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Fuzzy Topological Groups

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1. INTRODUCTION

In his classic paper [1] of 1965, Zadeh introduced the notion of fuzzy sets and fuzzy set operations. Subsequently, Chang [2], Wong [3], Lowen [4] and others applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. In an analogous application with groups, Rosenfeld [5] formulated the elements of a theory of fuzzy groups. In the present paper, we bring together the structure of a fuzzy topological space and that of a fuzzy group to form a combined structure, that of a fuzzy topological group. Homomorphic images and inverse images, quotients and products of fuzzy topological groups are also briefly examined. Notation for fuzzy sets follows that of Zadeh [1].

2. PRELIMINARIES

Let X be a set and I the unit interval [0, 1]. A fuzzy set A in X is characterized by a membership function μ_A which associates with each point $x \in X$ its "grade of membership" $\mu_A(x) \in I$.

DEFINITION 2.1. Let A and B be fuzzy sets in X. Then:

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \qquad \text{for all } x \in X;$$

$$A \subset B \Leftrightarrow \mu_A(x) \leqslant \mu_B(x) \qquad \text{for all } x \in X;$$

$$C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \qquad \text{for all } x \in X;$$

$$D = A \cap B \Leftrightarrow \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\} \qquad \text{for all } x \in X.$$

More generally, for a family of fuzzy sets $\mathscr{A} = \{A_j \mid j \in J\}$, the union, $C = \bigcup_{j \in J} A_j$, and the intersection, $D = \bigcap_{j \in J} A_j$, are defined by

$$\mu_C(x) = \sup_{j \in J} \mu_{A_j}(x), \qquad x \in X,$$

$$\mu_D(x) = \inf_{j \in J} \mu_{A_j}(x), \qquad x \in X.$$

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We denote by k_c the fuzzy set in X with membership function $\mu_{k_c}(x) = c$ for all $x \in X$. The fuzzy set k_1 corresponds to the set X and the fuzzy set k_0 to the empty set $\varnothing$.

DEFINITION 2.2. Let f be a mapping from a set X to a set Y. Let B be a fuzzy set in Y, with membership function μ_B. Then the *inverse image* of B, written $f^{-1}[B]$, is the fuzzy set in X with membership function defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \qquad \text{for all } x \in X.$$

Conversely, let A be a fuzzy set in X, with membership function μ_A. Then the *image* of A, written $f[A]$, is the fuzzy set in Y with membership function defined by

$$\begin{aligned}\mu_{f[A]}(y) &= \sup_{z \in f^{-1}(y)} \mu_A(z) \qquad &&\text{if} \quad f^{-1}(y) \text{ is nonempty,} \\ &= 0 &&\text{otherwise,}\end{aligned}$$

for all $y \in Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

PROPOSITION 2.1. *Let f be a mapping from a set X to a set Y, and let $\{A_j\}$, $j \in J$, be a family of fuzzy sets in X and $\{B_j\}$, $j \in J$, a family of fuzzy sets in Y. Then:*

(i) $f^{-1}[\bigcup_{j \in J} B_j] = \bigcup_{j \in J} f^{-1}[B_j]$,

(ii) $f^{-1}[\bigcap_{j \in J} B_j] = \bigcap_{j \in J} f^{-1}[B_j]$,

(iii) $f[\bigcup_{j \in J} A_j] = \bigcup_{j \in J} f[A_j]$,

(iv) $f[\bigcap_{j \in J} A_j] \subset \bigcap_{j \in J} f[A_j]$.

Proof. (i), (ii), and (iii) follow immediately from the definitions.

(iv) The membership function of $f[\bigcap_{j \in J} A_j]$ is given by

$$\begin{aligned}\mu_{f[\bigcap_{j \in J} A_j]}(y) &= \sup_{z \in f^{-1}(y)} \inf_{j \in J} \mu_{A_j}(z) \leqslant \inf_{j \in J} \sup_{z \in f^{-1}(y)} \mu_{A_j}(z) \\ &= \mu_{\bigcap_{j \in J} f[A_j]}\end{aligned}$$

for all $y \in Y$; hence the assertion. ∎

3. FUZZY TOPOLOGICAL SPACES AND SUBSPACES

The following definition of a fuzzy topological space is due to Lowen [4].

DEFINITION 3.1. A *fuzzy topology* on a set X is a family $\mathscr{T}$ of fuzzy sets in X which satisfies the following conditions:

(i) For all $c \in I$, $k_c \in \mathscr{T}$.

(ii) If $A, B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$.

(iii) If $A_j \in \mathscr{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathscr{T}$.

The pair $(X, \mathscr{T})$ is called a *fuzzy topological space,* or fts for short, and the members of $\mathscr{T}$ are called *$\mathscr{T}$-open fuzzy sets,* or, when there is no risk of confusion, simply *open fuzzy sets.*

In the definition of a fuzzy topology by Chang [2], the condition (i) is just

(i)′ $k_0, k_1 \in \mathscr{T}$.

The necessity for the inclusion in $\mathscr{T}$ of all fuzzy sets with constant membership functions will become apparent later.

DEFINITION 3.2. Let A be a fuzzy set in X and $\mathscr{T}$ a fuzzy topology on X. Then the *induced fuzzy topology* on A is the family of fuzzy subsets of A which are the intersections with A of $\mathscr{T}$-open fuzzy sets in X. The induced fuzzy topology is denoted by $\mathscr{T}_A$, and the pair $(A, \mathscr{T}_A)$ is called a *fuzzy subspace* of $(X, \mathscr{T})$.

Note that the induced fuzzy topology does not in general satisfy condition (i) of Definition 3.1. Condition (ii), however, is satisfied, and so is condition (iii). Thus, if $U_j' \in \mathscr{T}_A$ for all $j \in J$, then there exist $U_j \in \mathscr{T}$, $j \in J$, such that $U_j' = U_j \cap A$ for each $j \in J$. The union $U' = \bigcup_{j \in J} U_j' = \bigcup_{j \in J} (U_j \cap A)$ has membership function $\mu_{U'}$, given by

$$\mu_{U'}(x) = \sup_{j \in J} \mu_{U_j'}(x) = \sup_{j \in J} \min\{\mu_{U_j}(x), \mu_A(x)\} = \min\{\sup_{j \in J} \mu_{U_j}(x), \mu_A(x)\}$$

$$= \mu_{(\bigcup_{j \in J} U_j) \cap A}(x),$$

for all $x \in X$. Hence $U' = (\bigcup_{j \in J} U_j) \cap A$ and is therefore in $\mathscr{T}_A$.

DEFINITION 3.3. Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ be two fts's. A mapping f of $(X, \mathscr{T})$ into $(Y, \mathscr{U})$ is *fuzzy continuous* iff for each open fuzzy set V in $\mathscr{U}$ the inverse image $f^{-1}[V]$ is in $\mathscr{T}$. Conversely, f is *fuzzy open* iff for each open fuzzy set U in $\mathscr{T}$, the image $f[U]$ is in $\mathscr{U}$.

If $(A, \mathscr{T}_A)$, $(B, \mathscr{U}_B)$ are fuzzy subspaces of fts's $(X, \mathscr{T})$, $(Y, \mathscr{U})$ respectively, and f is a mapping of $(X, \mathscr{T})$ into $(Y, \mathscr{U})$, then we say that f is a mapping of $(A, \mathscr{T}_A)$ *into* $(B, \mathscr{U}_B)$ if $f[A] \subset B$.

DEFINITION 3.3′. Let $(A, \mathscr{T}_A)$, $(B, \mathscr{U}_B)$ be fuzzy subspaces of fts's $(X, \mathscr{T})$, $(Y, \mathscr{U})$ respectively. Then a mapping f of $(A, \mathscr{T}_A)$ into $(B, \mathscr{U}_B)$ is *relatively fuzzy continuous* iff for each open fuzzy set V' in $\mathscr{U}_B$, the intersection $f^{-1}[V'] \cap A$ is in $\mathscr{T}_A$. Conversely, f is *relatively fuzzy open* iff for each open fuzzy set U' in $\mathscr{T}_A$, the image $f[U']$ is in $\mathscr{U}_B$.

PROPOSITION 3.1. *Let* $(A, \mathscr{T}_A)$, $(B, \mathscr{U}_B)$ *be fuzzy subspaces of* fts's $(X, \mathscr{T})$, $(Y, \mathscr{U})$ *respectively, and let* f *be a fuzzy continuous mapping of* $(X, \mathscr{T})$ *into* $(Y, \mathscr{U})$ *such that* $f[A] \subset B$. *Then* f *is a relatively fuzzy continuous mapping of* $(A, \mathscr{T}_A)$ *into* $(B, \mathscr{U}_B)$.

Proof. Let V' be open in $\mathscr{U}_B$. Then there exists $V \in \mathscr{U}$ such that $V' = V \cap B$. The inverse image $f^{-1}[V]$ is open in $\mathscr{T}$. Hence, $f^{-1}[V'] \cap A = f^{-1}[V] \cap f^{-1}[B] \cap A = f^{-1}[V] \cap A$ is open in $\mathscr{T}_A$. ▮

A bijective mapping f of a fts $(X, \mathscr{T})$ into a fts $(Y, \mathscr{U})$ is a *fuzzy homeomorphism* iff it is fuzzy continuous and fuzzy open. A bijective mapping f of a fuzzy subspace $(A, \mathscr{T}_A)$ of $(X, \mathscr{T})$ into a fuzzy subspace $(B, \mathscr{U}_B)$ of $(Y, \mathscr{U})$ is a *relative fuzzy homeomorphism* iff $f[A] = B$ and f is relatively fuzzy continuous and relatively fuzzy open.

PROPOSITION 3.2. *Let* f *be a fuzzy continuous* (resp. *fuzzy open*) *mapping of a* fts $(X, \mathscr{T})$ *into a* fts $(Y, \mathscr{U})$, *and* g *a fuzzy continuous* (resp. *fuzzy open*) *mapping of* $(Y, \mathscr{U})$ *into a* fts $(Z, \mathscr{V})$. *Then the composition* $g \circ f$ *is a fuzzy continuous* (resp. *fuzzy open*) *mapping of* $(X, \mathscr{T})$ *into* $(Z, \mathscr{V})$.

Proof. Obvious.

PROPOSITION 3.2′. *Let* $(A, \mathscr{T}_A)$, $(B, \mathscr{U}_B)$, $(C, \mathscr{V}_C)$ *be fuzzy subspaces of* fts's $(X, \mathscr{T})$, $(Y, \mathscr{U})$, $(Z, \mathscr{V})$ *respectively. Let* f *be a relatively fuzzy continuous* (resp. *relatively fuzzy open*) *mapping of* $(A, \mathscr{T}_A)$ *into* $(B, \mathscr{U}_B)$ *and* g *a relatively fuzzy continuous* (resp. *relatively fuzzy open*) *mapping of* $(B, \mathscr{U}_B)$ *into* $(C, \mathscr{V}_C)$. *Then the composition* $g \circ f$ *is a relatively fuzzy continuous* (resp. *relatively fuzzy open*) *mapping of* $(A, \mathscr{T}_A)$ *into* $(C, \mathscr{V}_C)$.

Proof. Let W' be open in $\mathscr{V}_C$. Then $g^{-1}[W'] \cap B$ is open in $\mathscr{U}_B$ and $f^{-1}[g^{-1}[W'] \cap B] \cap A$ is open in $\mathscr{T}_A$. But $(g \circ f)^{-1}[W'] \cap A = f^{-1}[g^{-1}[W'] \cap B] \cap A$, since $f[A] \subset B$, and so $g \circ f$ is relatively fuzzy continuous. The proof in the case of relatively fuzzy open mappings is trivial. ▮

DEFINITION 3.4. Let $\mathscr{T}$ be a fuzzy topology on a set X. A subfamily $\mathscr{B}$ of $\mathscr{T}$ is a *base* for $\mathscr{T}$ iff each member of $\mathscr{T}$ can be expressed as the union of members of $\mathscr{B}$.

DEFINITION 3.4′. Let $\mathscr{T}$ be a fuzzy topology on a set X and $\mathscr{T}_A$ the induced fuzzy topology on a fuzzy subset A of X. A subfamily $\mathscr{B}'$ of $\mathscr{T}_A$ is a *base* for $\mathscr{T}_A$ iff each member of $\mathscr{T}_A$ can be expressed as the union of members of $\mathscr{B}'$.

Note that if $\mathscr{B}$ is a base for a fuzzy topology $\mathscr{T}$ on a set X, then $\mathscr{B}_A = \{U \cap A \mid U \in \mathscr{B}\}$ is a base for the induced fuzzy topology $\mathscr{T}_A$ on the fuzzy subset A.

PROPOSITION 3.3. *Let f be a mapping from a* fts $(X, \mathscr{T})$ *to a* fts $(Y, \mathscr{U})$. *Let* $\mathscr{B}$ *be a base for* $\mathscr{U}$. *Then f is fuzzy continuous iff for each B in $\mathscr{B}$ the inverse image* $f^{-1}[B]$ *is in* $\mathscr{T}$.

Proof. Straightforward.

PROPOSITION 3.3′. *Let* $(A, \mathscr{T}_A)$, $(B, \mathscr{U}_B)$ *be fuzzy subspaces of* fts's $(X, \mathscr{T})$, $(Y, \mathscr{U})$ *respectively. Let* $\mathscr{B}'$ *be a base for* $\mathscr{U}_B$. *Then a mapping f of* $(A, \mathscr{T}_A)$ *into* $(B, \mathscr{U}_B)$ *is relatively fuzzy continuous iff for each B' in $\mathscr{B}'$ the intersection* $f^{-1}[B'] \cap A$ *is in* $\mathscr{T}_A$.

Proof. Straightforward.

DEFINITION 3.5. Given two fuzzy topologies $\mathscr{T}_1$, $\mathscr{T}_2$ on the same set X, we say that $\mathscr{T}_1$ is *finer* that $\mathscr{T}_2$ (and that $\mathscr{T}_2$ is *coarser* that $\mathscr{T}_1$) if the identity mapping of $(X, \mathscr{T}_1)$ into $(X, \mathscr{T}_2)$ is fuzzy continuous.

DEFINITION 3.6. Let f be a mapping of a set X into a set Y, and let $\mathscr{U}$ be a fuzzy topology on Y. The coarsest fuzzy topology $\mathscr{T}$ on X for which f is fuzzy continuous is called the *inverse image under f of* $\mathscr{U}$. The $\mathscr{T}$-open fuzzy sets in X are the inverse images of $\mathscr{U}$-open fuzzy sets in Y.

DEFINITION 3.7. Let f be a mapping of a set X into a set Y, and let $\mathscr{T}$ be a fuzzy topology on X. The finest fuzzy topology $\mathscr{U}$ on Y for which f is fuzzy continuous is called the *image under f of* $\mathscr{T}$. A fuzzy set U in Y is $\mathscr{U}$-open iff $f^{-1}[U]$ is a $\mathscr{T}$-open fuzzy set in X.

DEFINITION 3.8. Given a family $\{(X_j, \mathscr{T}_j)\}$, $j \in J$, of fts's, we define their product $\prod_{j\in J} (X_j, \mathscr{T}_j)$ to be the fts $(X, \mathscr{T})$, where $X = \prod_{j\in J} X_j$ is the usual set product and $\mathscr{T}$ is the coarsest topology on X for which the projections p_j of X onto X_j are fuzzy continuous for each $j \in J$. The fuzzy topology $\mathscr{T}$ is called the *product fuzzy topology* on X, and $(X, \mathscr{T})$ a *product* fts.

PROPOSITION 3.4. *Let* $\{(X_j, \mathscr{T}_j)\}$, $j \in J$, *be a family of* fts's *and* $(X, \mathscr{T})$ *the product* fts. *The product fuzzy topology $\mathscr{T}$ on X has as a base the set of finite intersections of fuzzy sets of the form* $p_j^{-1}[U_j]$, *where* $U_j \in \mathscr{T}_j$, $j \in J$.

Proof. See [3].

Let $\{X_j\}$, $j = 1, 2,..., n$, be a finite family of fuzzy sets and for each $j = 1, 2,..., n$, let A_j be a fuzzy set in X_j. We define the *product* $A = \prod_{j=1}^n A_j$ of the family $\{A_j\}$, $j = 1, 2,..., n$, as the fuzzy set in $X = \prod_{j=1}^n X_j$ that has membership function given by

$$\mu_A(x_1, ..., x_n) = \min\{\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\},$$

for all $(x_1, \ldots, x_n) \in X$. Notice that for each $j = 1, 2, \ldots, n$, $p_j[A] \subset A_j$, since the membership function of $p_j[A]$ is given by

$$\begin{aligned}\mu_{p_j[A]}(x_j) &= \sup_{(z_1,\ldots,z_n)\in p_j^{-1}(x_j)} \mu_A(z_1, \ldots, z_n) \\ &= \sup_{(z_1,\ldots,z_n)\in p_j^{-1}(x_j)} \min\{\mu_{A_1}(z_1), \ldots, \mu_{A_n}(z_n)\} \\ &= \min\{\sup_{z_1\in X_1} \mu_{A_1}(z_1), \ldots, \mu_{A_j}(x_j), \ldots, \sup_{z_n\in X_n} \mu_{A_n}(z_n)\} \\ &\leqslant \mu_{A_j}(x_j),\end{aligned}$$

for all $x_j \in X_j$.

It follows from the above that if X_j has fuzzy topology $\mathscr{T}_j$, $j = 1, 2, \ldots, n$, the product fuzzy topology $\mathscr{T}$ on X has as a base the set of product fuzzy sets of the form $\prod_{j=1}^{n} U_j$, where $U_j \in \mathscr{T}_j$, $j = 1, 2, \ldots, n$.

PROPOSITION 3.4′. *Let* $\{(X_j, \mathscr{T}_j)\}$, $j = 1, 2, \ldots, n$, *be a finite family of* fts's *and* $(X, \mathscr{T})$ *the product* fts. *For each* $j = 1, 2, \ldots, n$, *let* A_j *be a fuzzy set in* X_j *and* A *the product fuzzy set in* X. *Then the induced fuzzy topology* $\mathscr{T}_A$ *on* A *has as a base the set of product fuzzy sets of the form* $\prod_{j=1}^{n} U_j'$, *where* $U_j' \in (\mathscr{T}_j)_{A_j}$, $j = 1, 2, \ldots, n$.

Proof. In accordance with the preceding remarks, $\mathscr{T}$ has a base

$$\mathscr{B} = \left\{\prod_{j=1}^{n} U_j \mid U_j \in \mathscr{T}_j, j = 1, 2, \ldots, n\right\}.$$

A base for $\mathscr{T}_A$ is therefore given by

$$\mathscr{B}_A = \left\{\left(\prod_{j=1}^{n} U_j\right) \cap A \mid U_j \in \mathscr{T}_j, j = 1, 2, \ldots, n\right\}.$$

But $(\prod_{j=1}^{n} U_j) \cap A = \prod_{j=1}^{n} (U_j \cap A)$. The proposition follows with $U_j' = U_j \cap A$, $j = 1, 2, \ldots, n$. ∎

By an abuse of notation, we denote the fuzzy subspace $(A, \mathscr{T}_A)$ by $\prod_{j=1}^{n} (A_j, (\mathscr{T}_j)_{A_j})$.

PROPOSITION 3.5. *Let* $\{(X_j, \mathscr{T}_j)\}$, $j \in J$, *be a family of* fts's, $(X, \mathscr{T})$ *the product* fts, *and* f *a mapping of a* fts $(Y, \mathscr{U})$ *into* $(X, \mathscr{T})$. *Then* f *is fuzzy continuous iff* $p_j \circ f$ *is fuzzy continuous for each* $j \in J$.

Proof. See [3].

COROLLARY. *Let* $\{(X_j, \mathscr{T}_j)\}$, $\{(Y_j, \mathscr{U}_j)\}$, $j \in J$, *be two families of* fts's *and* $(X, \mathscr{T})$, $(Y, \mathscr{U})$ *the respective product* fts's. *For each* $j \in J$, *let* f_j *be a mapping of*

$(X_j, \mathscr{T}_j)$ into $(Y_j, \mathscr{U}_j)$. Then the product mapping $f = \prod_{j \in J} f_j: (x_j) \to (f_j(x_j))$ of $(X, \mathscr{T})$ into $(Y, \mathscr{U})$ is fuzzy continuous if f_j is fuzzy continuous for each $j \in J$.

Proof. The mapping f can be written as $x \to (f_j(p_j(x))$, where $x = (x_j)$, and is therefore fuzzy continuous by Proposition 3.5. ∎

PROPOSITION 3.5′. *Let $\{(X_j, \mathscr{T}_j)\}$, $j = 1, 2,..., n$, be a finite family of* fts's *and $(X, \mathscr{T})$ the product* fts. *For each $j = 1, 2,..., n$, let A_j be a fuzzy set in X_j and A the product fuzzy set in X. Let $(Y, \mathscr{U})$ be a* fts, *B a fuzzy set in Y, and f a mapping of the fuzzy subspace $(B, \mathscr{U}_B)$ into the fuzzy subspace $(A, \mathscr{T}_A)$. Then f is relatively fuzzy continuous iff $p_j \circ f$ is relatively fuzzy continuous for each $j = 1, 2,..., n$.*

Proof. $(\Rightarrow)$ By Proposition 3.1, the fuzzy continuity of p_j implies the relative fuzzy continuity of p_j for each $j = 1, 2,..., n$. The composition $p_j \circ f$ is therefore relatively fuzzy continuous for each $j = 1, 2,..., n$.

$(\Leftarrow)$ Let $U' = U_1' \times \cdots \times U_n'$, where $U_j' \in (\mathscr{T}_j)_{A_j}$, $j = 1, 2,..., n$. By Proposition 3.4′, the set of such U' forms a base for $\mathscr{T}_A$. Since

$$f^{-1}[U'] \cap B = f^{-1}[p_1^{-1}[U_1'] \cap \cdots \cap p_n^{-1}[U_n']] \cap B = \bigcap_{j=1}^{n} ((p_j \circ f)^{-1} [U_j'] \cap B)$$

is open in $\mathscr{U}_B$, as $p_j \circ f$ is relatively fuzzy continuous for each $j = 1, 2,..., n$, it follows from Proposition 3.3′ that f is relatively fuzzy continuous. ∎

COROLLARY. *Let $\{(X_j, \mathscr{T}_j)\}$, $\{(Y_j, \mathscr{U}_j)\}$, $j = 1, 2,..., n$, be two finite families of* fts's *and $(X, \mathscr{T})$, $(Y, \mathscr{U})$ the respective product* fts's. *For each $j = 1, 2,..., n$, let A_j be a fuzzy set in X_j, B_j a fuzzy set in Y_j, and f_j a mapping of the fuzzy subspace $(A_j, (\mathscr{T}_j)_{A_j})$ into the fuzzy subspace $(B_j, (\mathscr{U}_j)_{B_j})$. Let $A = \prod_{j=1}^{n} A_j$, $B = \prod_{j=1}^{n} B_j$ be the product fuzzy sets in X, Y respectively. Then the product mapping $f = \prod_{j=1}^{n} f_j: (x_1, ..., x_n) \to (f_1(x_1),..., f_n(x_n))$ of the fuzzy subspace $(A, \mathscr{T}_A)$ into the fuzzy subspace $(B, \mathscr{U}_B)$ is relatively fuzzy continuous if f_j is relatively fuzzy continuous for each $j = 1, 2,..., n$.*

Proof. Analogous to the proof of the Corollary to Proposition 3.5.

PROPOSITION 3.6. *Let $\{(X_j, \mathscr{T}_j)\}$, $\{(Y_j, \mathscr{U}_j)\}$, $j = 1, 2,..., n$, be two finite families of* fts's *and $(X, \mathscr{T})$, $(Y, \mathscr{U})$ the respective product* fts's. *For each $j = 1, 2,..., n$, let f_j be a mapping of $(X_j, \mathscr{T}_j)$ into $(Y_j, \mathscr{U}_j)$. Then the product mapping $f = \prod_{j=1}^{n} f_j: (x_1, ..., x_n) \to (f_1(x_1),..., f_n(x_n))$ of $(X, \mathscr{T})$ into $(Y, \mathscr{U})$ is fuzzy open if f_j is fuzzy open for each $j = 1, 2,..., n$.*

Proof. Let U be open in $\mathscr{T}$. Then there exist open fuzzy sets $U_{jm} \in \mathscr{T}_j$, $m \in M$, $j = 1, 2,..., n$, such that

$$U = \bigcup_{m \in M} \prod_{j=1}^{n} U_{jm}.$$

The image $f[U]$ of U has membership function $\mu_{f[U]}$, where, for all $y \in Y$,

$$\mu_{f[U]}(y) = \mu_{\bigcup_{m\in M} f[\prod_{j=1}^n U_{jm}]}(y) = \sup_{m\in M} \sup_{z\in f^{-1}(y)} \mu_{\prod_{j=1}^n U_{jm}}(z)$$

$$= \sup_{m\in M} \sup_{z_1\in f_1^{-1}(y_1)} \cdots \sup_{z_n\in f_n^{-1}(y_n)} \min\{\mu_{U_{1m}}(z_1),\ldots, \mu_{U_{nm}}(z_n)\}$$

$$= \sup_{m\in M} \min\{ \sup_{z_1\in f_1^{-1}(y_1)} \mu_{U_{1m}}(z_1),\ldots, \sup_{z_n\in f_n^{-1}(y_n)} \mu_{U_{nm}}(z_n)\}$$

$$= \sup_{m\in M} \min\{\mu_{f_1[U_{1m}]}(y_1),\ldots, \mu_{f_n[U_{nm}]}(y_n)\}$$

$$= \mu_{\bigcup_{m\in M}\prod_{j=1}^n (f_j[U_{jm}])}(y).$$

Thus $f[U] = \bigcup_{m\in M} \prod_{j=1}^n (f_j[U_{jm}])$. Since f_j is fuzzy open for each $j = 1, 2,\ldots, n$, $f[U]$ is open in $\mathscr{U}$. ▮

PROPOSITION 3.6′. *Let* $\{(X_j, \mathscr{T}_j)\}$, $\{(Y_j, \mathscr{U}_j)\}$, $j = 1, 2,\ldots, n$, *be two finite families of* fts's *and* $(X, \mathscr{T})$, $(Y, \mathscr{U})$ *the respective product* fts's. *For each* $j = 1, 2,\ldots, n$, *let* A_j *be a fuzzy set in* X_j, B_j *a fuzzy set in* Y_j, *and* f_j *a mapping of the fuzzy subspace* $(A_j, (\mathscr{T}_j)_{A_j})$ *into the fuzzy subspace* $(B_j, (\mathscr{U}_j)_{B_j})$. *Let* $A = \prod_{j=1}^n A_j$, $B = \prod_{j=1}^n B_j$ *be the product fuzzy sets in* X, Y *respectively. Then the product mapping* $f = \prod_{j=1}^n f_j$: $(x_1,\ldots, x_n) \to (f_1(x_1),\ldots, f_n(x_n))$ *of the fuzzy subspace* $(A, \mathscr{T}_A)$ *into the fuzzy subspace* $(B, \mathscr{U}_B)$ *is relatively fuzzy open if* f_j *is relatively fuzzy open for each* $j = 1, 2,\ldots, n$.

Proof. Let U' be open in $\mathscr{T}_A$. By Proposition 3.4′, there exist open fuzzy sets $U'_{jm} \in (\mathscr{T}_j)_{A_j}$, $m \in M$, $j = 1, 2,\ldots, n$, such that

$$U' = \bigcup_{m\in M} \prod_{j=1}^n U'_{jm}.$$

As in the proof of Proposition 3.6, it follows that

$$f[U'] = \bigcup_{m\in M} \prod_{j=1}^n (f_j[U'_{jm}]).$$

Since f_j is relatively fuzzy open for each $j = 1, 2,\ldots, n$, $f[U']$ is open in $\mathscr{U}_B$. ▮

PROPOSITION 3.7. *Let* $(X_1, \mathscr{T}_1)$, $(X_2, \mathscr{T}_2)$ *be* fts's *and* $(X, \mathscr{T})$ *the product* fts. *Then for each* $a_1 \in X_1$, *the mapping* i: $x_2 \to (a_1, x_2)$ *of* $(X_2, \mathscr{T}_2)$ *into* $(X, \mathscr{T})$ *is fuzzy continuous.*

Proof. The constant mapping i_1: $x_2 \to a_1$ of $(X_2, \mathscr{T}_2)$ into $(X_1, \mathscr{T}_1)$ is fuzzy continuous, for, if U_1 is open in $\mathscr{T}_1$, the inverse image $i_1^{-1}[U_1]$ has membership function given by

$$\mu_{i_1^{-1}[U_1]}(x_2) = \mu_{U_1}(a_1) = \mu_{k_c}(x_2)$$

for all $x_2 \in X_2$, where k_c is the open fuzzy set in X_2 which has constant membership function with value $c = \mu_{U_1}(a_1)$. Since the identity mapping $i_2 \colon x_2 \to x_2$ of $(X_2, \mathscr{T}_2)$ into itself is fuzzy continuous, the mapping i is fuzzy continuous by Proposition 3.5. ∎

PROPOSITION 3.7′. *Let* $(X_1, \mathscr{T}_1)$, $(X_2, \mathscr{T}_2)$ *be* fts's *and* $(X, \mathscr{T})$ *the product* fts. Let A_1, A_2 *be fuzzy sets in* X_1, X_2 *respectively, and* A *the product fuzzy set in* X. *Then for each* $a_1 \in X_1$ *such that* $\mu_{A_1}(a_1) \geqslant \mu_{A_2}(x_2)$ *for all* $x_2 \in X_2$, *the mapping* $i \colon x_2 \to (a_1, x_2)$ *of the fuzzy subspace* $(A_2, (\mathscr{T}_2)_{A_2})$ *into the fuzzy subspace* $(A, \mathscr{T}_A)$ *is relatively fuzzy continuous.*

Proof. We see that $i[A_2] \subset A$, since the membership function of $i[A_2]$ is given by

$$\mu_{i[A_2]}(x_1, x_2) = \mu_{A_2}(x_2) \quad \text{if} \quad x_1 = a_1,$$
$$= 0 \quad \text{otherwise},$$

and that of A by

$$\mu_A(x_1, x_2) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2)\}$$
$$\geqslant \mu_{A_2}(x_2)$$

for all $(x_1, x_2) \in X$. The proof of the relative fuzzy continuity of i is analogous to the proof of the fuzzy continuity of i in Proposition 3.7. ∎

4. FUZZY GROUPS

DEFINITION 4.1. Let X be a group and G a fuzzy set in X with membership function μ_G. Then G is a *fuzzy group* in X iff the following conditions are satisfied:

(i) $\mu_G(xy) \geqslant \min\{\mu_G(x), \mu_G(y)\}$, for all $x, y \in X$;

(ii) $\mu_G(x^{-1}) \geqslant \mu_G(x)$, for all $x \in X$.

Note that Rosenfeld [5] refers to G as a *fuzzy subgroup*.

PROPOSITION 4.1. *G is a fuzzy group in X iff*

$$\mu_G(xy^{-1}) \geqslant \min\{\mu_G(x), \mu_G(y)\}, \qquad \text{for all } x, y \in X.$$

Proof. See [5].

PROPOSITION 4.2. *Let X, Y be groups and f a homomorphism of X into Y. Let G be a fuzzy group in Y. Then the inverse image $f^{-1}[G]$ of G is a fuzzy group in X.*

Proof. For all $x, y \in X$,

$$\begin{aligned}\mu_{f^{-1}[G]}(xy^{-1}) &= \mu_G(f(xy^{-1})) = \mu_G(f(x)\,(f(y))^{-1}) \\ &\geqslant \min\{\mu_G(f(x)), \mu_G(f(y))\} \\ &= \min\{\mu_{f^{-1}[G]}(x), \mu_{f^{-1}[G]}(y)\}. \quad \blacksquare\end{aligned}$$

For images, we need the following property [5]. A fuzzy set A in X is said to have the *sup property* if, for any subset $T \subset X$, there exists $t_0 \in T$ such that $\mu_A(t_0) = \sup_{t\in T} \mu_A(t)$.

PROPOSITION 4.3. *Let X, Y be groups and f a homomorphism of X into Y. Let G be a fuzzy group in X that has the sup property. Then the image $f[G]$ of G is a fuzzy group in Y.*

Proof. Let $u, v \in Y$. If either $f^{-1}(u)$ or $f^{-1}(v)$ is empty, then the inequality in Proposition 4.1 is trivially satisfied. Suppose neither $f^{-1}(u)$ nor $f^{-1}(v)$ is empty. Ler $r_0 \in f^{-1}(u)$, $s_0 \in f^{-1}(v)$ be such that

$$\mu_G(r_0) = \sup_{t\in f^{-1}(u)} \mu_G(t), \qquad \mu_G(s_0) = \sup_{t\in f^{-1}(v)} \mu_G(t).$$

Then,

$$\begin{aligned}\mu_{f[G]}(uv^{-1}) &= \sup_{w\in f^{-1}(uv^{-1})} \mu_G(w) \geqslant \min\{\mu_G(r_0), \mu_G(s_0)\} \\ &= \min\{\mu_{f[G]}(u), \mu_{f[G]}(v)\}. \quad \blacksquare\end{aligned}$$

We say that the membership function μ_G of a fuzzy group G in a group X is *f-invariant* [5] if, for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $\mu_G(x_1) = \mu_G(x_2)$. Clearly, a homomorphic image $f[G]$ of G is then a fuzzy group.

PROPOSITION 4.4. *If G is a fuzzy group in a group X, then $\mu_G(x^{-1}) = \mu_G(x)$ and $\mu_G(e) \geqslant \mu_G(x)$ for all $x \in G$, where e is the identity element of G.*

Proof. See [5].

Given a fuzzy group G in a group X, let G_e denote the set $\{x \mid \mu_G(x) = \mu_G(e)\}$. It follows that G_e is a subgroup of X. For $a \in X$, let $\rho_a\colon x \to xa$ and $\lambda_a\colon x \to ax$ denote, respectively, the right and left translations of X into itself.

PROPOSITION 4.5. *Let G be a fuzzy group in a group X. Then, for all $a \in G_e$, $\rho_a[G] = \lambda_a[G] = G$.*

Proof. (Compare [5]). Let $a \in G_e$. Then the membership function of $\rho_a[G]$ is given by

$$\begin{aligned}\mu_{\rho_a[G]}(x) &= \mu_G(xa^{-1}) \geqslant \min\{\mu_G(x), \mu_G(e)\} = \mu_G(x) \\ &= \mu_G(xa^{-1}a) \geqslant \min\{\mu_G(xa^{-1}), \mu_G(e)\} = \mu_G(xa^{-1}) = \mu_{\rho_a[G]}(x),\end{aligned}$$

for all $x \in X$. The proof for λ_a is similar. ∎

5. Fuzzy Topological Groups

Suppose G is a fuzzy group in a group X. Let α denote the mapping $(x, y) \to xy$ $X \times X$ into X, and β the mapping $x \to x^{-1}$ of X into itself. The image $\alpha[G \times G]$ of the product fuzzy set $G \times G$ has membership function $\mu_{\alpha[G\times G]}$, where

$$\begin{aligned}\mu_{\alpha[G\times G]}(x) &= \sup_{(z_1, z_2)\in\alpha^{-1}(x)} \mu_{G\times G}(z_1, z_2) \\ &= \sup_{(z_1, z_2)\in\alpha^{-1}(x)} \min\{\mu_G(z_1), \mu_G(z_2)\} \\ &\leqslant \sup_{(z_1, z_2)\in\alpha^{-1}(x)} \mu_G(z_1 z_2) = \mu_G(x),\end{aligned}$$

for all $x \in X$. Hence $\alpha[G \times G] \subset G$. By Proposition 4.4, $\mu_G(x) = \mu_G(x^{-1})$, for all $x \in X$. Hence $\beta[G] \subset G$. Next, note that if X is given a fuzzy topology $\mathscr{T}$, then G acquires an induced fuzzy topology $\mathscr{T}_G$. By definition, $(G, \mathscr{T}_G)$ is a fuzzy subspace of the fts $(X, \mathscr{T})$ and $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ a fuzzy subspace of the product fts $(X, \mathscr{T}) \times (X, \mathscr{T})$.

Definition 5.1. Let X be a group and $\mathscr{T}$ a fuzzy topology on X. Let G be a fuzzy group in X and let G be endowed with the induced fuzzy topology $\mathscr{T}_G$. Then G is a *fuzzy topological group* in X iff it satisfies the following two conditions:

(i) The mapping $\alpha: (x, y) \to xy$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous.

(ii) The mapping $\beta: x \to x^{-1}$ of $(G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous.

A fuzzy group structure and an induced fuzzy topology are said to be *compatible* if they satisfy (i) and (ii).

Proposition 5.1. *Let X be a group having fuzzy topology $\mathscr{T}$. A fuzzy group G in X is a fuzzy topological group iff the mapping $\gamma: (x, y) \to xy^{-1}$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous.*

Proof. ($\Rightarrow$) The mapping $(x, y) \to (x, y^{-1})$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into itself is relatively fuzzy continuous by the Corollary to Proposition 3.5′. Hence, the composition $(x, y) \to (x, y^{-1}) \to xy^{-1}$ is relatively fuzzy continuous.

($\Leftarrow$) By Proposition 4.4, $\mu_G(e) \geqslant \mu_G(x)$, for all $x \in X$, and therefore by Proposition 3.7′ the canonical injection $i: y \to (e, y)$ of $(G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ is relatively fuzzy continuous. Hence the composition $\beta: y \to (e, y) \to ey^{-1}$ is relatively fuzzy continuous. The mapping $\alpha: (x, y) \to xy$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous since it is the composition $(x, y) \to (x, y^{-1}) \to x(y^{-1})^{-1}$ of relatively fuzzy continuous mappings. ∎

If G is a fuzzy topological group in a group X carrying fuzzy topology $\mathscr{T}$, then, in general, the translations ρ_a, λ_a, $a \in X$, are not relatively fuzzy continuous mappings of $(G, \mathscr{T}_G)$ into itself. We do, however, have the following special case. Recall that $G_e = \{x \mid \mu_G(x) = \mu_G(e)\}$.

Proposition 5.2. *Let X be a group having fuzzy topology $\mathscr{T}$, and let G be a fuzzy topological group in X. For each $a \in G_e$, the translations ρ_a, λ_a are relative fuzzy homeomorphisms of $(G, \mathscr{T}_G)$ into itself.*

Proof. From Proposition 4.5, we note that $\rho_a[G] = G$ and $\lambda_a[G] = G$, for all $a \in G_e$. The mapping λ_a is the composition of the injection $i: y \to (a, y)$ and the mapping $\alpha: (x, y) \to xy$. Since $\mu_G(a) \geqslant \mu_G(y)$ for all $y \in Y$, it follows from Proposition 3.7′ that i is a relatively fuzzy continuous mapping of $(G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$. The mapping α is relatively fuzzy continuous by hypothesis. Hence, λ_a is relatively fuzzy continuous, and therefore $\lambda_a^{-1} = \lambda_{a^{-1}}$ also. The relative fuzzy continuity of ρ_a and ρ_a^{-1} is shown similarly. ∎

6. Homomorphisms

Suppose that X and Y are groups and that f is a homomorphism of X into Y. Let Y have fuzzy topology $\mathscr{U}$ and let G be a fuzzy topological group in Y. The mapping f gives rise to a fuzzy topology $\mathscr{T}$ on X, the *inverse image* under f of $\mathscr{U}$, and, by Proposition 4.2, it also gives rise to a fuzzy group in X, the inverse image $f^{-1}[G]$ of G. The following proposition shows that the induced fuzzy topology on $f^{-1}[G]$ and the fuzzy group structure are compatible.

Proposition 6.1. *Given groups X, Y, a homomorphism f of X into Y, and a fuzzy topology $\mathscr{U}$ on Y, let X have fuzzy topology $\mathscr{T}$, where $\mathscr{T}$ is the inverse image under f of $\mathscr{U}$, and let G be a fuzzy topological group in Y. Then the inverse image $f^{-1}[G]$ of G is a fuzzy topological group in X.*

Proof. We have to show that the mapping $\gamma_X\colon (x_1, x_2) \to x_1x_2^{-1}$ of $(f^{-1}[G], \mathscr{T}_{f^{-1}[G]}) \times (f^{-1}[G], \mathscr{T}_{f^{-1}[G]})$ into $(f^{-1}[G], \mathscr{T}_{f^{-1}[G]})$ is relatively fuzzy continuous. Let U' be an open fuzzy set in the induced fuzzy topology $\mathscr{T}_{f^{-1}[G]}$ on $f^{-1}[G]$. Note that since f is a fuzzy continuous mapping of $(X, \mathscr{T})$ into $(Y, \mathscr{U})$ it is, by Proposition 3.1, a relatively fuzzy continuous mapping of $(f^{-1}[G], \mathscr{T}_{f^{-1}[G]})$ into $(G, \mathscr{U}_G)$. Note also that there exists an open fuzzy set V' in $\mathscr{U}_G$ such that $f^{-1}[V'] = U'$. The membership function of $\gamma_X^{-1}[U']$ is given by:

$$\mu_{\gamma_X^{-1}[U']}(x_1, x_2) = \mu_{U'}(x_1x_2^{-1}) = \mu_{f^{-1}[V']}(x_1x_2^{-1}) = \mu_{V'}(f(x_1)\,(f(x_2))^{-1}),$$

for all $(x_1, x_2) \in X \times X$. By hypothesis, the mapping $\gamma_Y\colon (y_1, y_2) \to y_1y_2^{-1}$ of $(G, \mathscr{U}_G) \times (G, \mathscr{U}_G)$ into $(G, \mathscr{U}_G)$ is relatively fuzzy continuous, and, by the Corollary to Proposition 3.5′, so is the product mapping $f \times f$ of $(f^{-1}[G], \mathscr{T}_{f^{-1}[G]}) \times (f^{-1}[G], \mathscr{T}_{f^{-1}[G]})$ into $(G, \mathscr{U}_G) \times (G, \mathscr{U}_G)$. But,

$$\mu_{V'}(f(x_1)\,(f(x_2))^{-1}) = \mu_{\gamma_Y^{-1}[V']}(f(x_1), f(x_2)) = \mu_{(f\times f)^{-1}[\gamma_Y^{-1}[V']]}(x_1, x_2),$$

for all $(x_1, x_2) \in X \times X$. Hence, $\gamma_X^{-1}[U'] \cap (f^{-1}[G] \times f^{-1}[G]) = (f \times f)^{-1}[\gamma_Y^{-1}[V']] \cap (f^{-1}[G] \times f^{-1}[G])$ is open in the induced fuzzy topology on $f^{-1}[G] \times f^{-1}[G]$. ∎

The next proposition shows that for some homomorphic images a similar situation holds.

PROPOSITION 6.2. *Given groups X, Y, a homomorphism f of X into Y, and a fuzzy topology $\mathscr{T}$ on X, let Y have fuzzy topology $\mathscr{U}$, where $\mathscr{U}$ is the image under f of $\mathscr{T}$, and let G be a fuzzy topological group in X. If the membership function μ_G of G is f-invariant, then the image $f[G]$ of G is a fuzzy topological group in Y.*

Proof. In accordance with the remark following Proposition 4.3, $f[G]$ is a fuzzy group. We have to show that the mapping $\gamma_Y\colon (y_1, y_2) \to y_1y_2^{-1}$ of $(f[G], \mathscr{U}_{f[G]}) \times (f[G], \mathscr{U}_{f[G]})$ into $(f[G], \mathscr{U}_{f[G]})$ is relatively fuzzy continuous. Note that f is fuzzy open, for if $U \in \mathscr{T}$, then $f[U] \in \mathscr{U}$, since the inverse image $f^{-1}[f[U]]$ is the union of open fuzzy sets (Prop. 5.2) and thus open in $\mathscr{T}$. It follows that f is relatively fuzzy open, since if $U' \in \mathscr{T}_G$, there exists $U \in \mathscr{T}$ such that $U' = U \cap G$ and, by the f-invariance of μ_G, $f[U'] = f[U] \cap f[G] \in \mathscr{U}_{f[G]}$. By Proposition 3.6′, the product mapping $f \times f$ is a relatively fuzzy open mapping of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(f[G], \mathscr{U}_{f[G]}) \times (f[G], \mathscr{U}_{f[G]})$.

Let V' be an open fuzzy set in $\mathscr{U}_{f[G]}$. The membership function of $(f \times f)^{-1}[\gamma_Y^{-1}[V']]$ is given by

$$\mu_{(f\times f)^{-1}[\gamma_Y^{-1}[V']]}(x_1, x_2) = \mu_{V'}(f(x_1)\,(f(x_2))^{-1}) = \mu_{(\gamma_X^{-1}\circ f^{-1})[V']}(x_1, x_2),$$

for all $(x_1, x_2) \in X \times X$, where $\gamma_X\colon (x_1, x_2) \to x_1x_2^{-1}$. But, by hypothesis, γ_X is a relatively fuzzy continuous mapping of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$, and f

is a relatively fuzzy continuous mapping of $(G, \mathscr{T}_G)$ into $(f[G], \mathscr{U}_{f[G]})$. Hence, by the f-invariance of μ_G,

$$(f \times f)^{-1} [\gamma_Y^{-1}[V'] \cap (f[G] \times f[G])] = (f \times f)^{-1} [\gamma_Y^{-1}[V']] \cap (G \times G)$$

is open in the induced fuzzy topology on $G \times G$. As $f \times f$ is relatively fuzzy open,

$$(f \times f)\,(f \times f)^{-1} [\gamma_Y^{-1}[V'] \cap (f[G] \times f[G])] = \gamma_Y^{-1}[V'] \cap (f[G] \times f[G])$$

is open in the induced fuzzy topology on $f[G] \times f[G]$. ∎

7. Quotients and Products

Given a group X carrying a fuzzy topology $\mathscr{T}$, and G a fuzzy topological group in X, let N be a normal subgroup of X, and let ϕ be the canonical homomorphism of X onto the quotient group X/N. If the membership function μ_G of G is constant on N, then μ_G is ϕ-invariant, and the image $\phi[G]$ is accordingly a fuzzy group in X/N. We call $\phi[G]$ a *quotient fuzzy group* and denote it by G/N.

Proposition 7.1. *Let X be a group having fuzzy topology $\mathscr{T}$, G a fuzzy topological group in X and N a normal subgroup of X. Let the quotient group X/N be given the fuzzy topology which is the image of $\mathscr{T}$ under the canonical homomorphism ϕ. Then, if the membership function μ_G of G is constant on N, the quotient fuzzy group G/N is a fuzzy topological group in X/N.*

Proof. Apply Proposition 6.2. ∎

We refer to the above fuzzy topology on the quotient group X/N as the *quotient fuzzy topology* and to G/N as a *quotient fuzzy topological group.*

Proposition 7.2. *Let X, Y be groups, and f a homomorphism of X onto Y. Let $\mathscr{T}$ be a fuzzy topology on X, $\mathscr{U}$ a fuzzy topology on Y, and f both fuzzy continuous and fuzzy open. Let G be a fuzzy topological group in X such that its membership function μ_G is constant on the kernel $f^{-1}(e)$ of f. Let the quotient group $X/f^{-1}(e)$ have the quotient fuzzy topology. Then:*

(i) *The fuzzy groups $G/f^{-1}(e)$ and $f[G]$ are fuzzy topological groups in $X/f^{-1}(e)$ and Y respectively.*

(ii) *The canonical isomorphism $\bar{f}$ of $X/f^{-1}(e)$ onto Y is a relative fuzzy homeomorphism of $G/f^{-1}(e)$ onto $f[G]$.*

Proof. (i) That $G/f^{-1}(e)$ is a fuzzy topological group in $X/f^{-1}(e)$ follows immediately from Proposition 7.1. Since f is both fuzzy continuous and fuzzy open, the image under f of $\mathscr{T}$ coincides with $\mathscr{U}$. For, if V is a fuzzy set in Y such that $f^{-1}[V]$ is open in $\mathscr{T}$, $f[f^{-1}[V]] = V$ is open in $\mathscr{U}$, and, conversely, if V is open in $\mathscr{U}$, then $f^{-1}[V]$ is open in $\mathscr{T}$. By Proposition 6.2, the image $f[G]$ is therefore a fuzzy topological group in Y.

(ii) Let V' be an open fuzzy set in the induced fuzzy topology $\mathscr{U}_{f[G]}$ on $f[G]$, and let ϕ be the canonical homomorphism of X onto $X/f^{-1}(e)$. Then $f^{-1}[V'] = \phi^{-1}[\hat{f}^{-1}[V']]$ is open in $\mathscr{T}_G$, since f is relatively fuzzy continuous, and therefore $\hat{f}^{-1}[V']$ is open in the induced fuzzy topology on $G/f^{-1}(e)$, since ϕ is relatively fuzzy open. Conversely, if U' is an open fuzzy set in the induced fuzzy topology on $G/f^{-1}(e)$, then $\phi^{-1}[U'] = f^{-1}[\hat{f}[U']]$ is open in the induced fuzzy topology on G, which implies that $\hat{f}[U']$ is open in $\mathscr{U}_{f[G]}$ since f is relatively fuzzy open. ▮

We now briefly discuss products of fuzzy topological groups. Let $\{X_j\}$, $j = 1, 2,..., n$, be a finite family of groups, and X the product group. For each $j = 1, 2,..., n$, let X_j have fuzzy topology $\mathscr{T}_j$, and let G_j be a fuzzy topological group in X_j. The product fuzzy set $G = \prod_{j=1}^{n} G_j$ in X has membership function μ_G given by

$$\mu_G(x) = \min\{\mu_{G_1}(x_1),..., \mu_{G_n}(x_n)\},$$

where $x = (x_1 ,..., x_n)$. It follows that G is a fuzzy group in X, since, for all $x, y \in X$,

$$\begin{aligned} \mu_G(xy^{-1}) &= \mu_G(x_1 y_1^{-1},..., x_n y_n^{-1}) = \min\{\mu_{G_1}(x_1 y_1^{-1}),..., \mu_{G_n}(x_n y_n^{-1})\} \\ &\geqslant \min\{\min\{\mu_{G_1}(x_1), \mu_{G_1}(y_1)\},..., \min\{\mu_{G_n}(x_n), \mu_{G_n}(y_n)\}\} \\ &= \min\{\min\{\mu_{G_1}(x_1),..., \mu_{G_n}(x_n)\}, \min\{\mu_{G_1}(y_1),..., \mu_{G_n}(y_n)\}\} \\ &= \min\{\mu_G(x), \mu_G(y)\}. \end{aligned}$$

We call G the *product* of the fuzzy groups G_j, $j = 1, 2,..., n$.

The product group X has associated with it the product fuzzy topology. The next proposition shows that the induced fuzzy topology on G and the product fuzzy group structure are compatible.

PROPOSITION 7.3. *Let $\{X_j\}$, $j = 1, 2,..., n$, be a finite family of groups and, for each $j = 1, 2,..., n$, let $\mathscr{T}_j$ be a fuzzy topology on X_j, and G_j a fuzzy topological group in X_j. Let the product group $X = \prod_{j=1}^{n} X_j$ have the product fuzzy topology $\mathscr{T}$. Then the product fuzzy group $G = \prod_{j=1}^{n} G_j$ is a fuzzy topological group in X.*

Proof. The mapping $\gamma: (x, y) \to xy^{-1}$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ may be written as the composition of $\gamma_1: (x, y) = ((x_1 ,..., x_n), (y_1 ,..., y_n)) \to ((x_1 , y_1),..., (x_n , y_n))$ and $\gamma_2: ((x_1 , y_1),..., (x_n , y_n)) \to (x_1 y_1^{-1} ,..., x_n y_n^{-1})$. By Propositions 3.5 and 3.1, γ_1 is relatively fuzzy continuous, and, by the Corollary to Proposition 3.5′, γ_2 is relatively fuzzy continuous; hence γ is relatively fuzzy continuous. ▮

We refer to $G = \prod_{j=1}^{n} G_j$ as a *product fuzzy topological group*. The results of Propositions 7.1 and 7.3 may be combined to yield the following.

PROPOSITION 7.4. *Let $\{X_j\}$, $j = 1, 2,..., n$, be a finite family of groups and, for each $j = 1, 2,..., n$, let $\mathscr{T}_j$ be a fuzzy topology on X_j, N_j a normal subgroup of X_j, and G_j a fuzzy topological group in X_j such that its membership function μ_{G_j} is constant on N_j. Let the quotient groups X/N, where $N = \prod_{j=1}^{n} N_j$, and X_j/N_j, $j = 1, 2,..., n$, have the respective quotient fuzzy topologies, and the product groups $X = \prod_{j=1}^{n} X_j$ and $\prod_{j=1}^{n} (X_j/N_j)$ the respective product fuzzy topologies. Let $G = \prod_{j=1}^{n} G_j$ be the product fuzzy topological group in X. Then the canonical isomorphism ι of X/N onto $\prod_{j=1}^{n} (X_j/N_j)$ is a relative fuzzy homeomorphism of the quotient fuzzy topological group G/N onto the product fuzzy topological group $\prod_{j=1}^{n} (G_j/N_j)$.*

Proof. Let ϕ be the canonical homomorphism $x \to [x]$ of X onto X/N, and, for each $j = 1, 2,..., n$, let ϕ_j be the canonical homomorphism $x_j \to [x_j]$ of X_j onto X_j/N_j. Let $\prod_{j=1}^{n} \phi_j$ be the product mapping of X onto $\prod_{j=1}^{n} (X_j/N_j)$. Thus, $\iota \circ \phi = \prod_{j=1}^{n} \phi_j$. Note that for each $[x] \in X/N$,

$$\begin{aligned}\mu_{G/N}([x]) &= \mu_G(x) = \mu_{\prod_{j=1}^{n} G_j}(x_1, ..., x_n) \\ &= \min\{\mu_{G_1}(x_1), ..., \mu_{G_n}(x_n)\} \\ &= \min\{\mu_{G_1/N_1}([x_1]), ..., \mu_{G_n/N_n}([x_n])\} \\ &= \mu_{\prod_{j=1}^{n}(G_j/N_j)}(\iota([x])).\end{aligned}$$

By Propositions 7.1 and 7.3, G/N and $\prod_{j=1}^{n} (G_j/N_j)$ are fuzzy topological groups. Let V' be open in the induced fuzzy topology on $\prod_{j=1}^{n} (G_j/N_j)$. Then $\phi^{-1} \circ \iota^{-1}[V'] = (\prod_{j=1}^{n} \phi_j)^{-1} [V']$ is open in the induced fuzzy topology on G, since, by Propositions 3.5 and 3.1, $\prod_{j=1}^{n} \phi_j$ is relatively fuzzy continuous. Thus, $\iota^{-1}[V']$ is open in the induced fuzzy topology on G/N, since ϕ is relatively fuzzy open. Hence ι is relatively fuzzy continuous. Conversely, let U' be open in the induced fuzzy topology on G/N. Then $\phi^{-1}[U']$ is open in the induced fuzzy topology on G, and therefore $(\prod_{j=1}^{n} \phi_j) (\phi^{-1}[U']) = \iota[U']$ is open in the induced fuzzy topology on $\prod_{j=1}^{n} (G_j/N_j)$, since $\prod_{j=1}^{n} \phi_j$ is the product of relatively fuzzy open mappings and is therefore relatively fuzzy open by Proposition 3.6′. Hence ι is relatively fuzzy open. ∎

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