \tilde{E}_{z,est}^{n-1}(i, j_1, k)) \] (2.79)

estimation exchange between

\[ \tilde{E}_{z,est}^n(i, j_1, k) \rightarrow H_x^n(i, j_1 + \frac{1}{2}, k) \] (2.80)

results,

\[ H_x^n(i, j_1 + \frac{1}{2}, k) = H_x^n(i, j_1 + \frac{1}{2}, k) - \frac{\Delta t}{\mu_0 \Delta y_j} \tilde{E}_{z,est}^n(i, j_1, k) \] (2.81)

estimating \( \tilde{H}_x \),
\[ \hat{H}_{xest}^n(i,j_1+\frac{1}{2}k) = H_{x}^{n-1}(i,j_1-\frac{1}{2}k) \]  
\[ + \frac{\Delta t - \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j_1-\frac{1}{2}k)}}{\Delta t + \Delta y_j \sqrt{\mu_0 \epsilon_0 \epsilon(i,j_1+\frac{1}{2}k)}} \left( H_x^n(i,j_1-\frac{1}{2}k) - \hat{H}_{xest}^{n-1}(i,j_1+\frac{1}{2}k) \right) \]  

estimation exchange between

\[ \hat{H}_{xest}^n(i,j_1+\frac{1}{2}k) \rightarrow E_z^n(i,j_1,k) \]  

results,

\[ E_z^n(i,j_1,k) = E_z^n(i,j_1,k) - \frac{\Delta t}{\Delta y_j(i,j_1) + \Delta y_j(i,j_1+\frac{1}{2})} \hat{H}_{xest}^n(i,j_1+\frac{1}{2}k) \]  

for the \( k_{min} \) boundary, here HABC at

\[ (i_o \rightarrow i_1, j_o \rightarrow j_1, k_o \rightarrow k_1) \]  

HABC is place at \( k = k_o - \frac{1}{4} \) estimating \( \tilde{E}_x \),

\[ \hat{E}_{xest}^n(i+\frac{1}{2},j,k_o-\frac{1}{2}) = E_x^{n-1}(i+\frac{1}{2},j,k_o+\frac{1}{2}) \]  
\[ + \frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i+\frac{1}{2},j,k_o+\frac{1}{2})}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i+\frac{1}{2},j,k_o-\frac{1}{2})}} \left( E_x^n(i+\frac{1}{2},j,k_o+\frac{1}{2}) - \hat{E}_{xest}^{n-1}(i+\frac{1}{2},j,k_o-\frac{1}{2}) \right) \]  

estimation exchange between

\[ \hat{E}_{xest}^n(i+\frac{1}{2},j,k_o-\frac{1}{2}) \rightarrow H_y^n(i+\frac{1}{2},j,k_o-1) \]  

results in,

\[ H_y^n(i+\frac{1}{2},j,k_o-1) = H_y^n(i+\frac{1}{2},j,k_o-1) + \frac{\Delta t}{\mu_0 \Delta z_{k_o-\frac{1}{2}} + \Delta z_{k_o+\frac{1}{2}}} \hat{E}_{xest}^n(i+\frac{1}{2},j,k_o-\frac{1}{2}) \]  

estimating \( \hat{H}_y \),

\[ \hat{H}_{yest}^n(i+\frac{1}{2},j,k_o-1) = H_y^{n-1}(i+\frac{1}{2},j,k_o) \]
\[\frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i+1/2, j, k_o)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i+1/2, j, k_o-1)}} (H^n_y (i+1/2, j, k_o) - \tilde{H}^{n-1}_{y, ext} (i+1/2, j, k_o-1))\]

estimation exchange between

\[\tilde{H}^{n}_{y, ext} (i+1/2, j, k_o-1) \rightarrow E^n_x (i+1/2, j, k_o - 1/2)\]  (2.90)

results in,

\[E^n_x (i+1/2, j, k_o - 1/2) = E^n_x (i+1/2, j, k_o - 1/2) + \frac{\Delta t}{\epsilon_0 \epsilon (i+1/2, j, k_o-1/2) \Delta z_k} \tilde{H}^{n}_{y, ext} (i+1/2, j, k_o-1)\]  (2.91)

estimating \(\tilde{E}_y\),

\[\frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i,j+1/2, k_o+1/2)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i,j+1/2, k_o-1/2)}} (E^n_y (i,j+1/2, k_o+1/2) - \tilde{E}^{n-1}_{y, ext} (i,j+1/2, k_o-1/2))\]  (2.92)

estimation exchange between

\[\tilde{E}^{n}_{y, ext} (i,j+1/2, k_o-1) \rightarrow H^n_x (i,j+1/2, k_o-1)\]  (2.93)

results,

\[H^n_x (i,j+1/2, k_o-1) = H^n_x (i,j+1/2, k_o-1) - \frac{\Delta t}{\mu_0 (1/2 + 1/2) \Delta z_k} \tilde{E}^{n}_{y, ext} (i,j+1/2, k_o-1/2)\]  (2.94)

estimating \(\tilde{H}_x\),

\[\frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i,j+1/2, k_o)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon (i,j+1/2, k_o-1)}} (H^n_x (i,j+1/2, k_o) - \tilde{H}^{n-1}_{x, ext} (i,j+1/2, k_o-1))\]  (2.95)

estimation exchange between

\[\tilde{H}^{n}_{x, ext} (i,j+1/2, k_o-1) \rightarrow E^n_y (i,j+1/2, k_o - 1/2)\]  (2.96)
\[
E_y^n(i,j+\frac{1}{2},k_o-\frac{1}{2}) = \frac{\Delta t}{\epsilon_0 \epsilon(i,j+\frac{1}{2},k_o-\frac{1}{2}) \Delta z_k} \tilde{H}_{ext}^n(i,j+\frac{1}{2},k_o-1) \tag{2.97}
\]

for the \(k_{\text{max}}\) boundary, here HABC at
\[
(i_o \to i_1, j_o \to j_1, k_o \to k_1) \tag{2.98}
\]

HABC is place at \(k = k_1 + \frac{3}{4}\) estimating \(\tilde{E}_x\),
\[
\tilde{E}_x^n_{ext}(i+\frac{1}{2},j,k_1+\frac{1}{2}) = E^{-1}_x(i+\frac{1}{2},j,k_1-\frac{1}{2}) + \frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon(i+\frac{1}{2},j,k_1-\frac{1}{2})}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon(i+\frac{1}{2},j,k_1+\frac{1}{2})}} \left( E_x^n(i+\frac{1}{2},j,k_1-\frac{1}{2}) - \tilde{E}_x^{n-1}_{ext}(i+\frac{1}{2},j,k_1+\frac{1}{2}) \right) \tag{2.99}
\]
estimation exchange between
\[
\tilde{E}_x^n_{ext}(i+\frac{1}{2},j,k_1+\frac{1}{2}) \to H_y^n(i+\frac{1}{2},j,k_1+1) \tag{2.100}
\]
results,
\[
H_y^n(i+\frac{1}{2},j,k_1+1) = H_y^n(i+\frac{1}{2},j,k_1+1) - \frac{\Delta t}{\mu_0 \Delta z_{k_1-\frac{1}{2}} + \Delta z_{k_1+\frac{1}{2}}} \tilde{E}_x^n_{ext}(i+\frac{1}{2},j,k_1+\frac{1}{2}) \tag{2.101}
\]
estimating \(\tilde{H}_y\),
\[
\tilde{H}_y^n_{ext}(i+\frac{1}{2},j,k_1+1) = H^{-1}_y(i+\frac{1}{2},j,k_1) + \frac{\Delta t - \Delta z_k \sqrt{\mu_0 \epsilon(i+\frac{1}{2},j,k_1)} \sqrt{\mu_0 \epsilon(i+\frac{1}{2},j,k_1+1)}}{\Delta t + \Delta z_k \sqrt{\mu_0 \epsilon(i+\frac{1}{2},j,k_1+1)}} \left( H_y^n(i+\frac{1}{2},j,k_1) - \tilde{H}_y^{n-1}_{ext}(i+\frac{1}{2},j,k_1+1) \right) \tag{2.102}
\]
estimation exchange between
\[
\tilde{H}_y^n_{ext}(i+\frac{1}{2},j,k_1+1) \to E_x^n(i+\frac{1}{2},j,k_1+\frac{1}{2}) \tag{2.103}
\]
results in,
\[
E_x^n(i+\frac{1}{2},j,k_1+\frac{1}{2}) = E_x^n(i+\frac{1}{2},j,k_1+\frac{1}{2}) - \frac{\Delta t}{\epsilon_0 \epsilon(i+\frac{1}{2},j,k_1+\frac{1}{2}) \Delta z_k} \tilde{H}_y^n_{ext}(i+\frac{1}{2},j,k_1+1) \tag{2.104}
\]
estimating $\tilde{E}_y$,

$$
\tilde{E}_{ext}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) = E_{y}^{n-1}(i,j+\frac{1}{2},k_1-\frac{1}{2}) \quad (2.105)
$$

$$
+ \Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j+\frac{1}{2},k_1-\frac{1}{2})} \left( E_{y}^{n}(i,j+\frac{1}{2},k_1-\frac{1}{2}) - \tilde{E}_{ext}^{n-1}(i,j+\frac{1}{2},k_1+\frac{1}{2}) \right)
$$

estimation exchange between

$$
\tilde{E}_{yest}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) \rightarrow H_{x}^n(i,j+\frac{1}{2},k_1+1) \quad (2.106)
$$

results,

$$
H_{x}^n(i,j+\frac{1}{2},k_1+1) = H_{x}^n(i,j+\frac{1}{2},k_1+1) + \frac{\Delta t}{\mu_0 \left( \frac{\Delta z_{k_1-\frac{1}{2}}}{2} + \Delta z_{k_1+\frac{1}{2}} \right)} \tilde{E}_{yest}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) \quad (2.107)
$$

estimating $\tilde{H}_{x}$,

$$
\tilde{H}_{xest}^n(i,j+\frac{1}{2},k_1+1) = H_{x}^{n-1}(i,j+\frac{1}{2},k_1) \quad (2.108)
$$

$$
+ \Delta t - \Delta z_k \sqrt{\mu_0 \epsilon_0 \epsilon(i,j+\frac{1}{2},k_1)} \left( H_{x}^{n}(i,j+\frac{1}{2},k_1) - \tilde{H}_{xest}^{n-1}(i,j+\frac{1}{2},k_1+1) \right)
$$

estimation exchange between

$$
\tilde{H}_{xest}^n(i,j+\frac{1}{2},k_1+1) \rightarrow E_{y}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) \quad (2.109)
$$

results,

$$
E_{y}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) = E_{y}^n(i,j+\frac{1}{2},k_1+\frac{1}{2}) + \frac{\Delta t}{\epsilon_0 \epsilon(i,j+\frac{1}{2},k_1+\frac{1}{2}) \Delta z_k} \tilde{H}_{xest}^n(i,j+\frac{1}{2},k_1+1) \quad (2.110)
$$
Figure 2.4: Split representation all the fields in 3-D
Chapter 3

Introduction of HABC into LOD-FDTD method

3.1 LOD-FDTD Method

LOD method is an alternative method for the application of the FDTD method and it is by design implicit in nature. Implicit methods were introduced to overcome the time step limitation inherent in the conventional explicit methods. This implies that a larger time step can be used for the computational domain when compared to the normal explicit FDTD method. This results in the speed up of the overall simulation time, highly desirable when electromagnetic fields are to be determined for a large computational space or whenever objects having very fine details are to be modeled.

When compared with the Alternating Direction Implicit (ADI)-FDTD method, another implicit method introduced to overcome the time step limitation a speed up in computation time is observed because there are fewer terms on the right hand side of the equation.

3.2 Mathematical Derivation for 1D-LOD FDTD Method

An x-directed, y-polarized TEM wave in one dimension is given by,
3. INTRODUCTION OF HABC INTO LOD-FDTD METHOD

\[ \epsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} \quad (3.1) \]

\[ \mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} \quad (3.2) \]

Above equations when re-written in matrix form gives,

\[ \frac{\partial}{\partial t} \begin{pmatrix} E_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial x}{\epsilon \partial x} \\ -\frac{\partial x}{\mu \partial x} & 0 \end{pmatrix} \begin{pmatrix} E_y \\ H_z \end{pmatrix} \quad (3.3) \]

which when converted to the generalized form is written as,

\[ \frac{\partial U}{\partial t} = GU \quad (3.4) \]

Here,

\[ U = \begin{pmatrix} E_y \\ H_z \end{pmatrix} \quad G = \begin{pmatrix} 0 & -\frac{\partial x}{\epsilon \partial x} \\ -\frac{\partial x}{\mu \partial x} & 0 \end{pmatrix} \quad (3.5) \]

Using first-order approximations for equation \[ (3.4) \] gives us,

\[ \frac{\partial U}{\partial t} = \frac{U^{n+1} - U^n}{\Delta t} \]

and

\[ U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2} \]

Replacing these values in equation \[ (3.4) \] results in,
CHAPTER 3. INTRODUCTION OF HABC INTO LOD-FDTD METHOD

\[
\frac{U^{n+1} - U^n}{\Delta t} = G \left( \frac{U^{n+1} + U^n}{2} \right) \tag{3.6}
\]

\[
U^{n+1} - U^n = \frac{\Delta t}{2} G \left( U^{n+1} + U^n \right) \tag{3.7}
\]

\[
U^{n+1} - \frac{\Delta t}{2} G U^{n+1} = U^n + \frac{\Delta t}{2} G U^n \tag{3.8}
\]

\[
U^{n+1} \left( \mathbb{I} - \frac{\Delta t}{2} G \right) = U^n \left( \mathbb{I} + \frac{\Delta t}{2} G \right) \tag{3.9}
\]

\[
U^{n+1} = \frac{\left( \mathbb{I} + \frac{\Delta t}{2} G \right) U^n}{\left( \mathbb{I} - \frac{\Delta t}{2} G \right)} \tag{3.10}
\]

Here, equation 3.10 refers to the generalized form for LOD-FDTD equations. Expanding equation 3.8 while replacing the values of U and G from equation 3.5 into it results in the following reformulations,

\[
E_y^{n+1} + \frac{\Delta t}{2\varepsilon_0} \left( \frac{\partial H_z^{n+1}}{\partial x} \right) = E_y^n - \frac{\Delta t}{2\varepsilon_0} \left( \frac{\partial H_z^n}{\partial x} \right) \tag{3.11}
\]

\[
H_z^{n+1} + \frac{\Delta t}{2\mu_0} \left( \frac{\partial E_y^{n+1}}{\partial x} \right) = H_z^n - \frac{\Delta t}{2\mu_0} \left( \frac{\partial E_y^n}{\partial x} \right) \tag{3.12}
\]

Rearranging equations 3.11 and 3.12 gives us,

\[
E_y^{n+1} = E_y^n - \frac{\Delta t}{2\varepsilon_0} \left( \frac{\partial H_z^{n+1}}{\partial x} + \frac{\partial H_z^n}{\partial x} \right) \tag{3.13}
\]
CHAPTER 3. INTRODUCTION OF HABC INTO LOD-FDTD METHOD

Discretizing equations 3.13 and 3.14 yields,

\[
H_{z}^{n+1} = H_{z}^{n} - \frac{\Delta t}{2\mu_{0}} \left( \frac{\partial E_{y}^{n+1}}{\partial x} + \frac{\partial E_{y}^{n}}{\partial x} \right) \tag{3.14}
\]

Replacing the value of \(H_{z}^{n+1}\) and \(H_{z}^{n+1}(k+\frac{1}{2})\) in equation 3.15 yields,

\[
E_{y}^{n+1}(k) = E_{y}^{n}(k) - \frac{\Delta t}{2\varepsilon_{0} \Delta x} \left( \frac{H_{z}^{n+1}(k+\frac{1}{2}) - H_{z}^{n+1}(k-\frac{1}{2})}{\Delta x} + \frac{H_{z}^{n}(k+\frac{1}{2}) - H_{z}^{n}(k-\frac{1}{2})}{\Delta x} \right) \tag{3.15}
\]

\[
H_{z}^{n+1}(k+\frac{1}{2}) = H_{z}^{n}(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k+1) - E_{y}^{n+1}(k) + E_{y}^{n}(k+1) - E_{y}^{n}(k) \right)
\]

\[
E_{y}^{n+1}(k+1) = E_{y}^{n}(k) - \frac{\Delta t}{2\varepsilon_{0} \Delta x} \left[ H_{z}^{n}(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k+1) - E_{y}^{n+1}(k) \right) + \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k+1) - E_{y}^{n+1}(k-1) \right) + E_{y}^{n}(k) - E_{y}^{n}(k-1) + H_{z}^{n}(k+\frac{1}{2}) - H_{z}^{n}(k-\frac{1}{2}) \right]
\]

\[
E_{y}^{n+1}(k) = E_{y}^{n}(k) - \frac{\Delta t}{2\varepsilon_{0} \Delta x} \left[ H_{z}^{n}(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k+1) + \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k) \right) - \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k) \right) - H_{z}^{n}(k-\frac{1}{2}) + \frac{\Delta t}{2\mu_{0} \Delta x} \left( E_{y}^{n+1}(k+1) \right) - H_{z}^{n}(k-\frac{1}{2}) \right) \right]
\]
Moving the future E-field values on the left hand side of the above equation results in,

\[
\begin{align*}
E^{n+1}_y(k) &= E^n_y(k) - \frac{\Delta t}{2\epsilon_0\Delta x} (H^n_z(k+\frac{1}{2}) - H^n_z(k-\frac{1}{2})) + \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^{n+1}_y(k+1)) \\
&\quad + \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^{n+1}_y(k)) + \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^n_y(k+1) - E^n_y(k) - E^n_y(k-1)) \\
&\quad - \frac{\Delta t}{2\epsilon_0\Delta x} (H^n_z(k+\frac{1}{2}) - H^n_z(k-\frac{1}{2})) - \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^{n+1}_y(k-1)) \\
&\quad + \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^n_y(k-1))
\end{align*}
\]

The E and H field update equations for the 1D lod-fdtd method are

\[
\begin{align*}
H^{n+1}_z(k+\frac{1}{2}) &= H^n_z(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} (E^{n+1}_y(k+1) - E^n_y(k+1) + E^n_y(k) - E^{n+1}_y(k)) + \frac{\Delta t^2}{4\mu_0\epsilon_0\Delta x^2} (E^n_y(k+1) - 2E^n_y(k) + E^n_y(k-1))
\end{align*}
\]
3.3 Boundary conditions in LOD-FDTD Method

The LOD-FDTD update equations when implemented take the form of tridiagonal matrices which are then solved using subroutines packages like LAPACK etc. In practice, LOD-FDTD method is not immune to the boundary value problem encountered in the normal explicit FDTD method. Since at each time step a tridiagonal matrix system is being solved iteratively to calculate the $E$ field update values which are being used to calculate the $H$ field update values. The update equations at the boundaries possess the same problem and cannot be updated conventionally as their neighbouring node values and special conditions are introduced at the boundary of the region to either reflect the wave (PEC case) or to absorb the outgoing wave so that it won’t reflect into the interior domain and create noise.

3.3.1 Implementation of Mur Boundary Condition

Mur’s ABCs are one of the simplest and most used ABCs employed to cater for the boundary condition problem is FDTD. It is both easier to implement compared to others and versatile. Despite it advantages, it has less efficiency when compared to other ABCs used in the FDTD circles.

3.3.2 Implementation of HABC into LOD-FDTD

Considering Amperes law with the electric current density term and Faraday’s law with it’s equivalent magnetic current density terms as,

$$\varepsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y \quad (3.18)$$

$$\mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} - M_z \quad (3.19)$$

Here $J$ and $M$ can act as independent sources of of E- and H-field energy, provided the conductivity and loss terms are discarded. Again re-arranging the above
equations into matrix format,
\[
\frac{\partial}{\partial t} \begin{pmatrix} E_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial x}{\epsilon \partial x} \\ -\frac{\partial x}{\mu \partial x} & 0 \end{pmatrix} \begin{pmatrix} E_y \\ H_z \end{pmatrix} - \begin{pmatrix} J \\ \frac{e_0 M}{\mu_0} \end{pmatrix}
\] (3.20)

above equation is re-written using the generalized form as,
\[
\frac{\partial U}{\partial t} = GU - T
\] (3.21)

Using first-order approximations for (3.21) obtains
\[
\frac{\partial U}{\partial t} = \frac{U^{n+1} - U^n}{\Delta t}
\]

and
\[
U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2}
\]

Replacing these in (3.21) results in,
\[
\frac{U^{n+1} - U^n}{\Delta t} = G \left( \frac{U^{n+1} + U^n}{2} \right) - \left( \frac{T^{n+\frac{1}{2}}}{2} \right)
\] (3.22)

\[
U^{n+1} - U^n = \frac{\Delta t}{2} G (U^{n+1} + U^n) - \frac{\Delta t}{2} \left( T^{n+\frac{1}{2}} \right)
\] (3.23)

\[
U^{n+1} - \frac{\Delta t}{2} GU^{n+1} = U^n + \frac{\Delta t}{2} GU^n - \frac{\Delta t}{2} \left( T^{n+\frac{1}{2}} \right)
\] (3.24)

\[
U^{n+1} \left( I - \frac{\Delta t}{2} G \right) = U^n \left( I + \frac{\Delta t}{2} G \right) - \frac{\Delta t}{2} \left( T^{n+\frac{1}{2}} \right)
\] (3.25)
CHAPTER 3. INTRODUCTION OF HABC INTO LOD-FDTD METHOD

\[ U^{n+1} = \left( \frac{1 + \Delta t}{2} G \right) U^n - \frac{\Delta t}{2} \left( T^{n+\frac{1}{2}} \right) \] (3.26)

Expanding (3.24) results in the following reformulations,

\[ E^{n+1} = E^n - \frac{\Delta t}{2\varepsilon_0} \left( \frac{\partial H^{n+1}_z}{\partial x} + \frac{\partial H^n_z}{\partial x} \right) - \frac{\Delta t}{2\varepsilon_0} \left( J^{n+\frac{1}{2}} \right) \] (3.27)

\[ H^{n+1}_z = H^n_z - \frac{\Delta t}{2\mu_0} \left( \frac{\partial E^{n+1}_y}{\partial x} + \frac{\partial E^n_y}{\partial x} \right) - \frac{\Delta t}{2\mu_0} \left( M^{n+\frac{1}{2}} \right) \] (3.28)

discretizing (3.27) and (3.28) gives us,

\[ E^{n+1}_y(k) = E^n_y(k) - \frac{\Delta t}{2\varepsilon_0} \left( \frac{H^{n+1}_z(k+\frac{1}{2}) - H^{n+1}_z(k-\frac{1}{2})}{\Delta x} + \frac{H^n_z(k+\frac{1}{2}) - H^n_z(k-\frac{1}{2})}{\Delta x} \right) \]
\[ - \frac{\Delta t}{2\varepsilon_0} \left( J^{n+\frac{1}{2}}(k) \right) \] (3.29)

\[ H^{n+1}_z(k+\frac{1}{2}) = H^n_z(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0 \Delta x} \left( \frac{E^{n+1}_y(k+1) - E^{n+1}_y(k)}{\Delta x} + \frac{E^n_y(k+1) - E^n_y(k)}{\Delta x} \right) \]
\[ - \frac{\Delta t}{2\mu_0} \left( M^{n+\frac{1}{2}}(k+\frac{1}{2}) \right) \] (3.30)

\[ E^{n+1}_y(k) = E^n_y(k) - \frac{\Delta t}{2\varepsilon_0 \Delta x} \left( H^{n+1}_z(k+\frac{1}{2}) - H^{n+1}_z(k-\frac{1}{2}) + H^n_z(k+\frac{1}{2}) - H^n_z(k-\frac{1}{2}) \right) \]
\[ - \frac{\Delta t}{2\varepsilon_0} \left( J^{n+\frac{1}{2}}(k) \right) \] (3.31)

\[ H^{n+1}_z(k+\frac{1}{2}) = H^n_z(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0 \Delta x} \left( E^{n+1}_y(k+1) - E^{n+1}_y(k) + E^n_y(k+1) - E^n_y(k) \right) \]
\[ - \frac{\Delta t}{2\mu_0} \left( M^{n+\frac{1}{2}}(k+\frac{1}{2}) \right) \] (3.32)
Replacing the value of \( H_z^{n+1}(k+\frac{1}{2}) \) and \( H_z^{n+1}(k-\frac{1}{2}) \) in (3.31) yields,

\[
E_y^{n+1}(k) = E_y^n(k) - \frac{\Delta t}{2\epsilon_0\Delta x} \left[ H_z^n(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^{n+1}(k+1) - E_y^{n+1}(k) \right) \right. \\
+ \left. E_y^n(k+1) - E_y^n(k) - \frac{\Delta t}{2\mu_0} \left( M^{n+\frac{1}{2}}(k+\frac{1}{2}) \right) \right] - \frac{\Delta t}{2\mu_0\Delta x} \left( H_z^n(k-\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^{n+1}(k-1) - E_y^{n+1}(k) \right) \right) \\
+ \left( E_y^{n+1}(k) - E_y^n(k) - E_y^n(k-1) - \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^{n+1}(k-1) - E_y^{n+1}(k) \right) \right) \\
+ \left. H_z^n(k+\frac{1}{2}) - H_z^n(k-\frac{1}{2}) \right] - \frac{\Delta t}{2\epsilon_0} \left( J^{n+\frac{1}{2}}(k) \right)
\]

\[
E_y^{n+1}(k) = E_y^n(k) - \frac{\Delta t}{2\epsilon_0\Delta x} \left[ H_z^n(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^{n+1}(k+1) + \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^n(k+1) - E_y^n(k) \right) \right) \right. \\
+ \left. \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^n(k+1) - E_y^n(k) \right) \right] - \frac{\Delta t}{2\mu_0\Delta x} \left( H_z^n(k-\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^n(k-1) - E_y^n(k) \right) \right) \\
+ \left( M^{n+\frac{1}{2}}(k+\frac{1}{2}) - M^{n+\frac{1}{2}}(k-\frac{1}{2}) \right) - \frac{\Delta t}{2\epsilon_0\Delta x} \left( H_z^n(k+\frac{1}{2}) - H_z^n(k-\frac{1}{2}) \right) \\
+ \left. \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^n(k+1) + \frac{\Delta t}{2\mu_0\Delta x} \left( E_y^n(k+1) - E_y^n(k) \right) \right) \right) - \frac{\Delta t}{2\epsilon_0} \left( J^{n+\frac{1}{2}}(k) \right)
\]

\[
E_y^{n+1}(k) = E_y^n(k) - \frac{\Delta t}{\epsilon_0\Delta x} \left[ H_z^n(k+\frac{1}{2}) - \frac{\Delta t}{\mu_0\epsilon_0\Delta x^2} \left( E_y^{n+1}(k+1) \right) \right. \\
+ \left. \frac{\Delta t}{\mu_0\epsilon_0\Delta x^2} \left( E_y^n(k+1) - 2E_y^n(k) + E_y^n(k-1) \right) \right] - \frac{\Delta t}{2\mu_0\epsilon_0\Delta x^2} \left( E_y^{n+1}(k) \right) \\
+ \left( M^{n+\frac{1}{2}}(k+\frac{1}{2}) - M^{n+\frac{1}{2}}(k-\frac{1}{2}) \right) - \frac{\Delta t}{2\epsilon_0\Delta x} \left( J^{n+\frac{1}{2}}(k) \right) \\
+ \left. \frac{\Delta t}{\mu_0\epsilon_0\Delta x^2} \left( E_y^{n+1}(k-1) \right) \right)
\]
CHAPTER 3. INTRODUCTION OF HABC INTO LOD-FDTD METHOD

Moving the future E-field values to the left hand side of the above equation results in,

\[- \left( \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k-1) + \left( 1 + \frac{\Delta t^2}{2\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k) - \left( \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k+1) \]

\[= E_y^n(k) - \frac{\Delta t}{\varepsilon_0\Delta x} (H_z^n(k+\frac{1}{2}) - H_z^n(k-\frac{1}{2})) + \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} (E_y^n(k+1) - 2E_y^n(k) + E_y^n(k-1)) \]

\[+ \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x} (M^{n+\frac{1}{2}}(k+\frac{1}{2}) - M^{n+\frac{1}{2}}(k-\frac{1}{2})) - \frac{\Delta t}{2\varepsilon_0} (J^{n+\frac{1}{2}}(k)) \]

The E and H field update equations for the 1D lod-fdtd method are

\[- \left( \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k+1) + \left( 1 + \frac{\Delta t^2}{2\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k) - \left( \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \right) E_y^{n+1}(k-1) \]

\[= E_y^n(k) - \frac{\Delta t}{\varepsilon_0\Delta x} (H_z^n(k+\frac{1}{2}) - H_z^n(k-\frac{1}{2})) + \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} (E_y^n(k+1) - 2E_y^n(k) + E_y^n(k-1)) \]

\[+ \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x} (M^{n+\frac{1}{2}}(k+\frac{1}{2}) - M^{n+\frac{1}{2}}(k-\frac{1}{2})) - \frac{\Delta t}{2\varepsilon_0} (J^{n+\frac{1}{2}}(k)) \]

\[H_z^{n+1}(k+\frac{1}{2}) = H_z^n(k+\frac{1}{2}) - \frac{\Delta t}{2\mu_0\Delta x} (E_y^{n+1}(k+1) - E_y^{n+1}(k) + E_y^n(k+1) - E_y^n(k)) \]

\[- \frac{\Delta t}{2\mu_0} (M^{n+\frac{1}{2}}(k+\frac{1}{2})) \]  \hspace{1cm} (3.33)

Here \( J \) and \( M \) values in the above equation are substituted by the estimates of the E and H field values as dictated by the relation in \( (2.2) \). For related work on this topic please see [4]. Higdon operator is used for the computation of these estimates.

\[M^{n+\frac{1}{2}}(k+\frac{1}{2}) = -\frac{1}{\Delta x} \left( \tilde{E}_{y}^{n+1}(k+1) + \tilde{E}_{y}^{n}(k+1) \right) = -\frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \left( \tilde{E}_{y}^{n+1}(k+1) + \tilde{E}_{y}^{n}(k+1) \right) \]

\[M^{n+\frac{3}{2}}(k-\frac{1}{2}) = -\frac{1}{\Delta x} \left( \tilde{E}_{y}^{n+1}(k) + \tilde{E}_{y}^{n}(k) \right) = \frac{\Delta t^2}{4\mu_0\varepsilon_0\Delta x^2} \left( \tilde{E}_{y}^{n+1}(k) + \tilde{E}_{y}^{n}(k) \right) \]
\[
J^{n+\frac{1}{2}}(k) = -\frac{1}{\Delta x} \left( \hat{H}^{n+1}_z(k-\frac{1}{2}) + \hat{H}^n_z(k-\frac{1}{2}) \right) = \frac{\Delta t}{2\epsilon_0 \Delta x} \left( \hat{H}^{n+1}_z(k-\frac{1}{2}) + \hat{H}^n_z(k-\frac{1}{2}) \right)
\]

Solving the above equations as a tridiagonal matrix system would yield the update for \(E^{n+1}_y(k)\), which is then used to calculate the update for \(H^{n+1}_z(k+\frac{1}{2})\) values.
Chapter 4

Numerical Experiments and Results

In this chapter the accuracy and efficiency of Huygens Absorbing boundary condition and the Mur boundary condition is observed and compared when implemented in both the normal explicit FDTD method and the LOD FDTD method.

4.1 Simulation Scenarios

4.1.1 Implementation of Mur BC and HABC in Explicit FDTD

Both Mur boundary condition and the HABC were implemented to verify their accuracy. The system on which these computations were performed was a Intel Core 2 processors based system with 2 GigaBytes of Ram and an Ubuntu operating system. Compiler used for compilations was gfortran version 4.8.2. The FDTD space in both cases was filled with a homogeneous medium with an identical space of 100 cells and their performance was compared afterwards. The space was excited by a hard gaussian pulse placed at the centre of the FDTD space at 50 with a pulse width of 225 and a peak value of unity. The boundary conditions in both the cases were placed at 30 cells form the source in either direction and the grid was terminated with a PEC condition. The total simulation time was for 1000 timesteps.
As can be seen from Figure 4.1, both Mur and HABC produce the same result with no reflection from the boundary at all. Further comparing the performance of HABC at the observation points 30 and 10 on the left hand side boundary of HABC, it can be observed in Figure 4.2 the reflection from the HABC is miniscule when compared to the original pulse.
Figure 4.2: Comparison of wave propagation both inside the FDTD space and beyond the HABC boundary at the left hand side HABC boundary

4.1.2 Implementation of Mur BC in Explicit FDTD and LOD FDTD

In this case Mur boundary condition was applied to both an Explicit FDTD space and the one using LOD FDTD scheme of 300 cells. The space was excited by a soft gaussian pulse placed at the centre of the FDTD space at 150 with a pulse width of 225 and a peak value of unity. The boundary conditions in both the cases were placed at 50 cells form the boundary in either direction (50,250) and the grid was terminated with a PEC condition. This simulation was run for 3000 timesteps. From [L3] it can be observed that Mur boundary condition produces the same result for both Explicit FDTD and LOD FDTD.
4.1.3 Implementation of HABC in Explicit FDTD and LOD FDTD

Just like the case of Mur boundary condition, HABC was applied to both an Explicit FDTD space and the one using LOD FDTD scheme of 300 cells. The space was excited by a soft gaussian pulse placed at the centre of the FDTD space at 150 with a pulse width of 225 and a peak value of unity. The boundary conditions in both the cases were placed at 50 cells form the boundary in either direction (50,250) and the grid was terminated with a PEC condition. The simulation was run for 3000 timesteps.

As we can be seen from the results in 4.4, with the HABC placed inside the FDTD space there appears to be a large reflection of the HABC about 20 \% of the peak value of the pulse which decays further on. When the observation point is placed beyond the HABC boundary at 30 as in 4.5 the results are similar to the performance of HABC in explicit FDTD as can be seen in 4.2.
CHAPTER 4. NUMERICAL EXPERIMENTS AND RESULTS

Figure 4.4: Wave propagation inside the FDTD space within the HABC placed at 50,250/obs. point at 90

Figure 4.5: Wave propagation within the FDTD space outside the HABC placed at 50,250/obs. point at 30
4.1.4 Comparision between Implementations of Mur BC and HABC in LOD-FDTD

Both Mur boundary condition and the HABC were implemented in an identical LOD-FDTD space of 300 cells and their performance was compared afterwards. The space was excited by a soft gaussian pulse placed at the centre of the FDTD space at 150 with a pulse width of 225 and a peak value of unity. The boundary conditions in both the cases were placed at 100 cells form the source in either direction (50, 250) and the grid was terminated with a PEC condition. Again the simulation was run for 3000 timesteps. As can be seen from the figure the result show the wave propagation as it is observed inside the FDTD space at the point 90. Here Mur implementation on the LOD FDTD shows identical performance as seen earlier in [4.1] and [4.3]. HABC-LOD FDTD again shows the reflections that were observed first in [4.4]. Various attempts were made to change different parameters to ascertain this anomaly in HABC results including changing the time step size from magic time step and changing the estimation operator from Higdon operator to the elementary operator as can be seen in [4.7] which showed similar results.

Figure 4.6: Wave propagation inside the FDTD space at observation point 90
Figure 4.7: Comparison of HABC performance with Mur BC using both Higdon and elementary operators.
Chapter 5

Conclusion and Future Work

5.1 Conclusion

Huygens absorbing boundary conditions are versatile and easy to implement in software. Their efficiency has been studied and proven to be comparable to other high level absorbing boundary conditions with the added advantage of simplicity and reduced computational cost as well as low usage of memory. LOD FDTD method is one of the popular implicit schemes and introduction of HABC in LOD FDTD scheme is an attractive study area.

In this thesis integration of HABC with the LOD FDTD method was attempted and the results were compared with the Mur implementation on the LOD FDTD method for comparison. The results so far show that HABC might be ineffective in absorbing outgoing waves compared to Mur boundary conditions producing large reflections evident in the graphs.

5.2 Future Work

At the moment the case of introducing HABC’s into LOD FDTD method has only been attempted in one dimension. The results achieved until now are far less promising and render HABC inappropriate as a grid termination scheme of choice for LOD FDTD method. More rigorous testing of HABC needs to be carried out to determine the reason for this behaviour and judge its feasibility as a boundary condition for LOD FDTD method.
Bibliography


