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C^1 fuzzy manifolds

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Abstract: This paper introduces the notion of a C^1 fuzzy manifold as a natural development of the notions of a fuzzy topological vector space and of a fuzzy derivative of a fuzzy continuous mapping between fuzzy topological vector spaces. First, a fuzzy atlas of class C^1 on a set is constructed and shown to yield a fuzzy topology that is compatible with the fuzzy atlas. The structure of a C^1 fuzzy manifold on the set then follows. Next, it is shown that the product of two fuzzy manifolds is a fuzzy manifold, and that the composition of two fuzzy differentiable mappings between fuzzy manifolds is fuzzy differentiable. Finally, the notions of a tangent vector and of a tangent space at a point in a fuzzy manifold are formulated, and the tangent space is shown to be a vector space.

Keywords: Fuzzy topology; fuzzy differentiation; fuzzy manifold; tangent vector space.

1. Introduction

The notion of a differentiable manifold, that is, a set locally like Euclidean space, arose historically in many different mathematical disciplines, including the study of local differential geometry, projective geometry, algebraic geometry, and continuous or Lie transformation groups [1]. In its present formulation, it has led to the reformulation and generalization of these classical disciplines, and it has been essential to the modern development of applications such as the theory of dynamical systems. Although there are established theories of relevant fuzzy structures, including fuzzy topological spaces [2, 7, 9, 10], fuzzy topological vector spaces [5], and fuzzy derivatives [3], there have been few attempts at formulating a satisfactory structure for fuzzy differentiable manifolds. The purpose of this study is to bring together the notions of a fuzzy topological space and fuzzy differentiation between fuzzy topological vector spaces to form a general notion of a fuzzy differentiable manifold.

The principle of the approach is to take the definition [3] of a fuzzy derivative of a fuzzy continuous mapping between fuzzy topological vector spaces (ftvs's), and use it to endow a ftvs with a fuzzy differential structure. This differential structure is then extended to sets that are locally homeomorphic to ftvs's, thereby establishing the notion of a fuzzy manifold.

First, we propose a definition of a fuzzy atlas of class C^1 on a set and show that it is possible to define a fuzzy topology on the set from the ftvs's associated with the fuzzy charts. Then we define C^1 fuzzy manifolds and show that if X and Y are C^1 fuzzy manifolds then their product $X \times Y$ is a C^1 fuzzy manifold, and that the composition $g \circ f$ of two fuzzy differentiable mappings g, f is fuzzy differentiable. The notion of C^1 fuzzy diffeomorphisms between C^1 fuzzy manifolds follows naturally. Last, we define a tangent vector and a tangent space at a point in a C^1 fuzzy manifold, and show that this tangent space has the structure of a vector space.

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2. Preliminaries

For the sake of completeness, we briefly review here definitions and some properties of fuzzy topological spaces, fuzzy topological vector spaces, and fuzzy differentiation, and include some new technical results concerning bases for fuzzy topologies and products of fuzzy differentiable mappings between fuzzy topological vector spaces.

Definitions and notation for fuzzy sets follow Zadeh [11], and those for fuzzy points and neighbourhoods follow Pu and Liu [9]. Thus if x_c, $0<c\leq 1$, is a fuzzy point in X with membership function $\mu_{x_c}(y)$, $y\in X$, then $\mu_{x_c}(y)=c$ if $y=x$ and $\mu_{x_c}(y)=0$ otherwise. We denote by k_c the fuzzy set in X with the constant membership function $\mu_{k_c}(x)=c$ for all $x\in X$. The following definition of a fuzzy topological space is due to Chang [2].

2.1. Definition. A *fuzzy topology* on a set X is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions.

(i) $k_0, k_1\in\mathcal{T}$.

(ii) If $A, B\in\mathcal{T}$, then $A\cap B\in\mathcal{T}$.

(iii) If $A_j\in\mathcal{T}$ for all $j\in J$ (J some index set), then $\bigcup_{j\in J}A_j\in\mathcal{T}$.

In the definition of a fuzzy topology due to Lowen [7], the condition (i) is

(i′) For all $c\in I$ (I the unit interval), $k_c\in\mathcal{T}$.

If a fuzzy topology defined according to 2.1 also satisfies Lowen's definition, then we refer to it as a *proper fuzzy topology.*

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space,* or *fts* for short, and the members of $\mathcal{T}$ are called *$\mathcal{T}$-open fuzzy sets* or simply *open fuzzy sets.* In the following, definitions and propositions hold for proper and improper fuzzy topologies, unless stated otherwise.

2.2. Definition. A fuzzy topological space is called a *fuzzy T_1* space if every fuzzy point is a closed fuzzy set.

2.3. Definition. Let $\mathcal{T}$ be a fuzzy topology on a set X. A subfamily $\mathcal{B}$ of $\mathcal{T}$ is called a *base* for $\mathcal{T}$ if each member of $\mathcal{T}$ can be expressed as the union of members of $\mathcal{B}$.

2.1. Proposition. *A family $\mathcal{B}$ of fuzzy sets in X is a base for a proper fuzzy topology on X if it satisfies the following conditions.*

(i) $\sup_{B\in\mathcal{B}}\{\mu_B(x)\}=1$, *for all* $x\in X$.

(ii) *If* $B_1, B_2\in\mathcal{B}$ *then* $B_1\cap B_2\in\mathcal{B}$.

(iii) *For every* $0\leq c<1$ *and every* $B\in\mathcal{B}$, $k_c\cap B\in\mathcal{B}$.

Proof. Let $\mathcal{T}(\mathcal{B})$, or simply $\mathcal{T}$, be the family of fuzzy sets than can each be expressed as a union of elements of $\mathcal{B}$. From condition (i), $k_1\in\mathcal{T}$, and it is obvious that if $A_j\in\mathcal{T}$, $j\in J$, then $\bigcup_{j\in J}A_j\in\mathcal{T}$. Let $\{B_j\}$ and $\{B_l\}$ be subfamilies of $\mathcal{B}$ (j and l ranging in index sets J and L, respectively) and let $\mu_A(x)=\sup_j\{\mu_{B_j}(x)\}$ and $\mu_C(x)=\sup_l\{\mu_{B_l}(x)\}$, $x\in X$. Then

$$\min\{\mu_A(x), \mu_C(x)\}=\min\left\{\sup_j\{\mu_{B_j}(x)\}, \sup_l\{\mu_{B_l}(x)\}\right\}=\sup_{j,l}\{\min\{\mu_{B_j}(x), \mu_{B_l}(x)\}\}, \quad x\in X.$$

This shows that if $A, C\in\mathcal{T}$, then $A\cap C\in\mathcal{T}$. Finally, it is necessary to prove that k_c, $0\leq c<1$, belongs to $\mathcal{T}$. Condition (iii) implies that for each c, $0\leq c<1$, the fuzzy set with membership function

$$\sup_{B\in\mathcal{B}}\{\min\{\mu_{k_c}(x), \mu_B(x)\}\}, \quad x\in X,$$

belongs to $\mathcal{T}$. By condition (i), for each $x \in X$ and for each c, $0 \leq c < 1$, there exists a fuzzy set $B \in \mathcal{B}$ with membership value $\mu_B(x) \geq c$. Then $\sup_{B \in \mathcal{B}} \{\min\{\mu_{k_c}(x), \mu_B(x)\}\} = c$, for all $x \in X$, $0 \leq c < 1$, which shows that $k_c \in \mathcal{T}$. Thus the family generated by unions of $B \in \mathcal{B}$ is a proper fuzzy topology. □

Note that if a base for an improper fuzzy topology is considered, condition (iii) is unnecessary.

2.4. Definition. Let $\{(X_j, \mathcal{T}_j)\}_{j \in J}$ be a family of fts's and $(X, \mathcal{T})$ be the *product fts* [4, 10]. The *product fuzzy topology* $\mathcal{T}$ on X has as a base the set of finite intersections of fuzzy sets of the form $p_j^{-1}[A_j]$, where $p_j^{-1}[A_j]$ is the inverse image of $A_j \in \mathcal{T}_j$ under the projection p_j of X onto X_j, $j \in J$.

Let $\{X_j\}_{1 \leq j \leq n}$ be a finite family of sets and for each j let A_j be a fuzzy set in X_j. The *product* $A = \prod_{j=1}^n A_j$ of the family $\{A_j\}$ is defined as the fuzzy set in $X = \prod_{j=1}^n X_j$ that has membership function

$$\mu_A(x) = \min_{1 \leq j \leq n} \{\mu_{Aj}(p_j(x))\}, \quad x \in X.$$

If X_j has fuzzy topology $\mathcal{T}_j$, $1 \leq j \leq n$, the product fuzzy topology on X has as a base the set of product fuzzy sets of the form $\prod_{j=1}^n A_j$, $A_j \in \mathcal{T}_j$.

2.5. Definition. Let $(X, \mathcal{T})$, $(Y, \mathcal{V})$ be two fts's. A mapping f of $(X, \mathcal{T})$ into $(Y, \mathcal{V})$ is said to be *fuzzy continuous* if for each open fuzzy set V in $\mathcal{V}$ the inverse image $f^{-1}[V]$ is in $\mathcal{T}$. Conversely, f is said to be *fuzzy open* if for each open fuzzy set U in $\mathcal{T}$, the image $f[U]$ is in $\mathcal{V}$.

2.6. Definition. A bijective mapping f of a fts $(X, \mathcal{T})$ onto a fts $(Y, \mathcal{V})$ is called a *fuzzy homeomorphism* if it is fuzzy continuous and fuzzy open.

2.2. Proposition. *Let f be a fuzzy continuous (resp. fuzzy open) mapping of a fts $(X, \mathcal{T})$ into a fts $(Y, \mathcal{V})$ and g a fuzzy continuous (resp. fuzzy open) mapping of $(Y, \mathcal{V})$ into a fts $(Z, \mathcal{Z})$. Then the composition $g \circ f$ is a fuzzy continuous (resp. fuzzy open) mapping of $(X, \mathcal{T})$ into $(Z, \mathcal{Z})$.*

Proof. Obvious.

2.3. Proposition. *Let $\{(X_j, \mathcal{T}_j)\}_{j \in J}$, $\{(Y_j, \mathcal{V}_j)\}_{j \in J}$ be two families of fts's and $(X, \mathcal{T})$, $(Y, \mathcal{V})$ the respective product fts's. For each $j \in J$, let f_j be a mapping of $(X_j, \mathcal{T}_j)$ into $(Y_j, \mathcal{V}_j)$. Then the product mapping $f = \prod_{j \in J} f_j : (x_j) \to (f_j(x_j))$ of $(X, \mathcal{T})$ into $(Y, \mathcal{V})$ is fuzzy continuous if f_j is fuzzy continuous for each $j \in J$.*

Proof. See [4].

2.4 Proposition. *Let $\{(X_j, \mathcal{T}_j)\}_{1 \leq j \leq n}$, $\{(Y_j, \mathcal{V}_j)\}_{1 \leq j \leq n}$ be two finite families of fts's and let $(X, \mathcal{T})$, $(Y, \mathcal{V})$ be the respective product fts's. For each j, $1 \leq j \leq n$, let f_j be a mapping of $(X_j, \mathcal{T}_j)$ into $(Y_j, \mathcal{V}_j)$. Then the product mapping $f = \prod_{j=1}^n f_j : (x_j) \to (f(x_j))$ of $(X, \mathcal{T})$ into $(Y, \mathcal{V})$ is fuzzy open if f_j is fuzzy open for each j, $1 \leq j \leq n$.*

Proof. See [4].

The following definition of a fuzzy topological vector space is due to Katsaras and Liu [5]. Let E denote a vector space over the field K of real or complex numbers.

2.7. Definition. Let $\{A_j\}_{1 \leq j \leq n}$ be a finite family of fuzzy sets in a vector space E. The *sum* $A = \sum_{j=1}^n A_j$ of the family $\{A_j\}$ is the fuzzy set in E that has membership function

$$\mu_A(x) = \sup_{\sum_j x_j = x} \min_{1 \leq j \leq n} \{\mu_{A_j}(x_j)\}, \quad x \in E.$$

The *scalar product* αA of $\alpha \in K$ and A a fuzzy set in E is the fuzzy set in E that has membership function $\mu_{\alpha A}(x)$, $x \in E$, given by

$$\begin{aligned} \mu_{\alpha A}(x) &= \mu_A(x/a) \quad \text{for } \alpha \neq 0, \\ &= \mu_{0_\lambda}(x) \quad \text{for } \alpha = 0, \end{aligned}$$

where $\lambda = \sup_{y \in E} \mu_A(y)$.

2.8. Definition. A *fuzzy topological vector space*, or *ftvs* for short, is a vector space E over the field K of real or complex numbers, E equipped with a fuzzy topology $\mathcal{T}$ and K equipped with the usual topology $\mathcal{K}$, such that the two mappings

(i) $(x, y) \to x + y$ of $(E, \mathcal{T}) \times (E, \mathcal{T})$ into $(E, \mathcal{T})$,

(ii) $(\alpha, x) \to \alpha x$ of $(K, \mathcal{K}) \times (E, \mathcal{T})$ into $(E, \mathcal{T})$,

are fuzzy continuous.

Note that the fts E may be proper or improper but K is a special case of an improper fts. This was not made explicit in [3]. In the sequel E denotes a ftvs with scalar field K.

Let E, F be two fuzzy topological vector spaces and let ϕ be a mapping from E into F. Let $o(t)$ denote any function of a real variable t such that $\lim_{t \to 0} o(t)/t = 0$.

2.9. Definition. The mapping ϕ is said to be *tangent* to 0 if given a neighbourhood W of 0_δ, $0 < \delta \leq 1$, in F there exists a neighbourhood V of 0_λ, for every λ, $0 < \lambda < \delta$, in E such that

$$\phi[tV] \subset o(t)W,$$

for some function $o(t)$.

Note that in [3] the range of λ was $0 < \lambda \leq \delta$. The reason for the modification will become apparent in the proof of Proposition 2.6, which depends on Lemma 2.1. (The propositions of [3] are unaffected.) The following definitions of fuzzy differentiability and fuzzy derivative were proposed by Ferraro and Foster [3].

2.10. Definition. Let E, F be two ftvs's, each endowed with a T_1 fuzzy topology. Let $f: E \to F$ be a fuzzy continuous mapping. Then f is said to be *fuzzy differentiable at a point* $x \in E$ if there exists a linear fuzzy continuous mapping u of E into F such that

$$f(x + y) = f(x) + u(y) + \phi(y), \quad y \in E,$$

where ϕ is tanget to 0. The mapping u is called the *fuzzy derivative* of f at x. The fuzzy derivative of f at x is denoted by $f'(x)$; it is an element of $L(E, F)$, the set of all linear fuzzy continuous mappings of E into F. The mapping f is *fuzzy differentiable* if it is fuzzy differentiable at every point of E.

From this point on, suppose that each ftvs is equipped with a T_1 proper fuzzy topology. Some of the properties of fuzzy derivatives have been discussed in [3]. In particular there is the following.

2.5. Proposition. *Let E, F, G be ftvs's, f a fuzzy continuous mapping of E into F, and g a fuzzy continuous mapping of F into G. Let $x \in E$ and $y = f(x)$. If f is fuzzy differentiable at x and g is fuzzy differentiable at y, then the composition $g \circ f$ is fuzzy differentiable at x.*

Proof. See [3].

Further, the fuzzy derivative of $g \circ f$ at $x \in E$ is $g'(f(x)) \circ f'(x)$ [3].

2.1. Lemma. *Let $F = F_1 \times F_2$ be the product ftvs of two ftvs's F_1, F_2. Each neighbourhood W of 0_δ, $0 < \delta \leq 1$, in F contains a product $W_1 \times W_2$ of neighbourhoods W_1, W_2 of $0_{\delta'}$ in F_1, F_2, respectively, for every δ', $0 < \delta' < \delta$.*

Proof. Since W is a neighbourhood of 0_δ, there exists an open fuzzy set W' in F such that $0_\delta \in W' \subset W$. The set W' can be represented as the union of the fuzzy sets $U_i \times V_j$ where U_i, V_j (i and j ranging in some index sets) are open fuzzy sets in F_1, F_2 respectively. Hence

$$\mu_{W'}(0) = \sup_{i,j} \{\min\{\mu_{U_i}(0), \mu_{V_j}(0)\}\} \geqslant \delta.$$

Therefore, for each δ', $0 < \delta' < \delta$, there exist i', j' such that $\mu_{U_{i'}}(0)$, $\mu_{V_{j'}}(0) \geqslant \delta'$. □

2.6. Proposition. *Let E_1, E_2, F_1, F_2 be ftvs's. Suppose that $f: E_1 \to F_1$ and $g: E_2 \to F_2$ are fuzzy differentiable. Then $f \times g: E_1 \times E_2 \to F_1 \times F_2$ is fuzzy differentiable.*

Proof. By hypothesis,

$$f(x_1 + y_1) - f(x_1) = f'(x_1)(y_1) + \chi_1(y_1), \qquad x_1, y_1 \in E_1,$$
$$g(x_2 + y_2) - g(x_2) = g'(x_2)(y_2) + \chi_2(y_2), \qquad x_2, y_2 \in E_2,$$

where χ_1, χ_2 are functions tangent to zero. By Lemma 2.1, each neighbourhood W of 0_δ, $0 < \delta \leqslant 1$, in $F_1 \times F_2$ contains a product of neighbourhoods W_1, W_2 of $0_{\delta'}$ in F_1, F_2 for each δ', $0 < \delta' < \delta$. Suppose (by Proposition 4.2 of [3]) that W_1 and W_2 are each balanced [5]. Since f and g are fuzzy differentiable, there are, for each 0_λ, $0 < \lambda < \delta'$, in E_1, E_2, fuzzy neighbourhoods V_1, V_2 of 0_λ in E_1, E_2 such that $\chi_1[tV_1] \subset o_1(t)W_1$, $\chi_2[tV_2] \subset o_2(t)W_2$. Set $o = \max\{o_1, o_2\}$. The product $V_1 \times V_2$ is a fuzzy neighbourhood of 0_λ in $E = E_1 \times E_2$. Set $V = V_1 \times V_2$. Write [5] $\chi_1[tV_1] \times \chi_2[tV_2] = (\chi_1 \times \chi_2)[tV_1 \times tV_2]$; hence $\chi_1[tV_1] \times \chi_2[tV_2] = (\chi_1 \times \chi_2)[tV]$. Set $\chi_1 \times \chi_2 = \chi$. Then

$$\begin{aligned}\mu_{o(t)W}(z) &\geqslant \min\{\mu_{o(t)W_1}(z_1), \mu_{o(t)W_2}(z_2)\} \\ &\geqslant \min\{\mu_{\chi_1[tV_1]}(z_1), \mu_{\chi_2[tV_2]}(z_2)\} = \mu_{\chi[tV]}(z), \quad z = (z_1, z_2),\ z_1 \in F_1,\ z_2 \in F_2.\end{aligned}$$

That is, $\chi[tV] \subset o(t)W$, where $V \subset E_1 \times E_2$ and $W \subset F_1 \times F_2$; hence χ is tangent to zero. Furthermore $f'(x_1) \times g'(x_2)$ is fuzzy continuous [4] and linear. □

2.11. Definition. Let E, F be fuzzy topological vector spaces. A bijection f of E onto F is said to be a *fuzzy diffeomorphism of class C^1* (or *C^1 fuzzy diffeomorphism* for short) if it and its inverse f^{-1} are fuzzy differentiable, and f' and $(f^{-1})'$ are fuzzy continuous.

2.7. Proposition. *Let E, F, G be ftvs's. If f is a fuzzy diffeomorphism of class C^1 of E onto F and G is a fuzzy diffeomorphism of class C^1 of F onto G, then $g \circ f$ is a fuzzy diffeomorphism of class C^1 of E onto G.*

Proof. This follows from Proposition 2.5 and from Proposition 2.2. □

2.8. Proposition. *Let E_1, E_2, F_1, F_2 be ftvs's. Suppose that $f: E_1 \to F_1$ and $g: E_2 \to F_2$ are fuzzy diffeomorphisms of class C^1. Then $f \times g: E_1 \times E_2 \to F_1 \times F_2$ is a fuzzy diffeomorphism of class C^1.*

Proof. Obvious from Proposition 2.6 and from Propositions 2.3 and 2.4. □

3. Fuzzy manifolds

We now define fuzzy atlases and manifolds in a way that generalizes the classical definition. We prove that the fuzzy differential structure is compatible with a fuzzy topology and that fuzzy manifolds and fuzzy differentiable mappings between them retain some of the properties of classical manifolds and differentiable mappings.

The following is based on the definition of a classical atlas given by Lang [6].

3.1. Definition. Let X be a set. A *fuzzy atlas* $\mathscr{A}$ *of class* C^1 (or C^1 *fuzzy atlas* for short) on X is a collection of pairs (A_j, ϕ_j) (j ranging here and subsequently in some index set) which satisfies the following conditions.

(i) Each A_j is a fuzzy set in X and $\sup_j\{\mu_{A_j}(x)\} = 1$, for all $x \in X$.

(ii) Each ϕ_j is a bijection, defined on the support of A_j, $\{x \in X: \mu_{A_j}(x) > 0\}$, which maps A_j onto an open fuzzy set $\phi_j[A_j]$ in some ftvs E_j, and, for each l in the index set, $\phi_j[A_j \cap A_l]$ is an open fuzzy set in E_j.

(iii) The mapping $\phi_l \circ \phi_j^{-1}$, which maps $\phi_j[A_j \cap A_l]$ onto $\phi_l[A_j \cap A_l]$ is a C^1 fuzzy diffeomorphism for each pair of indices j, l.

Each pair (A_j, ϕ_j) is called a *fuzzy chart* of the fuzzy atlas. If a point $x \in X$ lies in the support of A_j then (A_j, ϕ_j) is said to be a *fuzzy chart at x*.

3.1. Proposition. *Let $\mathscr{A}$ be a C^1 fuzzy atlas, with charts (A_j, ϕ_j), on a set X_1, and let $\mathscr{B}$ be a C^1 fuzzy atlas, with charts (B_l, ψ_l), on a set X_2. Then the collection of pairs $(A_j \times B_l, \phi_j \times \psi_l)$ forms a C^1 fuzzy atlas on $X_1 \times X_2$.*

Proof. (i) For all $x_1 \in X_1$, $x_2 \in X_2$, $\sup_j\{\mu_{A_j}(x_1)\} = \sup_l\{\mu_{B_l}(x_2)\} = 1$, and by definition $\mu_{A_j \times B_l}(x) = \min\{\mu_{A_j}(x_1), \mu_{B_l}(x_2)\}$, $x = (x_1, x_2)$. Therefore

$$\sup_{j,l}\{\mu_{A_j \times B_l}(x)\} = \sup_{j,l}\{\min\{\mu_{A_j}(x_1), \mu_{B_l}(x_2)\}\} = \min\left\{\sup_j\{\mu_{A_j}(x_1)\}, \sup_l\{\mu_{B_l}(x_2)\}\right\} = 1.$$

(ii) $\phi_j[A_j]$ and $\psi_l[B_l]$ are open fuzzy sets in ftvs's E_j, E_l respectively. It follows that $\phi_j[A_j] \times \psi_l[B_l]$ is an open fuzzy set in $E_j \times E_l$ [5] and, since $\phi_j[A_j] \times \psi_l[B_l] = (\phi_j \times \psi_l)[A_j \times B_l]$ [5], $(\phi_j \times \psi_l)[A_j \times B_l]$ is open in $E_j \times E_l$. Next, consider the intersections $A_j \cap A_q$ and $B_l \cap B_r$ for each q, r. It is straightforward to show that $(A_j \cap A_q) \times (B_l \cap B_r) = (A_j \times B_l) \cap (A_q \times B_r)$. But it has just been proved that $(\phi_j \times \psi_l)[(A_j \cap A_q) \times (B_l \cap B_r)]$ is an open fuzzy set in $E_j \times E_l$ because $\phi_j[A_j \cap A_q]$ and $\psi_l[B_l \cap B_r]$ are open fuzzy sets in E_j and E_l respectively. Hence $(\phi_j \times \psi_l)[(A_j \times B_l) \cap (A_q \times B_r)]$ is an open fuzzy set in $E_j \times E_l$.

(iii) The last condition of Definition 3.1 is satisfied by virtue of Proposition 2.8. □

Next it is shown that the set X can be given a fuzzy topology such that each A_j in the C^1 fuzzy atlas on X is an open set and each ϕ_j is fuzzy continuous.

3.2. Proposition. *Let $\mathscr{A}$ be a C^1 fuzzy atlas with charts (A_j, ϕ_j) and suppose that for any open fuzzy set V in ftvs E_j, $(\phi_j^{-1}[V], \phi_j)$ is a fuzzy chart of $\mathscr{A}$. The family $\{A_j\}$ of fuzzy sets forms a base for a proper fuzzy topology on X, and in this topology the ϕ_j are fuzzy continuous.*

Proof. First, $\sup\{\mu_{A_j}(x)\} = 1$, for all $x \in X$. Next, if (A_l, ϕ_l), (A_m, ϕ_m) are fuzzy charts then $(A_l \cap A_m, \phi_l)$ is a fuzzy chart, since $\phi_l[A_l \cap A_m]$ is an open fuzzy set; this shows that if $A_l, A_m \in \{A_j\}$ then $A_l \cap A_m \in \{A_j\}$. Thus conditions (i) and (ii) of Proposition 2.1 are satisfied. Finally, for each l and c, $0 \leq c < 1$, let $A_l' = \phi_l^{-1}[k_c \cap \phi_l[A_l]]$; then (A_l', ϕ_l) is a fuzzy chart, and hence A_l' belongs to $\{A_j\}$. The membership function of A_l' is $\mu_{A_l'}(x) = \min\{\mu_{k_c}(x), \mu_{A_l}(x)\}$, $x \in X$; thus, for every $0 \leq c < 1$, $k_c \cap A_l \in \{A_j\}$. That $\{A_j\}$ is a base then follows from Proposition 2.1. Each mapping ϕ_j is fuzzy continuous since ϕ_j^{-1} takes an open fuzzy set onto an open fuzzy set. □

Let $(X, \mathscr{T})$ be a fuzzy topological space. Suppose there exist an open fuzzy set A in X and a fuzzy continuous bijective mapping ϕ defined on the support of A and mapping A onto an open fuzzy set V in some ftvs E. Then (A, ϕ) is said to be *compatible* with the C^1 atlas $\{(A_j, \phi_j)\}$ if each mapping $\phi_j \circ \phi^{-1}$ of $\phi[A \cap A_j]$ onto $\phi_j[A \cap A_j]$ is a fuzzy diffeomorphism of class C^1. Two C^1 fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas. It may be verified immediately that the relation of compatibility between C^1 fuzzy atlases is an equivalence relation. An equivalence class of C^1 fuzzy atlases on X is said to define a C^1 *fuzzy manifold* on X. In the sequel we refer simply to fuzzy manifolds.

3.3. Proposition. *Let X, Y be fuzzy manifolds; then the product $X \times Y$ is a fuzzy manifold.*

Proof. An obvious consequence of Proposition 3.1.

3.2. Definition. Let X, Y be fuzzy manifolds and let f be a mapping of X into Y. Then f is said to be *fuzzy differentiable at a point $x \in X$* if there is a fuzzy chart (U, ϕ) at $x \in X$ and a fuzzy chart (V, ψ) at $f(x) \in Y$ such that the mapping $\psi \circ f \circ \phi^{-1}$, which maps $\phi[U \cap f^{-1}[V]]$ into $\psi[V]$ is fuzzy differentiable at $\phi(x)$. The mapping f is *fuzzy differentiable* if it is fuzzy differentiable at every point of X; it is a C^1 *fuzzy diffeomorphism* if $\psi \circ f \circ \phi^{-1}$ is a C^1 fuzzy diffeomorphism.

3.4. Proposition. *Let X, Y, Z be fuzzy manifolds, f a mapping of X into Y and g a mapping of Y into Z. If f and g are fuzzy differentiable then $g \circ f$ is fuzzy differentiable.*

Proof. Let (U, ϕ), (V, ψ), (W, χ) be fuzzy charts at $x \in X$, $f(x) \in Y$, $g(f(x)) \in Z$, respectively. Then $\psi \circ f \circ \phi^{-1}$, which maps $\phi[U \cap f^{-1}[V]]$ into $\psi[V]$, and $\chi \circ g \circ \psi^{-1}$, which maps $\psi[V \cap g^{-1}[W]]$ into $\chi[W]$, are fuzzy differentiable. Hence $\chi \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1} = \chi \circ (g \circ f) \circ \phi^{-1}$, which maps $\phi[U \cap f^{-1}[V] \cap f^{-1}[g^{-1}[W]]]$ into $\chi[W]$, is fuzzy differentiable, by virtue of Proposition 2.5, and the assertion follows. $\square$

Corollary. *If f and g are C^1 fuzzy diffeomorphisms then the composition $g \circ f$ is a C^1 fuzzy diffeomorphism.*

4. Tangent vector spaces of a fuzzy manifold

The notion of a directional derivative in Euclidean space leads to the notion of a tangent vector of a differentiable manifold. Let X be a fuzzy manifold and let x be a (crisp) point in X. Consider triples (U, ϕ, v_λ), where (U, ϕ) is a fuzzy chart at x and v_λ is a fuzzy point of the ftvs in which $\phi[U]$ lies. Two such triples (U, ϕ, v_λ), (V, ψ, w_λ) are said to be *related*, written $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$, if the fuzzy derivative of $\psi \circ \phi^{-1}$ at $\phi(x)$ maps v_λ into w_λ. That is, $(\psi \circ \phi^{-1})'(\phi(x))v_\lambda = w_\lambda$.

4.1. Proposition. *The relation $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$ is an equivalence relation.*

Proof. Straightforward.

4.1. Definition. An equivalence class of triples (U, ϕ, v_λ) is called a *tangent vector* of the fuzzy manifold X at x, and the *tangent space* at x, denoted by $T_x(X)$, is defined as the set of all tangent vectors at x.

The set $T_x(X)$ can be given the structure of a vector space. Define the sum of two tangent vectors at $x \in X$ as $(U_1, \phi_1, v_{1\lambda}) + (U_2, \phi_2, v_{2\gamma}) = (U_2, \phi_2, (\phi_2 \circ \phi_1^{-1})'(\phi_1(x))v_{1\lambda} + v_{2\gamma})$. Define the product of a tangent vector with a scalar α as $\alpha \cdot (U, \phi, v_\lambda) = (U, \phi, \alpha v_\lambda)$. Compare [8].

4.2. Proposition. *If $(U_1, \phi_1, v_{1\lambda}) \sim (V_1, \psi_1, w_{1\lambda})$ and $(U_2, \phi_2, v_{2\gamma}) \sim (V_2, \psi_2, w_{2\gamma})$, then $(U_1, \phi_1, v_{1\lambda}) + (U_2, \phi_2, v_{2\gamma}) \sim (V_1, \psi_1, w_{1\lambda}) + (V_2, \psi_2, w_{2\gamma})$.*

Proof. Form the sums $(U_2, \phi_2, (\phi_2 \circ \phi_1^{-1})'(\phi_1(x))v_{1\lambda} + v_{2\gamma})$, $(V_2, \psi_2, (\psi_2 \circ \psi_1^{-1})'(\psi_1(x))w_{1\lambda} + w_{2\gamma})$. From the definition of related triples,

$$\begin{aligned}(\psi_2 \circ \phi_2^{-1})'(\phi_2(x))((\phi_2 \circ \phi_1^{-1})'(\phi_1(x))v_{1\lambda} + v_{2\gamma}) &= (\psi_2 \circ \phi_1^{-1})'(\phi_1(x))v_{1\lambda} + (\psi_2 \circ \phi_2^{-1})'(\phi_2(x))v_{2\gamma} \\ &= ((\psi_2 \circ \phi_1^{-1})'(\phi_1(x)) \circ (\phi_1 \circ \psi_1^{-1})'(\psi_1(x)))w_{1\lambda} + w_{2\gamma} \\ &= (\psi_2 \circ \psi_1^{-1})'(\psi_1(x))w_{1\lambda} + w_{2\gamma}. \qquad \square\end{aligned}$$

4.3. Proposition. *If* $(U, \phi, v_\lambda) \sim (V, \psi, w_\lambda)$ *then* $\alpha \cdot (U, \phi, v_\lambda) \sim \alpha \cdot (V, \psi, w_\lambda)$.

Proof. Straightforward.

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