Chapter 3

Visual Apparent Motion and the Calculus of Variations

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Abstract. The rapid sequential presentation of two distinct objects in the human visual field induces, under suitable conditions, the illusion of a single object undergoing a smooth continuous transformation from the first to the second form. It is suggested that in generating this illusion the visual system operates according to variational principles and chooses those impleting motions which, in some suitable space, have minimum energy. Implications of this hypothesis are discussed in relation both to experimental data on apparent motion and to the general problem of visual pattern recognition.

1. Introduction

An illusion of movement occurs when two suitably shaped and suitably timed, spatially resolvable flashes of light are presented in sequence to the eye (Exner, 1875; Wertheimer, 1912). When this apparent motion is visually indistinguishable from the perception of a real object undergoing movement, the illusion is called *optimal motion* or *beta* motion (Wertheimer, 1912; Kenkel, 1913; Kolers, 1972). A common demonstration of beta motion is given by cinematography and certain flashing neon displays. The illusion has particular importance in that it evidences an active figure-construction process by the visual system; for, as Kolers (1972, p. 18) has emphasized, the phenomenon does not involve a simple perceptual replication of one of the stimulus figures across the intervening space, but the generation of an illusory object that smoothly and continuously changes in both position *and form* to fit with the disparate stimuli.

Two of the classical theories of apparent movement are the excitation theory of Wertheimer (1912) and Köhler (1923), and the figural theory of Linke and Hillebrand (see Neff, 1936). In the excitation theory, the separate stimulation of regions of the retina is assumed to give rise to a spread of activity in the neural substrate which coincides with that occurring in real motion (Motokawa, 1970, Chapter 10); in the figural theory, it is supposed that motion is inferred by the system because of the disparity in the locations and form of the two stimuli which are perceived as being different representations of the same object. Neither theory adequately fits all the experimental data (Kolers, 1972, Chapter 11). In particular, the excitation theory is incompatible with the 'motion-in-depth' effects obtainable with some stimulus pairs (Neuhaus, 1930; Kolers and Pomerantz, 1971), and in its vector form due to Brown and Voth (1937), the excitation theory gives false predictions for the direction of motion which can be induced between certain single and multiple stimuli (Kolers, 1972, Chapter 4); on the other hand, the figural theory fails to explain why motion is more likely to be seen between patterns which are close together and 'different' than between patterns which are further apart and the same (Kolers and Pomerantz, 1971; Navon, 1976).

A composite model which accounts for several apparent movement phenomena, including the case of motion seen between 'different' patterns, has been described by Kolers (1972, Chapter 7). The model has two separate channels, one for motion and space generation and one for pattern generation, linked to each other by a correlator unit. Navon (1976) has suggested an alternative unified scheme, which involves a difference in processing time for shape and location determination. These models are, however, essentially organizational, and not of the form that enables, for example, the path of the motion between two given patterns to be predicted.

It is suggested in the present study that the distinction between object position and form that occurs in the above theories is, at least technically, unnecessary, and there are, as will be seen, advantages in treating the two variables on the same basis. Thus given an object A say, and some transformed version $\tau(A)$ of A, if the transformation $\tau = \tau_1$ comes from the group of translations of the plane, then the disparity may be viewed as one in location (Figure 1(a)), whereas if $\tau = \tau_2$ comes from the group of linear transformations of the plane, then the disparity may be viewed as one in shape (Figure 1(b)). But, if both the groups are embedded in a larger group, say the group of affine motions of the plane, then as mappings preserving linear structure, τ_1 and τ_2 have exactly the same status. The question of the disparity in A and $\tau(A)$ in Figure 1(a) and (b) may then be decided in terms of some suitable distance measure on the affine group. Specifying the distances of the transformations τ_1 and τ_2 from the identity transformation does not, in fact, fix the motion completely. For example, in Figure 2 a bar-shaped object A and transform $\tau(A)$ are shown with hypothetical beta motions following in (a) a shallow curve and in (b) a sharp curve. Which, if either of the motions is actually selected by the visual system ought, in accordance with Maupertuis, to correspond to the 'shortest' or 'least-energetic' of the possible paths in perceptual space. In general, the problem of finding those curves for which some appropriate energy function achieves a minimum value among all curves having the same end-points is dealt with by the calculus of variations (see, for example, Gelfand and Fomin, 1963; Milnor, 1963). In the present case, these curves are in the manifold formed by all the local transformations which can be applied to A. Depending on the model chosen, there is a naturally associated energy function, which, for a given transformation τ, gives rise to at least one energy-minimizing time-parameterized family of transformations, connecting the identity to τ. This family of transformations is usually unique.

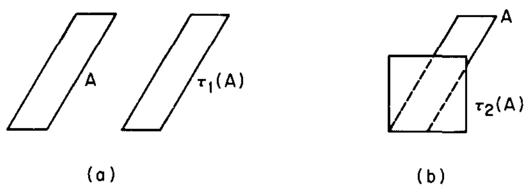

Figure 1. Stimulus pairs A, $\tau(A)$ for (a) $\tau = \tau_1$ a rigid motion and (b) $\tau = \tau_2$ a linear transformation

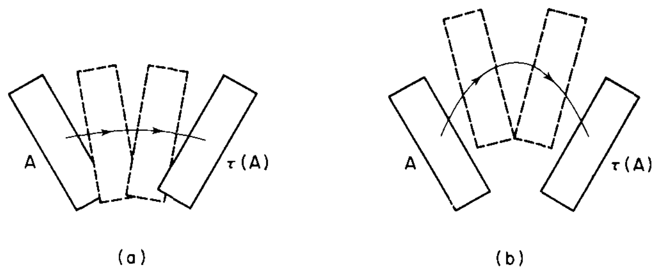

Figure 2. Two possible motions connecting objects A and $\tau(A)$

In what follows, two schemes for apparent motion are described. The first is oriented towards the excitation theory and the second towards the figural theory. Both make use of variational principles and both can be adjusted to fit most of the experimental data on the existence or otherwise of motion between various objects. It is in their predictions of the shape of the motion, however, that the models are found to differ. These predictions are compared with the corresponding data, and their significance discussed in relation to the general problem of visual pattern recognition.

2. Notation and Definitions

For simplicity we deal with the monocular situation, though there is little difficulty in extending the discussion to the binocular case. Let R denote the real line. Consider a fixed 2-dimensional plane R^2, perpendicular to the visual axis, and let R^2 be endowed with a fixed mapping C of R^2 into R, the *background field*, which assigns to each point in R^2, unless otherwise indicated, some specified luminance $C(x) \geqq 0$ (white-light stimuli, say). A visual *object* or *pattern* A on R^2 is (at least) a mapping of a non-empty subset U_A of R^2 into R such that $A(x) \geqq 0$ is the luminance of the object at the point $x \in U_A$. Neither background C nor objects A need be continuous functions and the *domain* U_A of A, which can coincide with R^2, need not be an open set. Depending upon the occasion, we may assign to an object a certain mathematical structure, for example, the metric structure arising from the standard metric structure on R^2, or the topological structure arising from the standard topological structure on R^2. Note that there is no loss in generality in using R^2 as background, instead of some fixed sphere centred at the eye, since we shall be concerned only with local properties of the visual field.

Let U be a subset of R^2. The action of an injective mapping $\tau: U \to R^2$, taking U into R^2, on an object A with domain $U_A = U$ is defined by:

$$(\tau(A))(p) = A(\tau^{-1}(p)) \quad \text{for all } p \in \tau(U).$$

The transformed object associates with each point p in its domain the luminance at its preimage $\tau^{-1}(p)$. The mapping τ and its inverse $\tau^{-1}:\tau(U) \to U$ will frequently be assumed differentiable (i.e. τ is a *diffeomorphism* into R^2). The class of a function, vector field, etc. and its domain of definition will always be understood to be such that everything is well-defined.

We now formulate the definition of beta motion in terms of the above quantities. It is convenient, though not strictly necessary, to consider beta motion as if it actually occurs on the plane R^2. Provided all sets and mappings defined on R^2 are understood to be specified only to within visual indistinguishability (Zeeman, 1962; Zadeh, 1965), the subjective illusory motion may certainly be replaced by an equivalent objective real motion. Accordingly, if F denotes the set of all objects on R^2, then given the sequential presentation to the visual system of some object A and transform $\tau(A)$ of A, beta motion between A and $\tau(A)$ is the generation by the visual system of a smooth time-parametrized curve ω in F joining these two objects. It is a smooth curve in the sense that we consider it arising from the action of a (differentiable) 1-parameter family of transformations $\psi:[0, 1] \times U_A \to R^2$, satisfying $\psi(0, p) = p$ and $\psi_1(p) = \tau(p)$ for all $p \in U_A$. The mapping $p \to \psi_t(p) = \psi(t, p)$ is a diffeomorphism of U_A onto $\psi_t(U_A)$. We use γ to denote the mapping $t \to \psi_t$ of $[0, 1]$ into the space of all diffeomorphisms of U_A into R^2. The curve ω in F (see Figure 3) may thus be written

$$\omega: t \in [0, 1] \to (\gamma(t))(A) \in F.$$

Usually A is referred to as the *initial object* and $\tau(A)$ as the *final object*. The

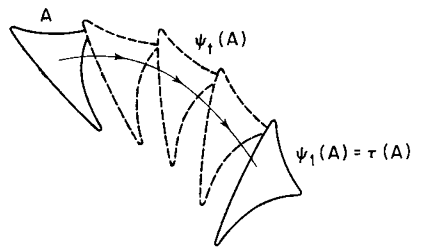

Figure 3. Action of a 1-parameter family of local transformations ψ_t taking object A into transform $\tau(A)$

symbol τ will always be reserved for the corresponding transformation. Note that the above definition automatically includes the 'plastic deformation' motion described by Kolers and Pomerantz (1971). By hypothesis, the curve ω or γ is chosen such that for some *Lagrangian L*, the *action* $S(\gamma)$ of γ, defined by

$$S(\gamma) = \int_0^1 L(\gamma(t), \gamma'(t))\, dt, \tag{1}$$

where $\gamma'(t)$ is the tangent vector to the curve γ at the point $\gamma(t)$, is minimized within the class of all paths joining $\gamma(0)$ to $\gamma(1)$. The integral $S(\gamma)$ is sometimes called the *energy* (Milnor, 1963).

3. A Figural Theory with Scalar Potential

As was pointed out in the introduction, the excitation theory of apparent motion due to Wertheimer and Köhler and its subsequent modifications have been shown to be inconsistent with certain experimental results. For example, the vector model has been shown by Kolers (1972, Chapter 5) to be false in that with the display of Figure 4(a), a vector addition of forces induced by the stimuli in perceptual space gives the motion of Figure 4(b), whereas it is the split motion of Figure 4(c) that is actually observed.

Nevertheless, with the introduction of a figural component into the excitation scheme, it is possible to account for many of the otherwise anomalous findings by Kolers (1972) and Navon (1976) concerning motion between multiple stimuli, and also for the data obtained by Foster (1975b) concerning the form of the path shapes. The scheme is as follows.

Suppose that there is in perceptual space an interaction between initial object A and final object $\tau(A)$ in such a way that a vector field $F: R^2 \to R^2$ is created in the vicinity of the two objects, and that this vector field arises in the same way as in electrostatics, that is,

$$F(r) = \int_{U_A} \frac{(r-p)}{|r-p|^3}\, ds(p) - \int_{\tau(U_A)} \frac{(r-q)}{|r-q|^3}\, ds(q),$$

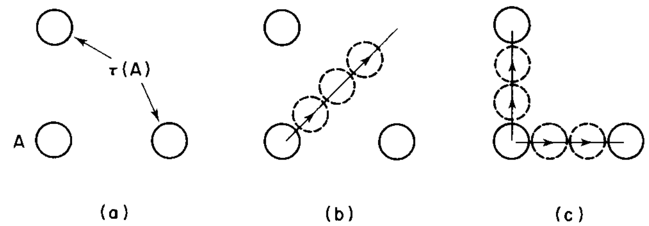

Figure 4. (a) Patterns used in test of vector model (Kolers, 1972, Chapter 5), (b) predicted motion, (c) experimentally observed motion

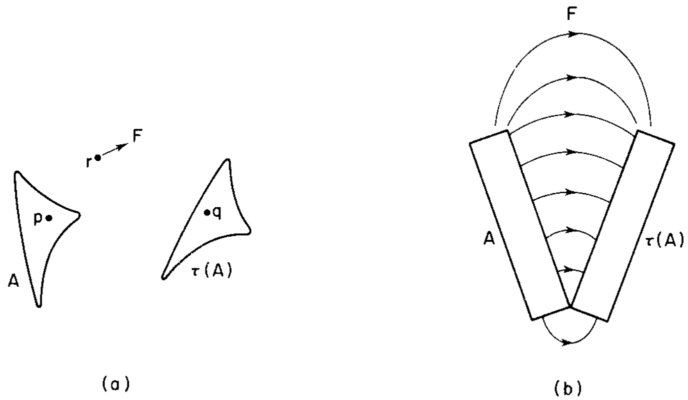

Figure 5. (a) Vector field F at r due to stimulus elements at p and q. (b) Field F due to two bar-shaped objects

where $p \in U_A$, $q \in \tau(U_A)$ (see Figure 5(a)) and ds is the usual surface measure on R^2. In the case that A is bar-shaped and τ is drawn from the group $E(2)$ of rigid motions of the plane, F has the form shown in Figure 5(b).

Associated with the vector field F there is a scalar potential function V for which $F = -\text{grad}\ V$. We suppose in accordance with the figural theory that the visual system introduces an illusory object in motion to reconcile the separate occurrence of A and $\tau(A)$. We further suppose that this illusory object is constrained to transform in such a way that it minimizes the action integral (1) for the Langrangian L of the form $T - V$, where T is the 'kinetic energy' assigned by the visual system to the object. It is the imposition of a figure-rationalizing constraint on the natural motion determined by the vector field that makes possible the satisfactory description of, for example, the split motion referred to earlier. The term T is obtained in the following way (Marsden et al. 1972). Let D denote the space of all diffeomorphisms of the domain U_A of A into R^2. Let $\rho \in D$ and let tangent vectors X, $Y \in T_\rho D$, the tangent space to D at ρ. For each point p in U_A, $X(p)$ and $Y(p)$ are in the tangent space to R^2 at $\rho(p)$. An inner product $(,)\rho$ on $T_\rho D$ may be defined thus:

$$(X, Y)_\rho = \int_{U_A} \langle X(p), Y(p) \rangle \, \mathrm{d}s(p),$$

where $\langle , \rangle$ is the usual inner product on R^2. For a smooth curve $\gamma : [0, 1] \to D$, T is then defined at time t by

$$T(t) = \tfrac{1}{2}(\gamma'(t), \gamma'(t))_{\gamma(t)}.$$

For the particular case of the bar-shaped objects of Figure 5(b), D may be replaced by $E(2)$, and the predicted action-minimizing motion $(\gamma(t))(A)$, $t \in [0, 1]$, is then found to be approximately circular. This motion is similar to, though not precisely the same as, the observed motion (Foster, 1975b) shown in Figure 6(a).

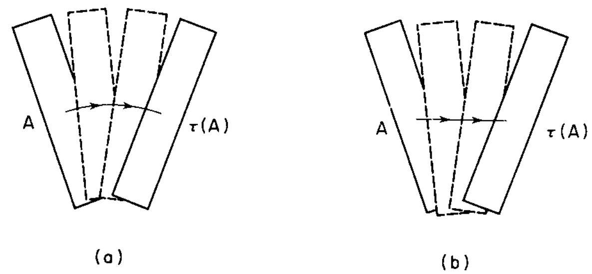

Figure 6. (a) Approximate motion between two bar-shaped objects predicted by scalar potential model. (b) Motion predicted by model with scalar potential component removed

The existence of the vector field is an essential requirement for the path to be curved. If the field is omitted, and the action-minimizing curves γ determined for $L = T$, the resulting motion is found (Foster, 1975b) to correspond to that of a free body, as in Figure 6(b).

The situation when the final object B is not the result of application to the object A of $\tau \in E(2)$, but of $\tau \in E(3)$, the group of rigid motions of R^3, necessarily requires some motion out of the plane R^2. Figure 7 shows the vector field for a pair of bar-shaped objects. (The field is similar to that in Figure 5(b).) If the action-minimizing curve γ is to remain within $E(3)$ for the configuration of Figure 7, the departure of $(\gamma(t))(A)$ from the plane must be minimal. In particular, the motion cannot take the form shown in Figure 8, i.e. a full semicircular rotation in depth. A semicircular motion is, however, obtained experimentally. Thus although motion in R^3 is not incompatible with the scalar potential model described here, the paths are not always of the right form.

A significant property of all the observed motions is that they appear to arise as the actions of segments of 1-parameter groups that is $\gamma(s)\gamma(t) = \gamma(s + t)$ for s, t, $s + t \in [0, 1]$. In the next section we simplify the model by dropping the scalar potential and changing the kinetic energy term in such a way that the local 1-parameter groups are precisely the paths which locally minimize action.

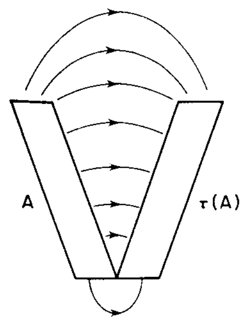

Figure 7. Vector fields for a pair of bar-shaped objects

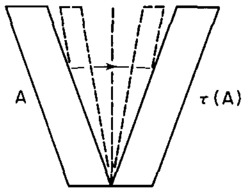

Figure 8. 'Motion-in-depth' between two bar-shaped objects. Top of illusory object appears to move through full semicircle, bottom remains fixed

4. A Pure Figural Theory

Consider again an arbitrary initial object A and final object B, and suppose that $B = \tau(A)$ for some local diffeomorphism of A such that τ may be embedded in a flow, that is, that $\tau = \psi_\tau$ for some local l-parameter group of local transformations $\psi:[0, 1] \times U \to R^2$; in fact such that $\tau = \psi_1$. (Although the collection of such embeddable τ may not be dense in the C^r manifold of all diffeomorphisms having the domain U, this is unlikely to be an important restriction in practice.)

We may then associate with τ a vector field X^ψ, namely the vector field on R^2 Induced by the local l-parameter group ψ_t thus:

$$X^\psi(p) = \frac{d\psi_t(u)}{dt} = (\gamma'(t))(u),$$

where $p = \psi_t(u)$ (see Figure 9).

Since every vector field gives rise to a flow, attention may be concentrated on the set of vector fields. So that motions in depth may be included, we shall, in fact, consider the set X of all vector fields defined on some suitably large (compact) subset M of R^3 containing the region of visual interest. The space X is then a Hilbert space with inner product

$$(X, Y) = \int_M \langle X(p), Y(p)\rangle dv(p).$$

The natural Lagrangian is then given by

$$L(X, Z(X)) = (Z(X), Z(X)),$$

where $Z(X)$ is the value of the vector field Z on $\mathcal{X}$ at the point $X \in \mathcal{X}$. Then paths $c:[0, 1] \to \mathcal{X}$ that minimize the action between fixed end-points $c(0) = 0$ and

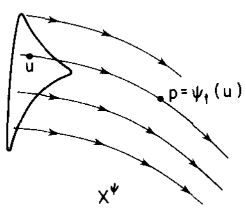

Figure 9. Action of local 1-parameter group of local transformations ψ_t

$c(1) = X,$

$$\int_0^1 L(c(t), c'(t))dt,$$

are precisely the rays in $\mathcal{X}$ emanating from the origin, i.e., the curves c for which

$$c(t) = tX, \quad t \in [0, 1].$$

We now return to consideration of the original collection of embeddable diffeomorphisms. The curve tX, $t \in [0, 1]$, defines a l-parameter family of embeddable transformations ϕ_t, $t \in [0, 1]$ by

$$\phi_t = \psi_1^{tX},$$

where ψ_1^{tX} is the local diffeomorphism defined by the flow of tX at time 1. But,

$$\psi_1^{tX} = \psi_t^X$$

i.e. the local diffeomorphism defined by the flow of X at time t. Note that ψ_1^0 is the identity. The curve tX, $t \in [0, 1]$, thus corresponds to a local l-parameter group connecting the identity to the diffeomorphism ψ_1^X.

Thus, as with the scheme of the preceding section, if the visual system resolves the disparity between patterns A and $\tau(A)$ by generating beta motion, and if it chooses those motions which minimize the above action integral, then it will effect a local 1-parameter group of local transformations. In the case that the metric structure is preserved throughout the motion, this model then predicts, correctly, the motions illustrated in Figures 6(a) and 8.

For beta motion in which the final object $B = \tau(A)$ is not a rigid transform of the initial object A, but for which τ is embeddable in a flow, the predicted action-minimizing motion is still of the form of a local 1-parameter group. Thus, in Figure 10, the curve should, and, experimentally, is observed to, go smoothly into the straight line. Other examples are easy to construct.

In the remainder of this study we shall be concerned solely with this pure figural model.

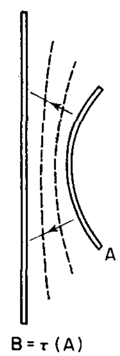

Figure 10. Motion between two objects A, τ (A), when τ is not a rigid transformation

5. Rigid Motion vs. Plastic Deformation

There is an inbuilt ill-definedness in the model which is a consequence of specifying not the pair of objects A, B inducing the beta motion, but their relationship $\tau: A \to \tau(A) = B$. The ambiguity arises in that in general there exists more than one diffeomorphism τ for which as subsets of the plane $\tau(A) = B$. Suppose τ_1 and τ_2 are two such transformations, i.e. $\tau_1(A) = \tau_2(A) = B$, with corresponding action-minimizing curves γ_1 and γ_2. Which of the motions is actually effected depends on the nature of the transformations τ_1 and τ_2 and on the relative actions or lengths of their associated paths γ_1 and γ_2. When τ_1 and τ_2 preserve the same structures (for example, the metric structure), then the motion should certainly correspond to the γ_i which has the smaller action. When τ_1 and τ_2 do not preserve the same structures, the outcome may depend on the precise form of the object A, and not just on the transformation τ. Kolers and Pomerantz (1971) have described two kinds of motion obtainable with the type of objects shown in Figure 11. Subjects reported seeing either a rigid 'motion-in-depth' or a 'plastic deformation', with the latter more probable at shorter interstimulus intervals. The 'motion-in-depth' presumably arises from the identification $\tau: A \to \tau(A) = B$, with $\tau \in E(3)$, and the 'plastic deformation' from the identification $\tau: A \to \tau(A) = B$, with $\tau \in A(2)$, the group of affine motions of the plane. The action integral associated with the 'plastic deformation' is smaller than that associated with the 'motion-in-depth'.

6. Motion Between Dissimilar Patterns

The analysis up to now has been concerned with beta motion between objects A and B for which $B = \tau(A)$, where τ is a local transformation embeddable in a differentiable flow, i.e. B is at least a smooth transform of A. But good beta motion is in fact obtainable between objects A and $\tau(A)$ where τ is only a homeomorphism, that is, τ preserves the connectivity of curves in A but not necessarily their smoothness. Figure 12(a) shows such a pair, where for suitable timings, the circle expands smoothly into the square (Kolers and Pomerantz, 1971).

By applying the analysis described at the end of Sec. 4 piece-by-piece, i.e., taking 1-parameter families like those of Figure 10 and suitably 'glueing' them together, we get the continuous (but not differentiable) flow shown in Figure 12(b). Note that although the mappings ψ_t are only homeomorphisms, the actual motion $t \to \psi_t(p)$ remains differentiable, i.e. smooth.

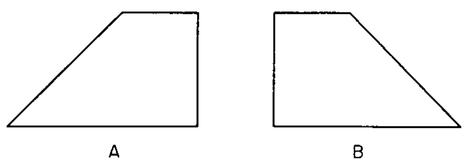

Figure 11. Patterns which yield either a rigid 'motion-in-depth' or a 'plastic deformation' (Kolers and Pomerantz, 1971; Kolers, 1972, Chapt. 6)

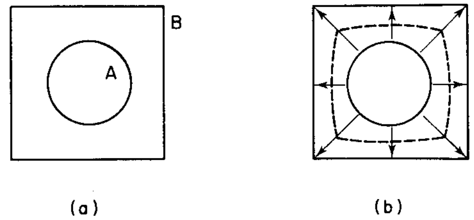

Figure 12. (a) Objects A and B for which B is not a differentiable transform of A. (b) Computed motion

An alternative to this 'glueing' together of piecewise-smooth flows is the 'replacement' of the homeomorphism τ by an embeddable diffeomorphism τ' and the effecting of a single smooth flow in the usual way. Provided τ' is close to τ, so, in terms of indistinguishability, $|\tau(x) - \tau'(x)| < \epsilon$, for all x in U_A, where ϵ is a measure of visual acuity, this substitution should be acceptable. When, however, there is no embeddable diffeomorphism close to the mapping τ relating A to B, then two possibilities arise: first, beta motion does not occur; second, beta motion does occur, but between A and the transform $\tau'(A)$ which is closest to B and for which τ' is embeddable. The second possibility is actually seen (Kolers, 1972, Chapter 4) in that subjects, when presented with patterns in the form of, for example, a hollow arrow and square, sometimes report 'a perception of plastic deformation in the course of the movement and sudden replacement at the terminus' (Kolers and Pomerantz, 1971).

What is meant by one pattern B being close to another pattern A has not been made precise for the general case. One measure might be

$$d(A, B) = \sup_{q \in B} \inf_{p \in A} |p - q|,$$

which simply records the greatest departure from overlap. This leads us to consider the following situation. Suppose patterns A and B are such that there exists

(1) an embeddable diffeomorphism τ such that $\tau(A) = B$ and such that the magnitude $\|X\| = (X, X)^{\frac{1}{2}}$ of the associated vector field $X(\psi_1^X = \tau)$ is large;

(2) an embeddable diffeomorphism τ' such that $d(B, \tau'(A))$ is small and such that the magnitude $\|X'\| = (X', X')^{\frac{1}{2}}$ of the associated vector field $X'(\psi_1^{X'} = \tau')$ is small.

In terms of actions, or distances, a motion from A to $\tau'(A)$ with subsequent replacement by B should then be the preferred solution. An example of such preferred motion has been described by Navon (1976). The patterns were letters formed into circles as shown in Figure 13.

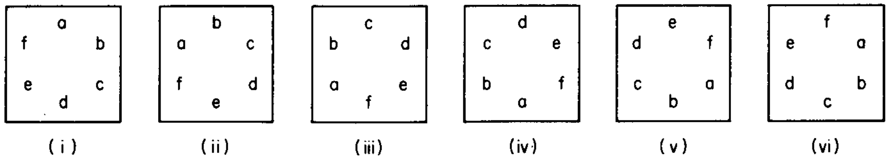

Figure 13. Irrelevance of 'figural identity' on motion. When arrays are presented in cyclic order, anticlockwise circular motion is only occasionally observed. The elements a, b, c, d, e, f can be either familiar or unfamiliar symbols (Navon, 1976)

The arrays were presented to the eye in cyclic order (i), (ii), . . . , (vi), (i), (ii), ... Of the two most likely apparent motion effects, namely (a) the elements revolving in an anticlockwise direction, and (b) the elements changing shape but staying in the same location, the 'stationary' interpretation was the most frequently observed.

7. Relationship of Apparent Movement to Form Perception

The principal objection to a figural theory of apparent movement is that motion is frequently obtainable between patterns which are not in the common sense the 'same', for example, circles and squares, and letters of the alphabet. Nevertheless, the observations by Sigman and Rock (1974) and by Corbin (1942), Rock and Ebenholtz (1962), and Attneave and Block (1973) indicate that there is necessarily some kind of intelligent rationalization involved in the production of apparent movement, and it is the perceived relationship of the stimuli, not the retinal distribution of their luminances, which is of importance.

Part of the problem centres on what is meant by the 'sameness' of patterns (Kolers, 1972, Chapter 4). Strictly, two objects A and B can only be the same if they define the same subsets of R^2, i.e., $A = B$. An equivalence $\sim$ of the form '$A \sim B$ if $B = \sigma(A)$ for some transformation σ belonging to the group of rigid motions $E(2)$' may be argued to be no different in form from an equivalence $\sim$ of the form '$A \sim B$ if $B = \sigma(A)$ for some transformation σ belonging to the group of homeomorphisms of the plane'. The first equivalence defines 'sameness' with respect to metric structure and the second with respect to topological structure. It is certainly possible, for some structures S and transformations σ to decide visually whether objects A and B can be associated as $\sigma: A \to \sigma(A) = B$, for σ preserving S, although not all transformations preserving S may be thus distinguished. For example, when S is the usual topological structure on R^2, one can recognize the difference in connectivity between an intact and broken circle, but not between the maze-like patterns cited by Julesz (1975).

The above considerations may be shown (Foster, 1975a) to lead to the notion that there is an underlying structure S_0 associated with the visual perception and identification of objects, and a set $\Gamma^*_{S_0}$ of transformations σ_0 which preserve S_0 and which can be visually effected. Visual recognition with respect to some arbitrarily fixed structure S may then be interpreted as being determined by what transformations preserving S also belong to $\Gamma^*_{S_0}$.

The figural resolution that is achieved in proper beta motion (when $B = \tau(A)$, $\tau = \psi_1$, see Sec. 6) and the incomplete figural resolution that occurs in 'motion-with-replacement' (when $B \neq \psi_1(A)$) are then both manifestations of this underlying identification, in which A is recognized as B with respect to S_0 and motion is induced to resolve the separate occurrences. Although in the case of apparent movement this underlying structure is weak (i.e., there exist many bijective mappings preserving it), it is not trivial (i.e., not all bijective mappings preserve it). Thus, Kolers (1972, Chapter 7) reports that the patterns of the form shown in Figure 14 usually give rise to a flicker effect when alternately presented, and 'motion-with-replacement' is only occasionally obtained. Navon (1976) has also shown that although the letters of Figure 13 are equivalent to one another (Sec. 6), this equivalence does not extend to a larger diamond-shaped figure, for when the latter is substituted for one of the letters in the array, circular motion is seen.

8. Conclusion

Of the two models for apparent movement that have been described here, it is the pure figural one that accounts best for the available experimental data. In this model each transformation τ is associated with a vector field X for which the flow ψ^X_t, $t \in [0, 1]$, is such that $\psi^X_1 = \tau$. The curve γ describing the motion between an object A and transform $\tau(A)$ is then associated with the evolution of the time-varying vector field tX. It is significant that the motion defined by γ corresponds with the flow generated in the ordinary way by a *time-invariant* vector field, *namely* X. This does not mean, however, that the assumption of variational principles could have been replaced by the assumption of the constancy of vector fields or the stationarity of flow, since there would then be no natural way to compare stationary flows that intersect or to relate the length of flows to the distance between patterns.

Although much use has been made here of the connection between vector fields and families of transformations, it should be emphasized that there are certain technical difficulties in expressing the approach within a Lie algebra–Lie group framework (Hoffman, 1966, 1968, 1970, 1978). The group of diffeomorphisms of a manifold does not generally admit an ordinary Lie group

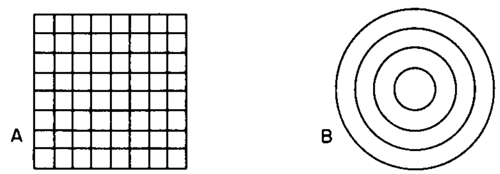

Figure 14. Displays that usually give rise to flicker instead of motion (Kolers, 1972, Chapt. 7)

structure and the exponential mapping may not cover a neighbourhood of the identity (see, for example, Ebin and Marsden, 1970).

In the pure figural model described in this study, there is no explicit mention of the time-dependence of apparent movement on interstimulus interval. For the model with scalar potential, time-dependence enters naturally in the setting-up of the vector field. For the pure figural model there are two possibilities. If the motion were generated at a constant rate, then clearly, the interstimulus interval would have to be chosen so that it is compatible with the impletion. Although consistent with the data of the 'motion-in-depth' vs. 'plastic deformation' experiments, Kolers (1972, Chapter 3) has shown that in other cases the 'velocity' of the flow increases linearly with pattern separation. An alternative explanation has been suggested by Sigman and Rock (1974). They argue that a motion rationalization scheme would not be expected to operate when the objects appeared simultaneously or when the delay between the two was so great that some stimulation in the intervening region would be expected to be detected in the real motion situation (Kaufman, et al., 1971). The dynamics are thus a consequence of the cognitive process.

If, as has been advocated here, apparent movement is intimately related to pattern perception, then, at least in the case of proper beta motion, the analysis of the forms of the paths effected may provide a method for the investigation of the mechanisms subserving visual recognition (Foster, 1972, 1973). It has been proposed (Kahneman, 1967) that apparent movement is connected with metacontrast, a shape-sensitive visual masking phenomenon (Alpern, 1952; Kolers, 1968; Kahneman, 1968). Although both phenomena can involve interactions between different photoreceptor systems (Foster and Idris, 1974; Foster, 1976) evidence relating to the space and time dependence of each (Kolers, 1972, Chapter 8) indicates that this is unlikely.

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