EXAMPLE SHEET FOR TOPIC 3 **AUTUMN 2023**

Q1.
Use dimensional analysis to investigate how the capillary rise, h, of a liquid in a tube varies with tube diameter, d, gravity, g, fluid density, ρ, surface tension, σ, and contact angle, θ.

Q2.
Find an expression for the rate of flow, Q, over a weir if it is assumed to be proportional to the width of the weir, b, and is a function of the upstream *freeboard* (i.e., height above the crest), H, and gravitational acceleration, g.

Q3.
The period of oscillation, t, of a simple pendulum is assumed to depend upon its length, l, bob mass, m, maximum displacement angle, $\theta_{\max}$, and gravitational acceleration, g.

(a) Perform a dimensional analysis to establish how t varies with these parameters.

(b) If the period of oscillation for a given pendulum on earth is 3 s, what will it be (for the same amplitude oscillations) on the moon ($g = 1.62\ \mathrm{m\ s^{-2}}$)?

(c) What other parameters do you think t might depend upon (weakly) in practice?

Q4.
Perform a dimensional analysis to establish how the phase speed, c, of waves on water depends on the wave amplitude, A, wavelength, λ, gravitational acceleration, g, water depth, h, density, ρ, and surface tension, σ.

What does this reduce to in the following special cases:

(a) small-amplitude waves on deep water caused by surface tension? (These are called "capillary waves" or "ripples".)

(b) small-amplitude long waves? (The restoring force is gravitational and the speed is independent of wavelength).

Q5.
A simply-supported beam of length, l, second moment of area, I, and Young's modulus, E, is subjected to a crossflow of velocity, V, density, ρ, and viscosity, μ. Conduct a dimensional analysis to establish how the centre deflection, δ, varies with these parameters.

What simplification is possible if it is known that E and I only appear in the combination EI?

Q6.
A viscous fluid is confined between two long, concentric cylinders and the torque required to rotate the inner cylinder is measured. The torque per unit length of cylinder, τ, is a function of angular velocity, ω, inner and outer radii, a and b, respectively, fluid density, ρ, and kinematic viscosity, ν.

(a) Conduct a formal dimensional analysis to write this relationship in a suitable dimensionless form.

(b) In laminar flow ρ and ν only appear in combination as the dynamic viscosity, $\mu = \rho\nu$. Without repeating the dimensional analysis from scratch, deduce the form of the dimensionless relationship in this case?

(c) In the laminar-flow regime, what happens to the torque if both a and b are doubled?

Q7.
In a channel with an erodible bed, the sediment of the bed is stirred into motion when the bed shear stress exceeds a threshold value τ_{crit}. This critical shear stress is known to be a function of sediment particle diameter, d, fluid density, ρ, kinematic viscosity, ν, and the reduced gravity $g' = (s-1)g$, where $s = \rho_s/\rho$ is the sediment-fluid density ratio.

(a) By conducting a dimensional analysis show that

$$\frac{\tau_{\text{crit}}}{\rho(s-1)gd} = f\left(\frac{(s-1)gd^3}{\nu^2}\right)$$

(b) For sufficiently large particles the dependence on molecular viscosity vanishes. In an experiment with water flow over anthracite (density 1500 kg m^{-3}, particle diameter 3 mm) the critical shear stress is found to be 0.80 N m^{-2}. Find the corresponding value of τ_{crit} for water flow over coarse sand (density 2650 kg m^{-3}, diameter 2 mm). Assume that both experiments are conducted in the viscosity-independent regime.

(c) By balancing forces, show that in a *wide* channel with small streamwise slope, S, uniform steady flow of depth h is obtained for a bed shear stress

$$\tau = \rho g h S$$

(d) Hence find the depth of water needed to initiate motion of the sand bed in (b) if the slope is 1 in 200.

Q8. (Exam 2019)

For a sediment bed exposed to a turbulent flow, the sediment entrainment rate, q, (volume of sediment passing unit length of surface per unit time) is a function of the particle density, ρ_p, the density of the fluid, ρ_f, the diameter of the sediment particles, d, the acceleration due to gravity, g, the kinematic viscosity of the fluid, ν, and the shear velocity of the flow, u_*.

(a) Conduct a formal dimensional analysis to show that

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$$

where

$$\Pi_1 = \frac{q}{\sqrt{gd^3}}, \qquad \Pi_2 = \frac{\rho_f}{\rho_p}, \qquad \Pi_3 = \frac{\sqrt{gd^3}}{\nu}, \qquad \Pi_4 = \frac{u_*^2}{gd}$$

For sufficiently high velocities, viscous effects can be neglected. Under this condition, two experiments are conducted with the parameters in the following table.

	Experiment 1	Experiment 2
Shear velocity, u_* (m s^{-1})	0.040	0.096
Fluid density, ρ_f (kg m^{-3})	1000	800
Particle density, ρ_p (kg m^{-3})	2600	?
Particle diameter, d (mm)	5	?

(b) Find the (i) particle density and (ii) diameter in experiment 2 if the two experiments are to be dynamically similar.

The following empirical relationship has been found to approximate the entrainment rate:

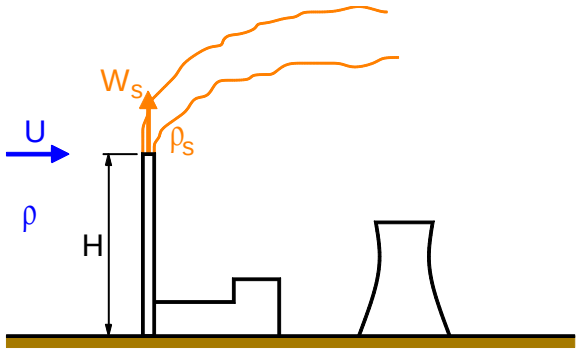
$$\Pi_1 = \frac{3500\sqrt{\Pi_4}}{1 - \Pi_2}[10\Pi_4 - 0.06(1 - \Pi_2)]$$

(c) Determine the entrainment rate, q, in experiment 1.

(d) A new experiment is carried out using sediment particles with diameter $d = 15$ mm. Assuming dynamical similarity with experiment 1, determine the entrainment rate in the new experiment.

Q9.

Wind-tunnel testing is to be used to determine the pollution levels expected from a new coal-fired power station. A model is constructed and a tracer gas is mixed with the stack emissions to determine local pollution concentrations.



The maximum ground-level concentration, $C_{\max}$, (volume of tracer per volume of air) is proportional to the source strength, Q, (volume of tracer emitted per second at standard temperature and pressure). The ground-level concentration (glc) is determined by the stack height, H, the crosswind velocity, U, the ambient air density, ρ, the source momentum flux, $\rho_s W_s^2$, and the buoyancy of stack gases, $(\rho - \rho_s)g$. (W_s is the vertical velocity of emissions, ρ_s is the density of stack gases and g is gravity).

(a) Show, by dimensional analysis, that

$$\frac{C_{\max} U H^2}{Q} = f\left(\frac{\alpha W_s^2}{U^2}, \frac{(1-\alpha) g H}{U^2}\right)$$

where $\alpha = \rho_s/\rho$ is the density ratio. (This is called "*buoyancy scaling*").

(b) Show that, if subscripts m and p denote model and prototype respectively, then

$$\frac{U_m}{U_p} = \left(\frac{1-\alpha_m}{1-\alpha_p}\right)^{1/2} l^{1/2} \qquad \text{and} \qquad \frac{(W_s/U_s)_m}{(W_s/U_s)_p} = \left(\frac{\alpha_p}{\alpha_m}\right)^{1/2}$$

where l is the ratio of lengths in the model to those in the prototype.

A simulation is to be conducted at 1:400 scale. In the prototype the stack gases are primarily hot air and the exit conditions are:

exit temperature:	80 °C
ambient air temperature:	15 °C
exit velocity:	3 m s^{-1}
crosswind velocity:	10 m s^{-1}
pollutant emission rate:	10 L s^{-1}

(c) Find the density ratio, α_p, in the prototype.

(d) Find the required wind-tunnel speed and stack exit velocity if the density ratio in the model, α_m, is the same as that in the prototype.

(e) Find the required wind-tunnel speed and stack exit velocity if the density ratio in the model, $\alpha_m = 0.2$.

(f) Using the density ratio from part (e), if the maximum glc in the model simulation is 1200 ppm for a tracer release rate of 10 ml s^{-1}, find the maximum glc at full-scale.

(g) Suggest a reason for using a model density ratio different from that at full scale. (There are hints in the question.)

(h) Suggest how highly-buoyant stack emissions may be produced safely in the laboratory.

Q10.
Subsea pipelines are usually laid from a barge onto the sea bed as shown in the figure below. A certain amount of tension is applied to the pipe at the barge during pipelaying. The length, l, of the pipe from the barge to the touchpoint on the sea bed depends on the depth of water, d, the submerged weight per unit length of the pipe, W_s, the tension force applied to the pipe, τ, and the flexural rigidity, EI, of the pipe; E is the Young's modulus of the pipe material and I is the second moment of area of the pipe.

(a) Show by dimensional analysis that

$$\frac{l}{d} = f\left(\frac{EI}{W_s d^3}, \frac{\tau}{W_s d}\right)$$

A model study is carried out in the laboratory in air to determine the length of the pipe from the barge to touchdown point on the sea bed. A solid flexible plastic pipe is used in the model to simulate the subsea pipe. The floor of the laboratory represents the sea bed. The following are the values of various quantities of the prototype:

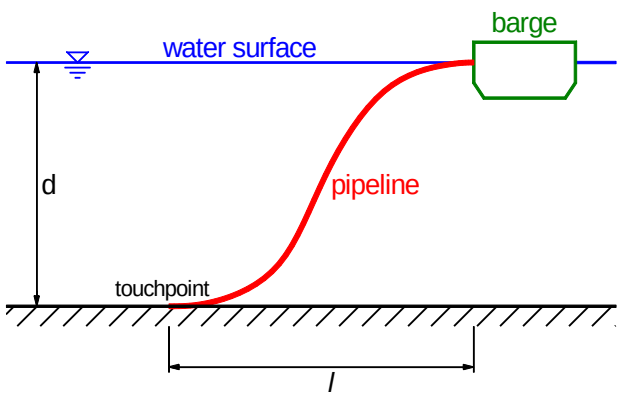
Young's modulus, $E = 200$ GN m^{-2};
Second moment of area, $I = 10^{-2}$ m^4;
Submerged weight of the pipe, $W_s = 1.0$ kN m^{-1};
Depth of water, $d = 250$ m;
Tension applied to the pipe = 10 kN.

The diameter of the plastic pipe chosen in the model is 10.0 mm. Its density is 1500 kg m^{-3} and its Young's modulus is 3.0 GN m^{-2}. The second moment of area of the pipe, I, is $\pi D^4/64$, where D is the diameter of the plastic pipe.

(b) What is the vertical height of the model pipe between the barge and the floor of the laboratory?

(c) Determine the tension that should be applied to the plastic pipe of the model for similarity.

Neglect the density of air in your calculations.



Q11. (Exam 2023)
A tidal barrage is to be constructed across a major river estuary. The power output, P, of each turbine is a function of the water-level difference between the two sides of the barrage, h, the acceleration due to gravity, g, the turbine diameter, D, its angular rotation rate, Ω, and the water density, ρ, and dynamic viscosity, μ.

(a) Using Buckingham's Pi Theorem, show that

$$\frac{P}{\rho(gh)^{3/2}D^2} = f\left(\frac{h}{D}, \frac{\Omega D}{\sqrt{gh}}, \frac{\rho D\sqrt{gh}}{\mu}\right)$$

(b) Under what conditions can the dependence on the last parameter above be neglected?

A physical model of the turbines and barrage is to be tested at 1/10th geometric scale in a laboratory.

(c) If the angular velocity of the prototype is 300 rpm, what should the angular velocity be at model scale?

(d) The power output at model scale is found to be 125 W. What would it be at full scale?

(e) State some major advantages and disadvantages of a tidal barrage for energy production.

Q12. (Exam 2020)
The power output, P, of a particular design of hydraulic turbine depends on the quantity of flow, Q, pressure drop, ρgH, angular velocity, N, runner diameter, D, fluid density, ρ and viscosity, μ.

(a) Conduct a formal dimensional analysis to show that

$$\frac{P}{\rho N^3 D^5} = f\left(\frac{Q}{ND^3}, \frac{gH}{N^2H^2}, \frac{\rho ND^2}{\mu}\right)$$

(b) Show that the non-dimensional relationship can also be written

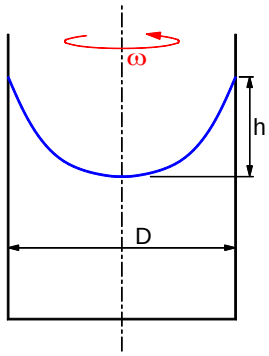
$$\frac{P}{\rho N^3 D^5} = f\left(\eta, \frac{gH}{N^2H^2}, \frac{\rho ND^2}{\mu}\right)$$

where η is the overall efficiency of the device.

(c) A proposed turbine is to operate under an available head of 80 m. A 1:5 scale model is tested under a constant head of 5 m and generates 800 W at a flow rate of 0.3 m^3 s^{-1} and angular velocity 200 rpm. Assuming Reynolds-number independence, calculate the corresponding angular velocity, quantity of flow and power output in the full-scale prototype.

Q13. (Exam 2018)

The vertical displacement, h, of a liquid in a rotating cylindrical vessel is a function of the diameter, D, and angular velocity, ω, of the vessel, the acceleration due to gravity, g, and the density, ρ, viscosity, μ, and surface tension, σ, of the fluid.



(a) Using dimensional analysis, show that

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$$

where

$$\Pi_1 = \frac{h}{D}, \qquad \Pi_2 = \frac{\rho\omega D^2}{\mu}, \qquad \Pi_3 = \omega\sqrt{\frac{D}{g}}, \qquad \Pi_4 = \frac{\rho\omega^2 D^3}{\sigma}$$

(b) Name the parameters Π_2 and Π_4 and interpret them in terms of forces.

Two laboratory experiments are conducted using two different fluids. In the first experiment a multigrade oil is used, with kinematic viscosity $3.1 \times 10^{-6}\ \mathrm{m^2\,s^{-1}}$ at the laboratory conditions. In the second experiment the working fluid is water, with kinematic viscosity $1.0 \times 10^{-6}\ \mathrm{m^2\,s^{-1}}$. The vessel diameter and rotational speed in the first experiment are 300 mm and 120 rpm, respectively.

(c) Assuming that in both experiments the effect of Π_4 can be considered negligible, determine (i) the vessel diameter, and (ii) the rotational speed in the second experiment if the flows are to be similar.

Q14.

In turbulent flow next to a heated wall the mean velocity gradient, $\mathrm{d}u/\mathrm{d}y$, is assumed to depend only on the distance, y, from the boundary, the wall shear stress, τ_w, the surface kinematic buoyancy flux, B, and the fluid density, ρ. (See the note below for a definition of B.)

(a) Perform a dimensional analysis to show that

$$\frac{y}{u_\tau}\frac{\mathrm{d}u}{\mathrm{d}y} = f\left(\frac{y}{L_{MO}}\right)$$

where $u_\tau = (\tau_w/\rho)^{1/2}$ is the friction velocity and $L_{MO} = u_\tau^3/(-\kappa B)$ is the Monin-Obukhov length. (κ is von Kármán's constant – just a number, albeit a famous one.)

(b) Derive the mathematical shape of the mean-velocity profile in the case of an adiabatic boundary layer ($B = 0$).

Note: B is related to the surface heat flux by $B = \alpha g q_H/\rho c_p$, where α is the coefficient of thermal expansion, g is the acceleration due to gravity, q_H is the surface heat flux per unit area and c_p is the specific heat capacity. The analysis is useful in describing the atmospheric boundary layer.

Q15.

In a heat exchanger the average heat flux per unit area, q_H, is a function of a characteristic length, L, the surface excess temperature, ΔT, the cross-flow velocity, V, and the fluid density, ρ, viscosity, μ, conductivity, k, and specific heat capacity, c_p.

(a) By conducting a formal dimensional analysis, show that there is a non-dimensional relationship of the form

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr}, \mathrm{Ec})$$

where

Nusselt number $\quad \mathrm{Nu} = \dfrac{q_H}{k\Delta T/L}$

Reynolds number $\quad \mathrm{Re} = \dfrac{\rho V L}{\mu}$

Prandtl number $\quad \mathrm{Pr} = \dfrac{\mu c_p}{k}$

Eckert number $\quad \mathrm{Ec} = \dfrac{V^2}{c_p \Delta T}$

(b) At sufficiently large Reynolds number the heat flux becomes independent of the molecular properties μ and k. By appropriate combination of the dimensionless groups above (and <u>not</u> by restarting the dimensional analysis from scratch), show that, in this regime, there is a non-dimensional relationship of the form

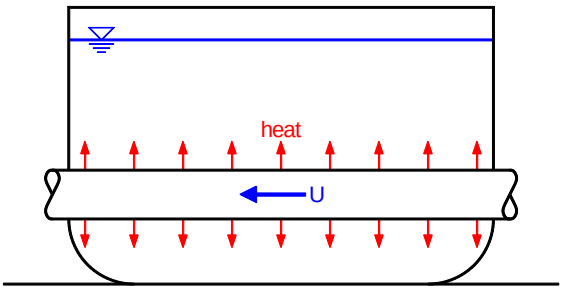
$$\frac{q_H}{\rho c_p V \Delta T} = f\left(\frac{V^2}{c_p \Delta T}\right)$$

(c) The crossflow velocity in a prototype heat exchanger is 10 m s^{-1} and operates with an excess temperature of 200 K. Assuming the fully-turbulent flow regime of part (b), what excess temperature should be used in a laboratory simulation using the same fluid at velocity 5 m s^{-1}? (Assume that the fluid properties ρ and c_p are unchanged).

(d) If the measured heat flux per unit area in the laboratory is 20 W m^{-2}, what would be the value in the prototype?

Q16. (Exam 2017)

A calorifer is a device in which heat is transferred from a liquid flowing in a tube to another liquid in a tank and to the outside through the walls of the tank (see figure). Under a few simplifying assumptions, the heat-transfer coefficient of a calorifer, α, is a function only of the fluid specific heat capacity, c, the fluid flow velocity, U, the pipe diameter, D, the fluid thermal conductivity, k, the fluid density, ρ, and the dynamic viscosity, μ.



(The heat-transfer coefficient is defined as the rate of heat transfer per unit area per unit temperature difference. You should know what the other thermal quantities are.)

(a) Using dimensional analysis, show that

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

where

$$\Pi_1 = \frac{\alpha D}{k}, \qquad \Pi_2 = \frac{c\mu}{k}, \qquad \Pi_3 = \frac{\rho U D}{\mu}$$

(b) Name the parameter Π_3 and interpret it in terms of forces.

A 1:10 scale model is to be used to predict the heat-transfer coefficient of a prototype calorifer with pipe diameter $D_p = 500$ mm and flow rate $Q_p = 80\ \mathrm{L\ s^{-1}}$. The fluid used in the model and the prototype is water.

(c) Determine the flow rate in the model if the flows are to be similar.

(d) The following empirical relationship has been determined from lab experiments:

$$\Pi_1 = 0.023(\Pi_2)^{0.4}(\Pi_3)^{0.8}$$

Calculate the heat-transfer coefficient in the prototype calorifer, α_p, if the thermal conductivity of water is $k = 0.6\ \mathrm{W\ m^{-1}\ K^{-1}}$, the specific heat capacity is $c = 4187\ \mathrm{J\ kg^{-1}\ K^{-1}}$, the density is $\rho = 1000\ \mathrm{kg\ m^{-3}}$, and the dynamic viscosity at the operating temperature is $\mu = 1 \times 10^{-3}$ Pa s.

Q17. (Exam 2021)
The safety of nuclear and chemical plants depends on the knowledge of the rise velocity of "slugs" of gas in liquid-containing pipes. The rise velocity, U, of such bubbles is assumed to depend on pipe diameter, D, gravitational acceleration, g, liquid density, ρ, viscosity, μ, and the surface tension, σ.

(a) Use dimensional analysis to show that

$$\mathrm{Fr} = f(\mathrm{Rb}, \mathrm{Eo})$$

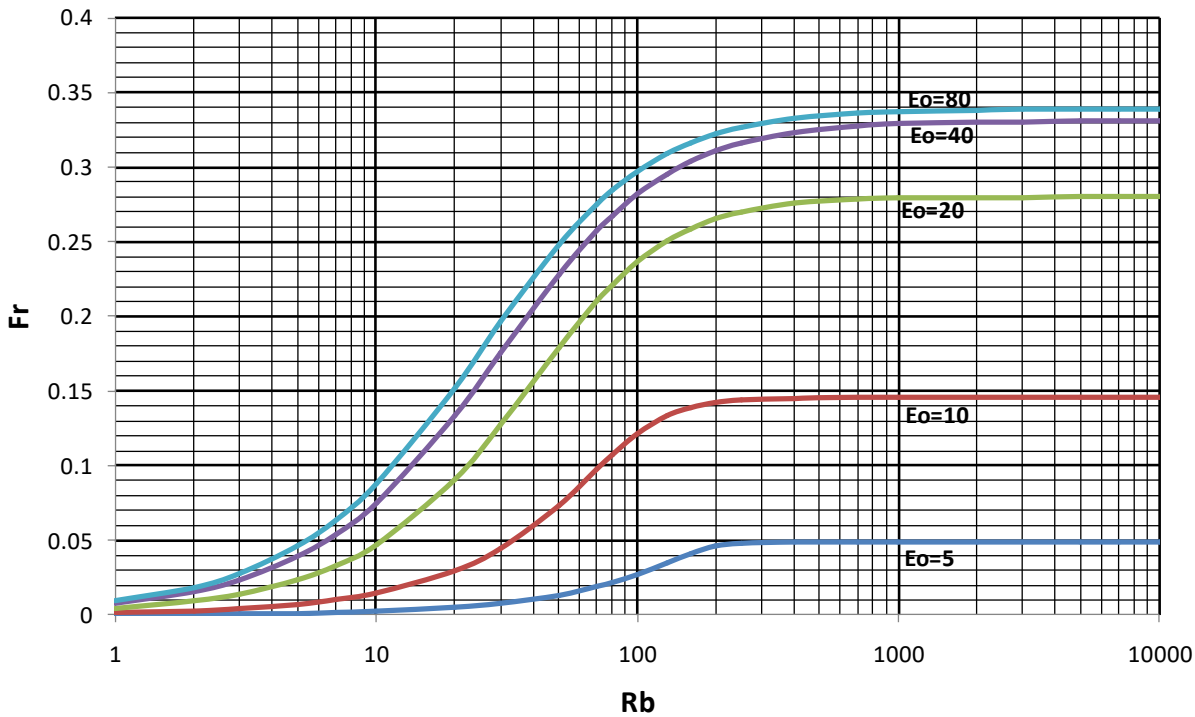
where

$$\mathrm{Fr} = \frac{U}{\sqrt{gD}} \qquad \text{(Froude number)}$$

$$\mathrm{Rb} = \frac{\rho\sqrt{gD^3}}{\mu} \qquad \text{(buoyancy Reynolds number)}$$

$$\mathrm{Eo} = \frac{\rho g D^2}{\sigma} \qquad \text{(Eötvös number)}$$

(b) Use the figure below (developed from the correlations of Viana et al., 2003, J. Fluid Mechanics, vol. 494, pp 379-398) to deduce the rise velocity of gas bubbles in pipes of diameter 13 mm carrying oil of density 950 kg m^{-3}, dynamic viscosity 0.022 Pa s and surface tension 0.0788 N m^{-1}.



Q18. (Exam 2022)
A tensioned wire strung between two fixed points will vibrate when subjected to a cross-flow of fluid. The frequency of vibration, f, is a function of the flow speed, U, fluid density, ρ, and dynamic viscosity, μ, and the wire diameter, D, length, L, tension, P, and solid density, ρ_s.

(a) Use Buckingham's Pi Theorem to show that

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4, \Pi_5)$$

where:

$$\Pi_1 = \frac{fD}{U}, \qquad \Pi_2 = \frac{\rho U D}{\mu}, \qquad \Pi_3 = \frac{\sqrt{P/\rho_s}}{UL}, \qquad \Pi_4 = \frac{L}{D}, \qquad \Pi_5 = \frac{\rho_s}{\rho}$$

A 1/10th-scale model is tested in the laboratory using the same fluid and wire material as proposed at full scale.

(b) If the cross-flow velocity is 3 m s^{-1} and the tension in the wire at full scale is 3 kN, what velocity and tension should be used at model scale for similarity?

(c) If the measured frequency of vibration at model scale is 800 Hz, what would be the frequency of vibration at full scale?

Q19.
A problem contains 5 variables: f (frequency), L and D (both lengths), U and W (both velocities). This gives $n = 5$ variables, $m = 2$ dimensions (length and time), so there must be 3 independent dimensionless groups.

A good solution to this problem ends up with non-dimensional groups fD/U, L/D, U/W. However, a legitimate (if not very good) solution is:

$$\Pi_1 = \frac{fD}{U}, \qquad \Pi_2 = \frac{fL}{W}, \qquad \Pi_3 = \frac{U}{W} \qquad (*)$$

(a) A student claims that, since L and D have the same dimensions, and W and U have the same dimensions, he can simply swap L for D and U for W in Π_2 in the solution (*). What is wrong with this argument?

(b) Another student claims that, since L and D have the same dimensions, she can simply swap L for D in Π_2 in solution (*). What is wrong with this argument?