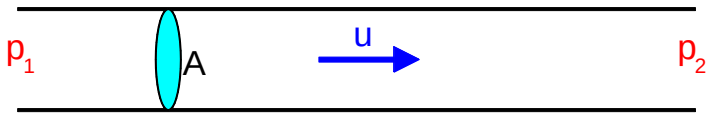
Objectives

(1) Recognise the potential for large **pressure transients** when pipe flow is stopped abruptly.
(2) Predict pressure rise and the speed of **water-hammer** waves in rigid and non-rigid pipes.
(3) Predict the **time series** of events at any point in a pipeline following sudden closure.
(4) Derive and use the **unsteady incompressible pipe-flow equation** to analyse the pressure-relieving behaviour of **surge tanks** and **pump bypasses**.

1. Unsteady Flow in Pipes

1.1 Introduction – Pressure Transients

A long pipeline contains a large mass of water. If this were all to be brought to rest simultaneously by the rapid closure of a valve then the pressure rise would be enormous. For example, the pressure difference, Δp, required to reduce the entire fluid momentum in the pipe to zero in time t may be estimated from:



$$\text{force} = \text{mass} \times \text{acceleration}$$

$$(p_1 - p_2)A = (\rho AL) \times \left(\frac{0-u}{t}\right)$$

$$\Rightarrow \qquad \Delta p = \frac{\rho L u}{t}$$

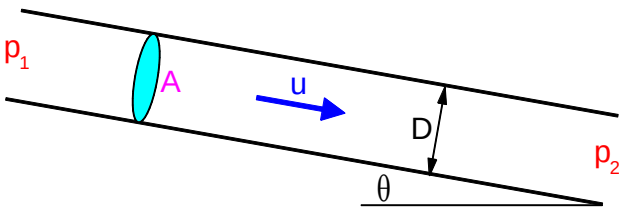
If the closure time, t, is small this leads to very large pressures. For example, if a 1 m s^{-1} flow of water in a 1000 m pipeline is shut off in 1 s, then the incompressible assumption leads to a pressure difference of $1000 \times 1000 \times 1/1 = 10^6$ Pa, or about 10 atmospheres!

Such large pressure transients can lead to severe pipe or valve damage, so should be designed against. Means of achieving a more gradual change in velocity are discussed in Section 3.

However, large pressure transients can sometimes be useful. In the *ram pump* or *hydraulic ram* the periodic closure of a valve by a relatively low-head flow creates short bursts of high pressure which can deliver water to a much greater height (albeit it at a smaller mean flow rate). Since the device requires only a steady flow of water and not a power supply it is very useful for raising water in remote regions.

1.2 Unsteady Incompressible Pipe-Flow Equation

To include the effects of both friction and slope consider the flow in a length L of pipe. For incompressible flow all the fluid in the pipe moves at the same speed u.



$$\text{mass} \times \text{acceleration} = \text{force}$$

$$m\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{p_1 A - p_2 A}_{\substack{\text{net pressure}\\ \text{force}}} + \underbrace{mg\sin\theta}_{\substack{\text{streamwise component}\\ \text{of weight}}} - \underbrace{\tau_w \times \pi D L}_{\text{wall friction}}$$

Hence, with $\tau_w = c_f \times (\text{dynamic pressure})$, and $c_f = \lambda/4$ (see Topic 2):

$$(\rho AL)\frac{\mathrm{d}u}{\mathrm{d}t} \quad = \quad (p_1 - p_2)A \quad + \quad (\rho AL)g\frac{z_1 - z_2}{L} \quad - \quad \underbrace{\frac{\lambda}{4} \times \frac{1}{2}\rho u|u| \times \pi DL}_{\text{accounting for flow direction}}$$

Dividing by ρAg (where $A = \pi D^2/4$), and writing $p^* = p + \rho gz$ for the piezometric pressure:

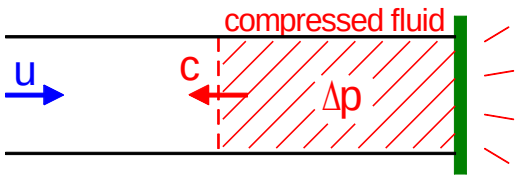
$$\frac{L}{g}\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{\frac{p_1^* - p_2^*}{\rho g}}_{\text{piezometric head}} - \underbrace{\lambda\frac{L}{D}\frac{u|u|}{2g}}_{\text{frictional head loss}}$$

By considering infinitesimal lengths of pipe this can be shown to be correct whether the slope is constant or not. Moreover, since the velocity is the same anywhere in the pipe, the difference in piezometric head is the same as the difference in total head.

Unsteady pipe-flow equation:

$$\frac{L}{g}\frac{\mathrm{d}u}{\mathrm{d}t} = H_1 - H_2 - \lambda\frac{L}{D}\frac{u|u|}{2g} \qquad (1)$$

For **steady** flow the LHS is zero. The equation then says simply that the drop in head ($H_1 - H_2$) equals the head loss due to friction from the Darcy-Wesibach equation.



For **rapid** changes of velocity the large acceleration or deceleration can only be accommodated by a large change in pressure, which, in turn, causes a large change of density. In this *rapid-closure* problem the pressure transient is huge, the incompressibility assumption breaks down and elastic properties of the fluid (and the pipe) must be considered. The whole of the fluid in the pipe does not respond to the valve closure simultaneously; instead, a pressure discontinuity (*shock*) propagates back along the pipe: the phenomenon of *water hammer*. We cover rapid closure in Section 2.

For **slow** changes of velocity, pressure changes are small enough for the incompressible approximation to remain valid, and this equation can be solved to give u as a function of time. This is the *slow-closure* problem which will be used to analyse surge tanks and pressure-relief valves in Section 3.

A more precise definition of what constitutes "slow" or "rapid" closure will be given later.

2. Water Hammer

The near-instantaneous stopping of all the water in a long pipeline can only be brought about by huge pressures.

In practice, the fluid adjacent to the valve is compressed and a positive pressure pulse propagates back along the pipe at speed c (rather akin to the build-up of cars in a motorway pile-up). The propagating front is referred to as a *shock*, and the phenomenon in pipelines as *water hammer*.

Although the phenomenon is most often associated with valve closure, large *negative* pressure pulses may occur if the valve is opened rapidly; this may lead to cavitation, which should be avoided at all costs.

2.1 Speed of Pressure Waves in Rigid Pipes

The elastic properties of a fluid are determined by its *bulk modulus*, K, which is the ratio of the change in pressure, Δp, to the volumetric strain (fractional change in volume):

$$\Delta p = K\frac{(-\Delta V)}{V}$$

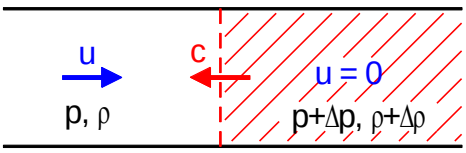
or in terms of the fractional change in density:

$$\boxed{\Delta p = K\frac{\Delta\rho}{\rho} \qquad (2)}$$

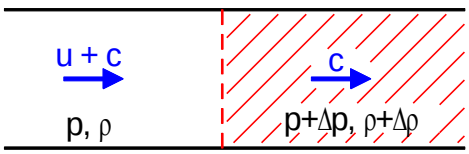
The bulk modulus of water is about 2.2 GPa (2.2×10^9 Pa).

Consider a discontinuity propagating at speed c (the *celerity*) to the left in response to a valve closure. In front of it is fluid with velocity u, which as yet has not felt the effect of the closure.

Fluid properties (pressure, density) change from (p, ρ) to $(p + \Delta p, \rho + \Delta\rho)$ across the shock. In the first instance we assume that the pipe is *rigid* – i.e. the cross-sectional area, A, is unchanged.



The problem is easier to analyse in the reference frame of the shock. To get this simply add the same **right**-directed velocity c to all velocities in the diagram:



Continuity:

$$(\text{mass flux})_{\text{in}} = (\text{mass flux})_{\text{out}}$$

$$\rho(c+u)A = (\rho + \Delta\rho)cA$$

Dividing by ρcA:

$$1 + \frac{u}{c} = 1 + \frac{\Delta\rho}{\rho}$$

Hence,

$$\frac{u}{c} = \frac{\Delta\rho}{\rho} \qquad (3)$$

By assumption, both sides are much less than 1.

Momentum

$$\text{force} = (\text{mass flux}) \times (\text{change in velocity})$$

$$pA - (p + \Delta p)A = \rho(c+u)A \times (-u)$$

Hence,

$$\Delta p = \rho(c+u)u$$

Since $u \ll c$ this gives:

Pressure rise across a shock:

$$\Delta p = \rho c u \qquad (4)$$

Substituting (3) and (4) into the compressibility relationship $\Delta p = K(\Delta\rho/\rho)$ gives

$$\rho c u = K\frac{u}{c}$$

Hence,

Pressure-wave speed in a rigid pipe:

$$c = \sqrt{\frac{K}{\rho}} \qquad (5)$$

where K is bulk modulus, ρ is density. This is effectively the speed of sound in the fluid.

Example.
Water (density 1000 kg m^{-3}, bulk modulus 2.2 GPa) is flowing at 0.5 m s^{-1} in a pipe when the flow is suddenly halted. Assuming that the walls of the pipe are sufficiently thick for it to be approximated as rigid find:

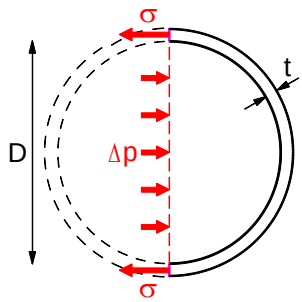
(a) the speed of water hammer waves;
(b) the pressure rise.

Answer: (a) 1480 m s^{-1}; (b) 7.42×10^5 Pa

2.2 Speed of Pressure Waves in Non-Rigid Pipes

In practice, the pressure rise may be sufficient to deform the pipe, increasing its cross-section. The pipe itself absorbs strain energy and reduces the speed of the pressure wave.

To include the area-change effects in the continuity equation we need to relate the change in cross-sectional area to the pressure rise. Formally, the internal pressure is balanced by an increased circumferential ("hoop") stress, which is related to the change in diameter and thence the change in area by the elastic properties of the pipe.



From the diagram, an increase in pressure Δp induces a hoop stress σ. If D is the internal diameter of the pipe and t is the wall thickness then, equating forces per unit length:

$$2\sigma t = \Delta p\, D$$

$$\Rightarrow \qquad \sigma = \frac{D}{2t}\Delta p \tag{6}$$

But from the constitutive relation:

$$\text{stress} = \text{Young's modulus} \times \text{strain}$$

$$\sigma = E \times \frac{\pi \Delta D}{\pi D} \qquad = E\frac{\Delta D}{D}$$

so that, equating these two expressions for σ,

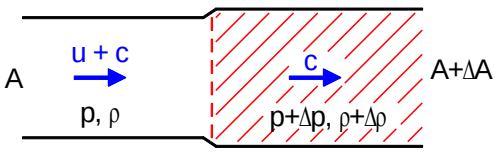
$$\frac{\Delta D}{D} = \frac{D}{2Et}\Delta p \tag{7}$$

This is the fractional change in diameter. We require the fractional change in area. From the geometry,

$$A = \frac{\pi D^2}{4} \qquad \Rightarrow \qquad \Delta A \approx \frac{\mathrm{d}A}{\mathrm{d}D}\Delta D \quad = \frac{2\pi D}{4}\Delta D \qquad \Rightarrow \qquad \frac{\Delta A}{A} = 2\frac{\Delta D}{D}$$

Hence,

$$\frac{\Delta A}{A} = \frac{D}{Et}\Delta p \tag{8}$$



The pressure change across the shock is still given by $\Delta p = \rho c u$ (e.g. consider the acceleration of fluid on the centreline), but continuity must account for the change of cross-sectional area:

$$\rho(c+u)A = (\rho+\Delta\rho)c(A+\Delta A)$$

Dividing by ρcA:

$$1+\frac{u}{c} = \left(1+\frac{\Delta\rho}{\rho}\right)\left(1+\frac{\Delta A}{A}\right) \quad = 1+\frac{\Delta\rho}{\rho}+\frac{\Delta A}{A}+2^{\text{nd}}-\text{order terms}$$

$$\Rightarrow \qquad \frac{u}{c} = \frac{\Delta\rho}{\rho}+\frac{\Delta A}{A}$$

Hence, using $u/c = \Delta p/\rho c^2$ (from momentum), $\Delta\rho/\rho = \Delta p/K$ (from compressibility) and $\Delta A/A = (D/Et)\Delta p$ (from elasticity):

$$\frac{\Delta p}{\rho c^2} = \frac{\Delta p}{K}+\frac{D\Delta p}{Et}$$

whence:

$$\frac{1}{\rho c^2} = \frac{1}{K}+\frac{D}{Et}$$

For convenience, and by comparison with the rigid-pipe limit, we write this as

$$\frac{1}{\rho c^2} = \frac{1}{K'}$$

in terms of an effective bulk modulus K'.

Pressure-wave speed in non-rigid pipes:

$$c = \sqrt{\frac{K'}{\rho}} \tag{9}$$

where the *effective bulk modulus* K' is given by

$$\frac{1}{K'} = \frac{1}{K}+\frac{D}{Et} \tag{10}$$

The terms on the RHS correspond to the elastic properties of the *fluid* and the *pipe*, respectively.

Example.
Repeat the example at the end of Section 2.1 for pipes of internal diameter $D = 200$ mm and wall thickness 5 mm made of:
(i) steel ($E = 210$ GPa);
(ii) PVC ($E = 2.6$ GPa).

Answer: (i) 1250 m s^{-1} and 6.23×10^5 Pa; (ii) 251 m s^{-1} and 1.26×10^5 Pa

2.3 Time Series of Events Following Sudden Closure

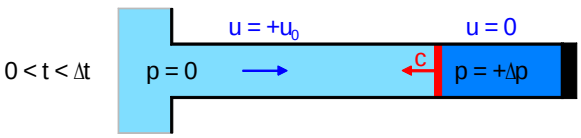
Consider flow from a large reservoir (constant pressure; excess pressure $p = 0$) at speed u_0. If a valve at the end of the pipeline is suddenly closed, pressure waves travel back and forth along the pipe. The time taken for pressure waves to travel from one end of the pipe to the other is

$$\Delta t = \frac{L}{c} \tag{11}$$

The sequence of events is as follows.

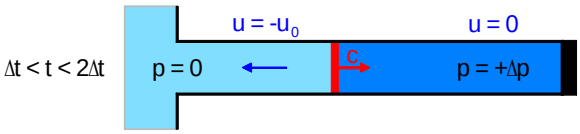
1. At $t = 0$ the valve is closed. The water immediately next to the valve is compressed to an excess pressure $+\Delta p$ and a pressure wave starts to propagate back along the pipe.

   For $0 < t < \Delta t$ the propagating wave moves into unaffected fluid ($u = u_0$ and $p = 0$). Behind the shock is stationary, compressed fluid ($u = 0$, $p = +\Delta p$)



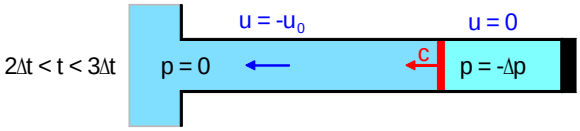
2. At $t = \Delta t$ the wave reaches the reservoir. All the fluid in the pipe is at rest; however, it is compressed to a higher pressure than the reservoir, so begins to drive a flow u_0 back toward the reservoir. The water-hammer wave is *reflected*.

   For $\Delta t < t < 2\Delta t$ the wave propagates back toward the valve, gradually decompressing the pipe.



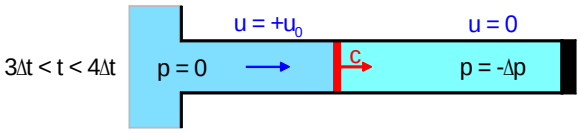
3. At $t = 2\Delta t$ the wave arrives back at the valve. The entire pipe is now decompressed; however the fluid in it is all moving backwards and cannot be brought to rest immediately.

   For $2\Delta t < t < 3\Delta t$ a negative pressure wave travels toward the reservoir, leaving behind lower pressure fluid, $p = -\Delta p$.



4. At $t = 3\Delta t$ the wave reaches the reservoir. All of the fluid in the pipe is now at rest; however it is at lower pressure than the reservoir, so a forward-moving flow ($u = u_0$) begins to rush in. The wave is again reflected.

   For $3\Delta t < t < 4\Delta t$ the pressure wave travels back toward the valve, restoring the initial conditions in the pipe.



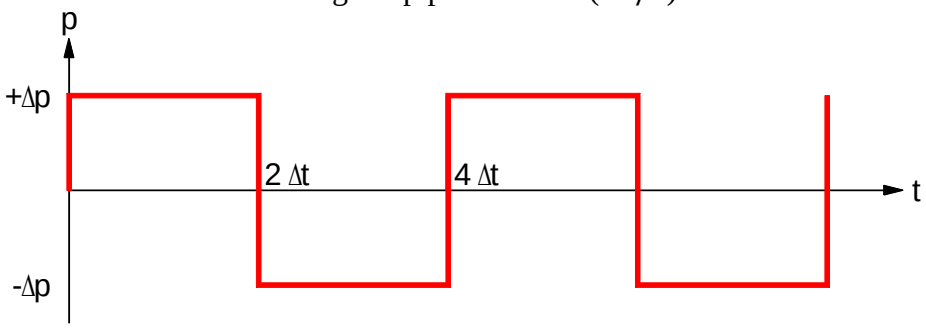
In the absence of friction the whole cycle would repeat with period $4\Delta t = 4L/c$.

Notes.

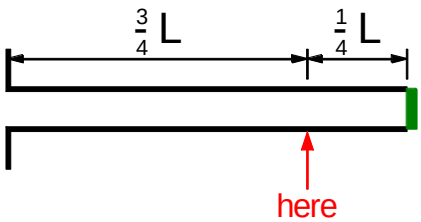
(1) At the open boundary (reservoir):
- the pressure is always 0;
- the velocity reflects with change of phase (positive to negative and vice versa).

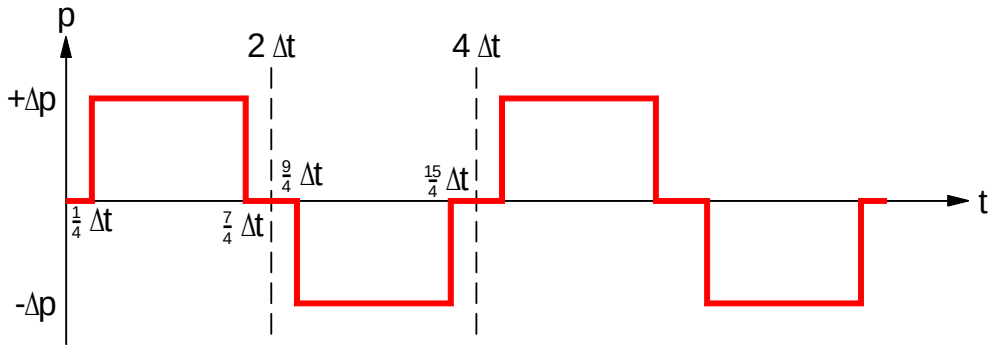
At the solid boundary (valve):
- the pressure reflects with change of phase;
- the velocity is always 0.

(2) The sequence of pressures at the valve are as shown. Pressures stay at $\pm\Delta p$ for the time it takes a wave to travel along the pipe and back $(2L/c)$



The sequence of pressures at an intermediate point – here, 1/4 of the way along the pipe from the valve – is along the pipeline are shown below. The length of the positive or negative pulses is determined by the fraction of the cycle that this point is on the **valve** (i.e. pressure-varying) side of the wave.





(c) In reality, the "square-wave" cycle of pressures is gradually attenuated by friction. Moreover, the negative pulse cannot take the **absolute** pressure below zero.

Example.
A steel pipeline of length 1155 m discharges water at velocity 2 m s^{-1} to atmosphere through a valve. The pipe has diameter 500 mm and wall thickness 10 mm. The bulk modulus of water is 2.0 GPa and the Young's modulus of the pipe material is 200 Gpa.

If a sudden closure of the valve occurs,
(a) determine the speed of water hammer waves;
(b) show pressure variations in time at the points immediately next to the valve and 866 m upstream of the valve.

Neglect friction in the pipe.

Answer: (a) 1155 m s^{-1}; (b) $\Delta p = 2.31 \times 10^6$ Pa; $\Delta t = 1$ s

2.4 "Slow" and "Rapid" Closure

Instantaneous valve closure is impossible in reality. Conventionally, the closure is regarded as "rapid" if it takes much less time than that for the pressure wave to travel along the pipe and back. i.e.

Rapid closure	$\Leftrightarrow$	$t_{\text{closure}} \ll 2\dfrac{L}{c}$	(use water-hammer theory)
Slow closure	$\Leftrightarrow$	$t_{\text{closure}} \gg 2\dfrac{L}{c}$	(use unsteady-incompressible-flow theory)

In the intermediate case, if the closure time is of the same order as $2L/c$ then the pipe flow behaviour has to be dealt with by computational techniques ("method of characteristics"), which are beyond the scope of this course.

3. Slow Closure: Unsteady Incompressible Pipe Flow

In practice, water hammer is an undesirable phenomenon and it is common to incorporate devices to alleviate rapid and large fluctuations in pressure. Two such are *surge tanks* (often an integral part of hydropower stations) and *pump-bypass valves* (used to protect pumps).

The subsequent development of the flow involves the unsteady pipe-flow equation:

$$\frac{L}{g}\frac{\mathrm{d}u}{\mathrm{d}t} = H_1 - H_2 - \underbrace{c\frac{u|u|}{2g}}_{\text{losses}}$$

where H_1, H_2 are the heads at the ends of the pipe, u is the velocity from end 1 to end 2 and $c = \lambda L/D$ for pipe friction alone, but may also include other forms of head losses.

If the head difference $H_1 - H_2$ along the pipeline is constant and we consider only that part of any subsequent motion where u is positive then this equation can be rearranged as

$$\frac{2L}{c}\frac{\mathrm{d}u}{\mathrm{d}t} = \pm a^2 - u^2\,, \qquad \text{where} \qquad \pm\, a^2 = \frac{2g(H_1 - H_2)}{c}$$

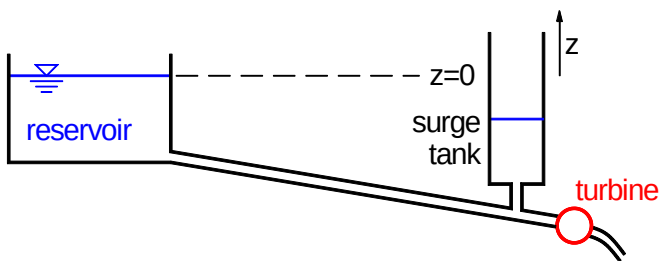
This may be solved for the two cases of retarding head ($H_2 > H_1$) or driving head ($H_1 > H_2$) respectively by separating variables and noting the standard integrals

$$\int_0^U \frac{\mathrm{d}u}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{U}{a}$$

$$\int_0^U \frac{\mathrm{d}u}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\frac{U}{a} \quad = \frac{1}{2a}\ln\left(\frac{a+U}{a-U}\right) \qquad (|U| < a)$$

We examine first in Section 3.1, however, a case where the head difference is not constant.

3.1 Surge Tank



A reservoir supplies a turbine with water at volumetric flow rate Q. The system is protected by a surge tank, which acts to absorb large changes to the flow when the turbine valve is opened or closed rapidly. Assuming negligible energy losses in the surge tank itself, the flow velocity u in the pipeline and the level of water z in the surge tank (relative to that in the reservoir) are given by the following coupled differential equations.

Unsteady pipe-flow equation (i.e. momentum equation):

$$\frac{L}{g}\frac{\mathrm{d}u}{\mathrm{d}t} = -z - \lambda\frac{L}{D}\frac{u|u|}{2g} \tag{12}$$

Continuity (from the difference between inflow and outflow at the surge-tank junction):

$$A_s \frac{\mathrm{d}z}{\mathrm{d}t} = uA - Q \tag{13}$$

where D is the pipe diameter, L is the pipe length and A and A_s are the cross-sectional areas of the pipe and surge tank, respectively.

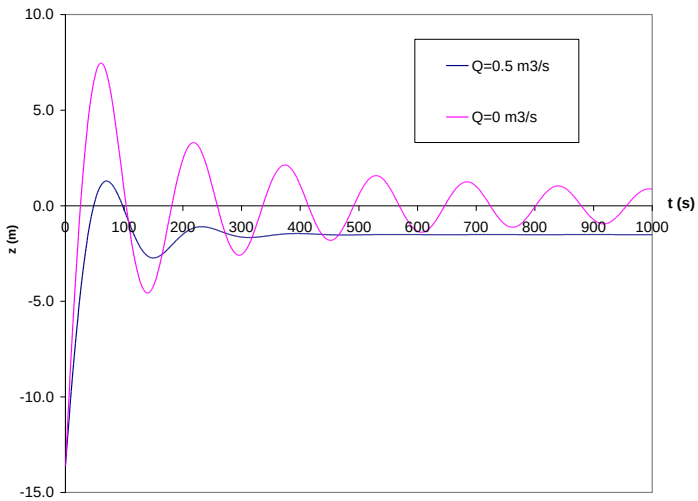
Under steady operation ($Q = Q_0$) the LHS of the equations is zero and the steady flow velocity and level of water in the surge tank are given by

$$u_0 = \frac{Q_0}{A} \tag{14}$$

$$z_0 = -\lambda \frac{L}{D} \frac{u_0^2}{2g} \tag{15}$$

corresponding to the usual discharge and head-loss formulae. If the discharge to the turbine varies, z and u will evolve according to equations (12) and (13).

In general, equations (12) and (13) must be solved numerically as a pair of coupled differential equations. Examples of the resulting damped oscillation in water level are shown right for complete and partial closures.



A useful estimate of oscillation period and, more importantly, a bound on maximum rise, may be obtained by neglecting pipe friction. The equations then yield simple harmonic motion about the level of water in the reservoir:

$$\left.\begin{aligned} \frac{L}{g}\frac{\mathrm{d}u}{\mathrm{d}t} &= -z \\ A_s \frac{\mathrm{d}z}{\mathrm{d}t} &= uA - Q \end{aligned}\right\} \quad \Rightarrow \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\left(\frac{g}{L}\frac{A}{A_s}\right) z$$

where Q is any remaining flow to the turbine (often zero). Hence,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\omega^2 z \qquad \text{where} \qquad \omega = \sqrt{\frac{g}{L}\frac{A}{A_s}}$$

with boundary conditions

$$z = 0, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{Q_0 - Q}{A_s} \quad \text{at} \quad t = 0$$

This has solution

$$z = z_{\max} \sin \omega t$$

where the period is

$$T = \frac{2\pi}{\omega}$$

and the maximum surge height is

$$z_{\max} = \frac{Q_0 - Q}{A_s \omega}$$

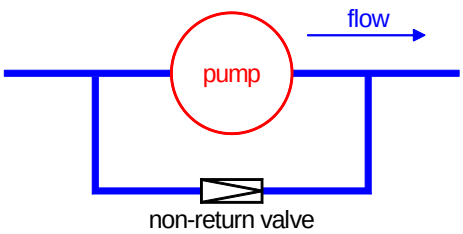
.

Example.
A pipe (length L = 500 m, diameter D = 1.5 m) is used to deliver water from a reservoir to a turbine at a volumetric flow rate of 2 $m^3\ s^{-1}$. The turbine is protected by a cylindrical surge tank of inside diameter 5 m. If friction losses can be neglected find the maximum surge in the surge tank and the period of oscillation if:
(a) the entire flow to the turbine is shut off;
(b) the flow to the turbine is halved.

Answer: (a) 150 s and 2.42 m; (b) 150 s and 1.21 m

3.2 Pump Bypass

If a pump were to trip suddenly, a large and damaging **negative** pressure would occur on the downstream side, as it takes a finite time to arrest the flow of water away from the pump. To prevent this, a length of pipe with a **non-return** valve is used to bypass the pump.



Under normal operation the pressure is higher on the downstream side of the pump, keeping the non-return valve closed. If the pump were to stop suddenly, the drop in pressure would open the valve (automatically), sucking water through to prevent a void. The flow would then continue until eventually stopped by friction and the downstream head.

The bypass arrangement is also useful when there are several pumps in series along a pipeline. During low demand any pump can then be removed from the system for maintenance without halting the flow.

Example.
Sewage, the density and viscosity of which differ little from those of clean water, has to be pumped from a tank at the end of an interceptor sewer to the first tank of a treatment works by means of a pump situated at the upstream end. The difference in liquid levels is 32 m; the pipeline is 5.2 km long, 1.5 m diameter and has a friction factor of 0.014. The steady-flow discharge is 4.5 $m^3\ s^{-1}$.

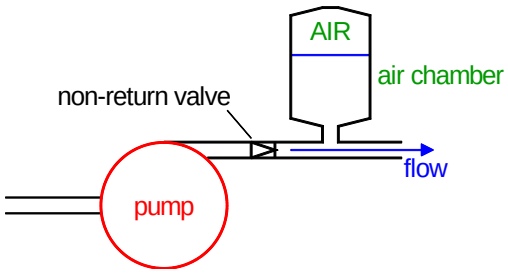
(a) Calculate the steady flow velocity in the pipe and the pump power.

(b) Explain, briefly, what would happen if the inlet to the pipeline were suddenly completely blocked, assuming that no protective devices are installed.

(c) It is proposed to install a by-pass, incorporating a non-return valve, to the pump so as to prevent the pressure just downstream of it falling below atmospheric. Calculate the time taken for the water to come to rest after a blockage.

Answer: (a) 2.55 m s^{-1} and 2.12 MW (!);
(b) large negative pressure pulse and cavitation; (c) 36.7 s

An alternative to the bypass arrangement is an *air-inlet valve*, which opens to allow air into the system in the event of failure and maintains a near-constant, slightly-sub-atmospheric pressure until the water has come to a halt.



Example.
Water is pumped from one reservoir to another in which the water level is 12 m above the pumping station, through a horizontal pipe 1500 m long and 0.5 m diameter at a rate of 0.4 $m^3\ s^{-1}$. The head loss due to pipe friction at this velocity is 8 m.

(a) Calculate the bulk velocity during normal operating conditions and the friction factor in the pipe.

(b) To protect the pump an air inlet valve is fitted to the pipe just downstream of the pump. This valve is designed to allow air to flow into the pipe when the pressure falls to 5 m of water below atmospheric pressure. Assuming a constant friction factor, calculate the time for the water to come to rest if the pump intake is suddenly and completely blocked.

Answer: (a) $u_0 = 2.04$ m s^{-1}; $\lambda = 0.0126$; (b) 16.1 s.