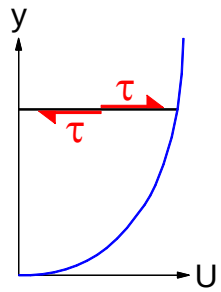
8. TURBULENCE MODELLING[1] SPRING 2024

The *Reynolds-averaged Navier-Stokes* (RANS) equations are transport equations for the *mean* variables in a turbulent flow. These equations contain net fluxes due to turbulent fluctuations. Turbulence models are needed to specify these fluxes.

8.1 Eddy-Viscosity Models

8.1.1 The Eddy-Viscosity Hypothesis



The mean shear stress has both viscous and turbulent parts. In *simple shear* (i.e. where $\partial U/\partial y$ is the only non-zero mean gradient):

$$\tau = \underbrace{\mu \frac{\partial U}{\partial y}}_{\text{viscous}} \quad \underbrace{-\rho\overline{uv}}_{\text{turbulent}} \tag{1}$$

The most popular type of turbulence model is an *eddy-viscosity model* (EVM) which assumes that turbulent stress is proportional to mean-velocity gradient in a manner similar to viscous stress. In simple shear (see later for the general case):

$$-\rho\overline{uv} = \mu_t \frac{\partial U}{\partial y} \tag{2}$$

μ_t is called an *eddy viscosity* or *turbulent viscosity*. The overall mean shear stress is then

$$\tau = \mu_{\text{eff}} \frac{\partial U}{\partial y} \tag{3}$$

where the total effective viscosity

$$\mu_{\text{eff}} = \mu + \mu_t \tag{4}$$

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[1] More advanced descriptions of turbulence and its modelling can be found in:
Leschziner, M.A., 2015, *Statistical turbulence modelling for fluid dynamics - demystified: an introductory text for graduate engineering students*, World Scientific.
Wilcox, D.C., 2006, *Turbulence Modelling for CFD*, 3rd Edition, DCW Industries.
Pope, S.B., 2000, *Turbulent flows*, Cambridge University Press.
Schlichting, H. and Gersten, K., 1999, *Boundary layer theory*, 8th English Edition, Springer-Verlag.

Note:
(1) This is a model!
(2) μ is a *physical* property of the *fluid* and can be measured;
μ_t is a *hypothetical* property of the *flow* and must be modelled.
(3) μ_t varies with position.
(4) At high Reynolds numbers, $\mu_t \gg \mu$ throughout much of the flow.

Eddy-viscosity models are widely used and popular because:
- they are easy to implement in existing viscous solvers;
- extra viscosity aids stability;
- they have some theoretical foundation in simple shear flows (see below).

However:
- there is little theoretical foundation in complex flows;
- modelling turbulent transport is reduced to a single scalar, μ_t, and, hence, at most one Reynolds stress can be represented accurately.

8.1.2 The Eddy Viscosity in the Log-Law Region

In the log-law region of a turbulent boundary layer it is assumed that:
(a) (i) total stress is constant (and equal to that at the wall);
(ii) viscous stress is negligible compared to turbulent stress:

$$\tau^{(\text{turb})} = \tau_w \equiv \rho u_\tau^2$$

(b) the mean velocity profile is logarithmic;

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

The eddy viscosity is then

$$\mu_t \equiv \frac{\tau^{(\text{turb})}}{\partial U/\,\partial y} \quad = \frac{\rho u_\tau^2}{u_\tau/\kappa y} \quad = \rho(\kappa u_\tau y)$$

Hence, in the log-law region, with $\nu_t = \mu_t/\rho$ as the *kinematic eddy viscosity*,

$$\boxed{\nu_t = \kappa u_\tau y \qquad (5)}$$

In the log-law region the eddy viscosity is *proportional to distance from the boundary*.

8.1.3 General Stress-Strain Relationship

The stress-strain relationship (2) applies only in simple shear and cannot hold in general because the LHS is symmetric in x and y components but the RHS is not. The appropriate generalisation gives representative shear and normal stresses (from which others can be obtained by "pattern-matching"):

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \tag{6}$$

$$-\rho\overline{u^2} = 2\mu_t\frac{\partial U}{\partial x} \quad -\frac{2}{3}\rho k \tag{7}$$

The $-\frac{2}{3}\rho k$ part (k is turbulent kinetic energy) in (7) ensures the correct sum of normal stresses:

$$-\rho\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) = -2\rho k$$

because the mean-velocity terms from this sum would be zero in incompressible flow:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

Using suffix notation both shear and normal stresses can be summarised in a single formula:

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij} \tag{8}$$

8.1.4 Other Turbulent Fluxes

Reynolds' *analogy* assumes a gradient-diffusion relationship between the turbulent flux of any conserved quantity and the gradient of the corresponding mean concentration; i.e.

$$-\rho\overline{v\phi} = \Gamma_t\frac{\partial \Phi}{\partial y} \tag{9}$$

with *turbulent diffusivity* Γ_t proportional to the eddy viscosity:

$$\Gamma_t = \frac{\mu_t}{\sigma_t} \tag{10}$$

σ_t is called a *turbulent Prandtl number*. Since the same turbulent eddies are responsible for transporting momentum and other scalars, σ_t is approximately 1.0 for most variables.

8.1.5 Specifying the Eddy Viscosity

The *kinematic* eddy viscosity has dimensions of [velocity] × [length], which suggests that it be modelled as

$$\nu_t = u_0 l_0 \tag{11}$$

Physically, u_0 should reflect the magnitude of velocity fluctuations and l_0 the size of turbulent eddies. For example, in the log-law region, $\nu_t = \kappa u_\tau y$, or

$$\nu_t = \text{velocity } (u_\tau) \ \times \ \text{length } (\kappa y)$$

For wall-bounded flows a candidate for u_0 is the friction velocity, $u_\tau = \sqrt{\tau_w/\rho}$. However, this

is not a *local* scale, since it depends on where the nearest wall is, and it is ambiguous near corners, since it can be equidistant from two (or more) walls. A more appropriate velocity scale in general is $k^{1/2}$, where k is the turbulent kinetic energy.

For simple wall-bounded flows, l_0 is proportional to distance from the boundary (e.g. $l_0 = \kappa y$). For simple free shear flows (e.g. jet, wake, mixing layer) l_0 is proportional to the width of the shear layer. However, these values are geometry-dependent and not conducive to complex geometries. For greater generality, we need to relate l_0 to local turbulence properties.

Common practice is to solve transport equations for one or more turbulent quantities (usually k + one other), from which μ_t can be derived on dimensional grounds. The following classification of eddy-viscosity models is based on the number of transport equations.

zero-equation models:
– constant-eddy-viscosity models;
– mixing-length models: l_0 specified geometrically; u_0 from mean flow gradients.
one-equation models:
– l_0 specified geometrically; transport equation to derive u_0;
two-equation models:
– transport equations for quantities from which u_0 and l_0 can be derived.

Of these, by far the most popular in general-purpose CFD are two-equation models: notably the $k - \varepsilon$ and $k - \omega$ models.

8.1.6 Mixing-Length Models (Prandtl, 1925).

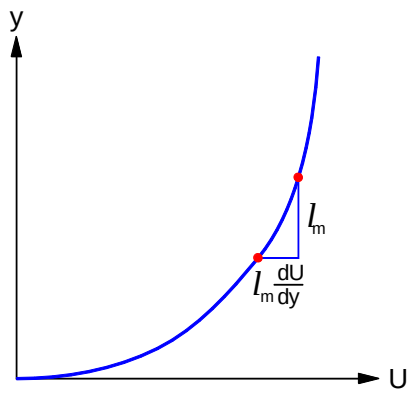
Eddy viscosity:

$$\mu_t = \rho \nu_t, \qquad \text{where} \qquad \nu_t = u_0 l_m \tag{12}$$

The *mixing length* l_m is specified geometrically and the velocity scale u_0 is then determined from the mean-velocity gradient. In simple shear:

$$u_0 = l_m \left| \frac{\partial U}{\partial y} \right| \tag{13}$$

The model is based on the premise that if a turbulent eddy displaces a fluid particle by distance l_m its velocity will differ from its surrounds by an amount $l_m |\partial U / \partial y|$. (Any constant of proportionality can be absorbed into the definition of l_m.)



The resulting turbulent shear stress is (assuming positive velocity gradient):

$$\tau^{(\text{turb})} = \mu_t \frac{\partial U}{\partial y} \quad = \rho u_0 l_m \left(\frac{\partial U}{\partial y} \right) \quad = \rho l_m^2 \left(\frac{\partial U}{\partial y} \right)^2 \tag{14}$$

The mixing length l_m depends on the type of flow.

Log Layer

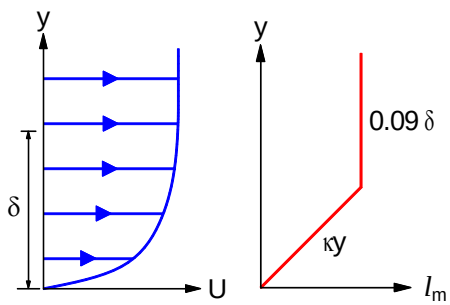
In the log layer,

$$\tau^{(\mathrm{turb})} = \rho u_\tau^2 \qquad \text{and} \qquad \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$$

Equation (14) then implies that

$$l_m = \kappa y$$

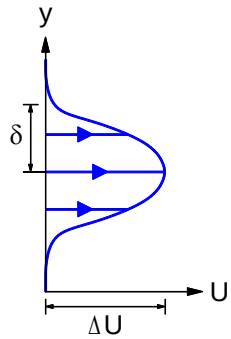
General Wall-bounded flows



In general, l_m is limited to a certain fraction of the boundary-layer depth δ. Cebeci and Smith (1974) suggest:

$$l_m = \min(\kappa y, 0.09\delta) \tag{15}$$

Free shear flows



For *free* shear layers (no wall boundary), l_m is assumed proportional to the local shear-layer half-width δ. Wilcox (2006) suggests:

$$\frac{l_m}{\delta} = \begin{cases} 0.071 & \text{(mixing layer)} \\ 0.098 & \text{(plane jet)} \\ 0.080 & \text{(round jet)} \\ 0.180 & \text{(plane wake)} \end{cases} \tag{16}$$

Mixing-length models work well in near-equilibrium boundary layers or very simple free-shear flows. However, although generalisations of the stress-strain relationship (13) exist for arbitrary velocity fields, it is difficult to specify the mixing length l_m for complex flows, which do not fit tidily into one of the above geometrically-simple categories.

8.1.7 The $k - \varepsilon$ Model

This is probably the most common type of turbulence model in use today. It is a two-equation eddy-viscosity model with the following specification:

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{17}$$

k is the turbulent kinetic energy and ε is its rate of dissipation. In the standard model C_μ is a constant (with value 0.09).

k and ε are determined by solving transport equations. For the record (i.e. you don't have to learn them) they are given here in conservative differential form, including implied summation over repeated index i.

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_i}\left(\rho U_i k - \Gamma^{(k)} \frac{\partial k}{\partial x_i} \right) = \rho(P^{(k)} - \varepsilon)$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_i}\left(\rho U_i \varepsilon - \Gamma^{(\varepsilon)} \frac{\partial \varepsilon}{\partial x_i} \right) = \rho(C_{\varepsilon 1} P^{(k)} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} \qquad (18)$$

rate of change — advection — diffusion — source

$P^{(k)}$ is the rate of production of turbulent kinetic energy k (see below).

The diffusivities of k and ε are related to the molecular and turbulent viscosities:

$$= \mu + \frac{\mu_t}{\sigma_k}\ , \qquad \Gamma^{(\varepsilon)} = \mu + \frac{\mu_t}{\sigma_\varepsilon}$$

and, in the standard model (Launder and Spalding, 1974), model constants are:

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3 \qquad (19)$$

Notes.

(1) The $k - \varepsilon$ model is not a single model but a class of different schemes. Variants have different coefficients, some including dependence on molecular-viscosity effects near boundaries ("low-Reynolds-number $k - \varepsilon$ models") and/or mean velocity gradients (e.g. "*realisable*" $k - \varepsilon$ models). Others have a different ε equation.

(2) Apart from the diffusion term, the k equation is that derived from the Navier-Stokes equation. The ε equation is, however, heavily modelled.

(3) Although k is a logical choice (as it has a physical definition and can be measured), use of ε as a second scale is not universal and other combinations such as $k - \omega$ (ω is a frequency), $k - \tau$ (τ is a timescale) or $k - l$ (l is a length) may be encountered. A popular hybrid of $k - \omega$ and $k - \varepsilon$ models is the SST $k - \omega$ model of Menter (1994).

<u>Rate of Production of Turbulent Kinetic Energy</u>

The source term in the k equation is a balance between *production* $P^{(k)}$ and *dissipation* ε. The rate of production of turbulent kinetic energy (per unit mass) $P^{(k)}$ is given in simple shear by

$$P^{(k)} = -\overline{uv}\frac{\partial U}{\partial y} = \nu_t \left(\frac{\partial U}{\partial y}\right)^2 \qquad (20)$$

or, in general, by

$$P^{(k)} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \qquad (21)$$

with implied summation over the repeated indices i and j. Under the eddy-viscosity assumption for the Reynolds stress, $P^{(k)}$ is invariably positive. (*Exercise*: prove this).

A flow for which $P^{(k)} = \varepsilon$ (production equals dissipation) is said to be in *local equilibrium*.

Values of the Model Constants

Some of the constants in the $k-\varepsilon$ model may be chosen for consistency with the log law and available experimental data.

In the fully-turbulent region the Reynolds stresses are assumed to dominate the total stress ($\tau = -\rho\overline{uv}$), whilst in a fully-developed boundary layer the total stress is constant and equal to that at the wall ($\tau = \tau_w \equiv \rho u_\tau^2$). Hence, the kinematic shear stress is

$$-\overline{uv} = u_\tau^2 \tag{22}$$

In the log-law region, the mean-velocity gradient is

$$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y} \tag{23}$$

so that, from equation (20) the rate of production of turbulent kinetic energy is

$$P^{(k)} = -\overline{uv}\frac{\partial U}{\partial y} \quad = u_\tau^2 \times \frac{u_\tau}{\kappa y} \quad = \frac{u_\tau^3}{\kappa y} \tag{24}$$

In the log-law region, we have already established that the kinematic eddy viscosity is

$$\nu_t = \kappa u_\tau y \tag{25}$$

so that, with the further assumption of local equilibrium, $P^{(k)} = \varepsilon$, equations (24) and (25) give

$$\nu_t = \frac{u_\tau^4}{\varepsilon}$$

Comparing this with the $k-\varepsilon$ eddy-viscosity formula (17):

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

leads to

$$C_\mu = \frac{u_\tau^4}{k^2} \quad = \left(\frac{-\overline{uv}}{k}\right)^2 \qquad \text{or} \qquad u_\tau = C_\mu^{1/4} k^{1/2} \tag{26}$$

A typical experimentally-measured ratio is $-\overline{uv}/k = 0.3$, giving the standard value $C_\mu = 0.09$.

In addition, the high-Reynolds-number (viscosity μ negligible) form of the ε equation (18) is consistent with the log law provided the constants satisfy (see the examples overleaf):

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2 \tag{27}$$

In practice, the standard constants do not quite satisfy this, but have values calibrated to give better agreement over a wide range of flows.

Classroom Example 1

(a) The $k-\varepsilon$ turbulence model forms an eddy viscosity μ_t from fluid density ρ, the turbulent kinetic energy (per unit mass) k and its dissipation rate ε. Write down the basic physical dimensions of μ_t, ρ, k and ε in terms of the fundamental dimensions of mass M, length L and time T, and hence show, on purely dimensional grounds, that any expression for μ_t in terms of the other variables must be of the form

$$\mu_t = \text{constant} \times \rho \frac{k^2}{\varepsilon}$$

(b) The $k-\omega$ turbulence model forms an eddy viscosity from ρ, k and a quantity ω which has dimensions of frequency (i.e. T^{-1}). Show, on dimensional grounds, that any expression for μ_t in terms of the other variables must be of the form

$$\mu_t = \text{constant} \times \rho \frac{k}{\omega}$$

Classroom Example 2

A modeled scalar-transport equation for ε is

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + \left(C_{\varepsilon 1}P^{(k)} - C_{\varepsilon 2}\varepsilon\right)\frac{\varepsilon}{k}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative, $P^{(k)}$ is the rate of production of k and the summation convention is implied by the repeated index i. σ_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are constants.

In a fully-developed turbulent boundary layer,

$$P^{(k)} = \varepsilon = \frac{u_\tau^3}{\kappa y} \qquad \text{and} \qquad k = C_\mu^{-1/2} u_\tau^2$$

where κ is von Kármán's constant, u_τ is the friction velocity and y is the distance from the boundary. Show that this implies the following relationship between coefficients:

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2$$

Classroom Example 3

In grid-generated turbulence there is no mean velocity shear, and hence no turbulence production and minimal diffusion. The k and ε transport equations reduce to

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\varepsilon, \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

where t is the travel time downstream of the grid (distance/mean velocity). By substituting into these equations, show that they admit a solution of the form

$$k = k_0 t^{-m} \qquad \varepsilon = \varepsilon_0 t^{-n}$$

where k_0, ε_0, m and n are constants, and find $C_{\varepsilon 2}$ in terms of m alone. (This rate of decay for k provides a means of determining $C_{\varepsilon 2}$ experimentally.)

8.2 Advanced Turbulence Models

Eddy-viscosity models are popular because:

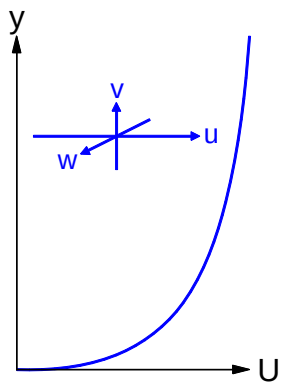
- they are simple to code;
- extra viscosity aids stability;
- they are supported theoretically in some simple but common types of flow;
- they are very effective in many engineering flows.

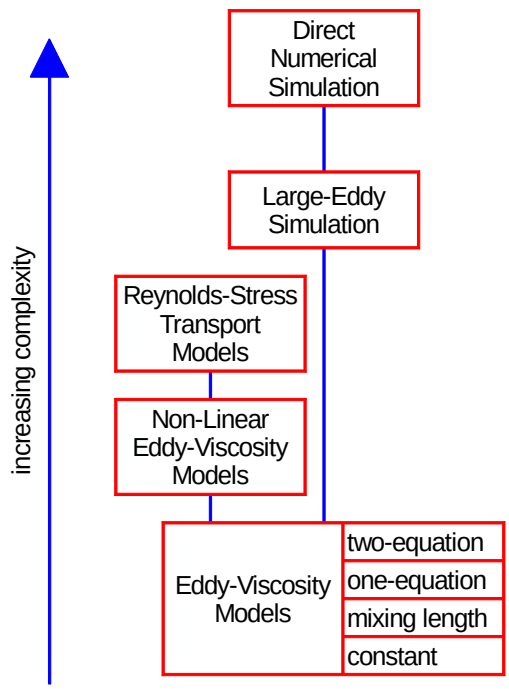
However, the dependence of a turbulence model on a single scalar μ_t is clearly untenable when more than one stress component has an effect on the mean flow. The eddy-viscosity model fails to represent turbulence physics, particularly in respect of the different rates of production of the different Reynolds stresses and the resulting anisotropy.

A classic example occurs in a simple fully-developed boundary-layer where, in the logarithmic region, the various normal stresses are typically in the ratio

$$\overline{u^2}:\overline{v^2}:\overline{w^2} = 1.0:0.4:0.6 \qquad (28)$$

An eddy-viscosity model would, however, predict all of these to be equal (to $\frac{2}{3}k$).





More advanced types of turbulence model (some of which have a proud history at the University of Manchester) are shown left, with a brief overview below. (A more advanced description can be found in the references at the end of this section and in the optional Section 10.)

8.2.1 Reynolds-Stress Transport Models (RSTM)[2]

Also known as *second-order closure* or *differential stress models* these solve transport equations for *all* stresses, $\overline{u^2}$, $\overline{uv}$ etc., rather than just turbulent kinetic energy k.

Exact equations for stresses $\overline{u_i u_j}$ can be derived from the Navier-Stokes equations and are of the usual canonical form:

rate of change + advection + diffusion = source

but certain terms have to be modelled. The most important balance is in the "source" term, which, for $\overline{u_i u_j}$, consists of parts that can be identified as:

production of energy by mean-velocity gradients, P_{ij};
redistribution of energy amongst different components by pressure fluctuations, Φ_{ij};
dissipation of energy by viscosity, ε_{ij}.

The important point is that, in this type of model, both advection and production terms are exact. Thus, the terms that supply energy to a particular Reynolds-stress component don't need modelling. For example, the rate of production of $\overline{u^2}$ per unit mass is:

$$P_{11} = -2\left(\overline{u^2}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z}\right)$$

Assessment.
For:

- The "energy-in" terms (advection and production) are exact, not modelled; thus, RSTMs should take better account of turbulence physics (in particular, anisotropy) than eddy-viscosity models.

Against:

- Models are very complex;
- Many important terms (notably redistribution and dissipation) require modelling;
- Models are computationally expensive (6 turbulent transport equations) and tend to be less stable; (only the small molecular viscosity contributes to any sort of gradient diffusion).

[2] The classic reference here is:
Launder, B.E., Reece, G.J. and Rodi, W., 1975, Progress in the development of a Reynolds-stress turbulence closure, J. Fluid Mech., 68, 537-566.

8.2.2 Non-Linear Eddy-Viscosity Models (NLEVM)[3]

A "half-way house" between eddy-viscosity and Reynolds-stress transport models, the idea behind this type of model is to extend the simple proportionality between Reynolds stresses and mean-velocity gradients:

$$\text{stress} \propto (\text{velocity gradient})$$

to a non-linear constitutive relation:

$$\text{stress} = C_1(\text{velocity gradient}) + C_2(\text{velocity gradient})^2 + C_3(\text{velocity gradient})^3 + \ldots$$

(The actual relationship is tensorial and highly mathematical – see the optional Section 10).

Models can be constructed so as to reproduce the correct anisotropy (28) in simple shear flow, as well as a qualitatively-correct response of turbulence to certain other types of flow: e.g. curved flows. Experience suggests that a cubic stress-strain relationship is optimal.

Assessment.
For:

- produce qualitatively-correct turbulent behaviour in certain important flows;
- only slightly more computationally expensive than linear eddy-viscosity models.

Against:

- don't accurately represent the real production and advection processes;
- little theoretical foundation in complex flows.

8.2.3 Large-Eddy Simulation (LES)

Resolving a full, time-dependent turbulent flow at large Reynolds number is impractical as it would require huge numbers of control volumes, all smaller than the tiniest scales of motion. *Large-eddy simulation* solves the time-dependent Navier-Stokes equations for the instantaneous (mean + turbulent) velocity that it can resolve on a moderate size of grid and models the subgrid-scale motions. The model for the latter is usually very simple, typically a mixing-length-type model with l_m proportional to the mesh size.

8.2.4 Direct Numerical Simulation (DNS)

This is not a turbulence model. It is an accurate solution of the complete time-dependent, Navier-Stokes equations without any modelled terms!

This is prohibitively expensive at large Reynolds numbers as huge numbers of grid nodes would be needed to resolve all scales of motion. Nevertheless, supercomputers have extended the Reynolds-number range to a few thousand for simple flows and these results have assisted greatly in the understanding of turbulence physics and calibration of simpler models.

[3] For some of the mathematical theory see:
Apsley, D.D. and Leschziner, M.A., 1998, A new low-Reynolds-number nonlinear two-equation turbulence model for complex flows, Int. J. Heat Fluid Flow, 19, 209-222

8.3 Wall Boundary Conditions

At walls the no-slip boundary condition applies, so that both mean and fluctuating velocities vanish. At high Reynolds numbers this presents three problems:

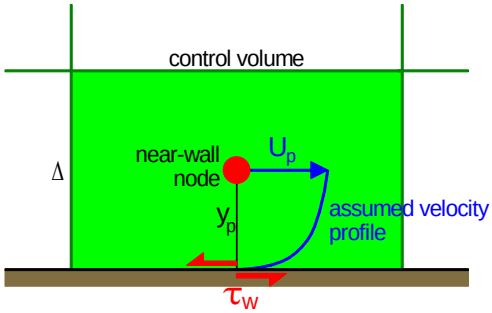
- there are large velocity gradients (which require many grid points to resolve);
- wall-normal fluctuations are selectively damped (not accounted for in many models);
- viscous and turbulent stresses are of comparable magnitude (breaking an assumption used to derive many models).

There are two main ways of handling this in turbulent flow.

(1) *Low-Reynolds-number turbulence models*
Resolve the flow right up to solid boundaries (typically to $y^+ < 1$). This requires a very large number of cells in the direction perpendicular to the boundary and special viscosity-dependent modifications to the turbulence model.

(2) *Wall functions*
Don't resolve the near-wall flow completely, but assume theoretical profiles between the near-wall node and the surface. This doesn't require as many cells, but the theoretical profiles used are really only justified in near-equilibrium boundary layers.

8.3.1 Wall Functions[4]



The momentum balance for the near-wall cell requires the wall shear stress τ_w $(= \rho u_\tau^2)$. Because the near-wall region isn't resolved, this requires some assumption about what goes on between the near-wall node and the surface.

If the near-wall node lies in the logarithmic region of a smooth wall then

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(E y_P^+), \qquad y_P^+ = \frac{y_P u_\tau}{\nu}, \qquad \nu_t = \kappa u_\tau y \tag{29}$$

Subscript P denotes the near-wall node.

Given U_P and y_P this could be solved (iteratively) for u_τ and hence the wall stress τ_w. This is fine if the boundary layer is near equilibrium. However, it will fail or give $u_\tau = 0$ near a separation or reattachment point, where $U_p = 0$, even though turbulence is actually quite large there. This does not pose a major problem for the mean velocity, but is poor for the k equation and disastrous in heat-transfer calculations. If a transport equation is being solved for k, a better approach when the turbulence is not in equilibrium (e.g. near separation or reattachment points) is to estimate an "effective" friction velocity from the relationship that holds in the log-layer (equation (26)):

[4] For an advanced discussion of wall functions (including rough- rather than smooth-walled boundaries) see: Apsley, D.D., 2007, *CFD calculation of turbulent flow with arbitrary wall roughness*, Flow, Turbulence and Combustion, 78, 153-175.

$$u_0 = C_\mu^{1/4} k_P^{1/2} \tag{30}$$

and derive the relationship between U_p and τ_w by assuming a kinematic eddy viscosity

$$\nu_t = \kappa u_0 y \tag{31}$$

Then, assuming constant stress ($\tau = \tau_w$):

$$\tau = (\rho \nu_t) \frac{\partial U}{\partial y}$$

$$\Rightarrow \qquad \tau_w = \rho(\kappa u_0 y) \frac{\partial U}{\partial y}$$

This can be rearranged and integrated for :

$$\frac{\partial U}{\partial y} = \left(\frac{\tau_w/\rho}{\kappa u_0}\right) \frac{1}{y}$$

$$\Rightarrow \qquad U = \frac{\tau_w/\rho}{\kappa u_0} \ln(Cy)$$

Applying this at the near-wall node P, and making sure that it is consistent with (29) in the equilibrium case where $u_0 = u_\tau$ and $\tau_w/\rho = u_\tau^2$, fixes constant of integration C and leads to

$$U_P = \frac{\tau_w/\rho}{\kappa u_0} \ln\left(E \frac{y_P u_0}{\nu}\right)$$

or, rearranging for the wall shear stress:

$$\tau_w = \rho \frac{\kappa u_0 U_P}{\ln\left(E \frac{y_P u_0}{\nu}\right)} \tag{32}$$

Since the code will discretise the velocity gradient at the boundary as U_P/y_P this is conveniently implemented via an *effective wall viscosity* μ_w, such that

$$\tau_w = \mu_w \frac{U_P}{y_P} \tag{33}$$

where

$$\mu_w = \frac{\rho(\kappa u_0 y_P)}{\ln\left(E \frac{y_P u_0}{\nu}\right)}, \qquad u_0 = C_\mu^{1/4} k_P^{1/2} \tag{34}$$

Amendments also have to be made to the turbulence equations, based on assumed profiles for k and ε. In particular, the production of turbulence energy is a *cell-averaged* quantity, determined by integrating across the cell and the value of ε is specified at the centre of the near-wall cell, not at the boundary.

To use these equilibrium profiles effectively, it is desirable that the grid spacing be such that the near-wall node lies within the logarithmic layer; ideally,

$$30 < y_P^+ < 150$$

This has to be relaxed somewhat in practice, but it means that when using wall functions the

grid can not be made arbitrarily small in the vicinity of solid boundaries. In practice, many commercial codes use an "all-y^+" wall treatment, and blend low-Re and wall-function treatments, depending on the size of the near-wall cell.

Summary

- A *turbulence model* is a means of specifying the Reynolds stresses (and other turbulent fluxes), so closing the mean flow equations.

- The most popular types are *eddy-viscosity models*, which assume that the Reynolds stress is proportional to the mean strain; e.g. in simple shear:

$$\tau^{(\mathrm{turb})} \equiv -\rho\overline{uv} \quad = \mu_t \frac{\partial U}{\partial y}$$

- The eddy viscosity μ_t may be specified geometrically (e.g. mixing-length models) or by solving additional transport equations. A popular combination is the $k - \varepsilon$ model.

- More advanced turbulence models include:
  – *Reynolds-stress transport models* (RSTM; solve transport equations for all stresses)
  – *non-linear eddy-viscosity models* (NLEVM; non-linear stress-strain relationship)
  – *large-eddy simulation* (LES; time-dependent calculation; model sub-grid scales)

- Wall boundary conditions require special treatment because of large flow gradients and selective damping of wall-normal velocity fluctuations. The main options are low-Reynolds-number models (fine grids) or wall functions (coarse grids).

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Appendix (Optional): Derivation of the Turbulent Kinetic Energy Equation

For simplicity, restrict to constant-density, constant-viscosity fluids, with no body forces. A more advanced derivation for individual Reynolds stresses, including body forces, is given in the optional Section 10.

The *summation convention* (implied sum over a repeated index) is used throughout.

<u>Continuity</u>

Instantaneous: $$\frac{\partial u_j}{\partial x_j} = 0$$

Average → $$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

Subtract → $$\frac{\partial u_j'}{\partial x_j} = 0$$

Important consequences:

(1) both mean ($\bar{u}_j$) and fluctuating(u_j') velocities satisfy incompressibility;

(2) u_j' and $\partial/\partial x_j$ commute when there is an implied summation; i.e.

$$\frac{\partial(u_j'\phi)}{\partial x_j} = u_j'\frac{\partial \phi}{\partial x_j} \qquad \text{for any } \phi$$

These results will be used repeatedly in what follows.

<u>Momentum</u>

Instantaneous: $$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial u_i}{\partial x_j}\right)$$

Average → $$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j}\right)$$

Subtract → $$\frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \bar{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} - \overline{u_j'\frac{\partial u_i'}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial u_i'}{\partial x_j}\right)$$

Multiply by u_i', sum over i and average, noting that, for any derivative, $u_i'\,\partial u_i' = \partial(\frac{1}{2}u_i'u_i')$:

$$\frac{\partial(\frac{1}{2}\overline{u_i'u_i'})}{\partial t} + \bar{u}_j\frac{\partial(\frac{1}{2}\overline{u_i'u_i'})}{\partial x_j} + \overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{\tfrac{1}{2}u_i'u_i'u_j'}) - 0$$
$$= -\frac{\partial}{\partial x_i}\left(\frac{\overline{p'u_i'}}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right) - \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

where we have used the commuting of u_j' and $\partial/\partial x_j$ from (2) above wherever necessary and rearranged the viscous term to start with the outer derivative of a product. Recognising $\frac{1}{2}\overline{u_i'u_i'}$ as k, changing the dummy summation index from i to j in the pressure term, and rearranging, gives:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{\nu \frac{\partial k}{\partial x_j} - \frac{\overline{pu_j'}}{\rho} - \frac{1}{2}\overline{u_i'u_i'u_j'}\right\} - \overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

Identifying individual physical processes:

$$\boxed{\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial d_i^{(k)}}{\partial x_i} + P^{(k)} - \varepsilon}$$

where:

$$d_j^{(k)} = \nu \frac{\partial k}{\partial x_j} - \frac{\overline{pu_j'}}{\rho} - \frac{1}{2}\overline{u_i'u_i'u_j'}$$ *diffusion* (viscous, pressure and triple correlation)

$$P^{(k)} = -\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$$ *production* (by mean velocity gradients)

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$ *dissipation* (by viscosity)

Examples

Q1.
In high-Reynolds-number turbulent boundary-layer flow over a flat surface the mean shear stress is made up of viscous and turbulent parts:

$$\tau = \mu \frac{\partial U}{\partial y} - \rho \overline{uv}$$

where μ is the molecular viscosity. In the lower part of the boundary layer the shear stress is effectively constant and equal to the wall shear stress τ_w.

(a) Define the friction velocity u_τ.

(b) Show that, sufficiently close to a smooth wall, the mean velocity profile is linear, and write down an expression for U in terms of τ_w, μ and the distance from the wall, y.

(c) At larger distances from the wall the viscous stress can be neglected, whilst the turbulent stress can be represented by a mixing-length eddy-viscosity model:

$$-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y}$$

where

$$\mu_t = \rho u_0 l_m \,, \qquad l_m = \kappa y, \qquad u_0 = l_m \frac{\partial U}{\partial y}$$

and κ (≈ 0.41) is a constant. Again, assuming that $\tau = \tau_w$, show that this leads to a logarithmic velocity profile of the form

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln(E \frac{y u_\tau}{\nu})$$

where E is a constant of integration.

(d) Write the velocity profiles in parts (b) and (c) in *wall units*.

(e) In simple shear flow, the rate of production of turbulent kinetic energy per unit mass is

$$P^{(k)} = -\overline{uv} \frac{\partial U}{\partial y}$$

Using the results of (c), prove that, in the logarithmic velocity region,

$$P^{(k)} = \frac{u_\tau^3}{\kappa y}$$

and explain what is meant by the statement that the turbulence is in local equilibrium.

Q2.
On dimensional grounds, an eddy viscosity μ_t can be written as

$$\mu_t = \rho u_0 l_0$$

where u_0 is some representative magnitude of turbulent velocity fluctuations and l_0 is a turbulent length scale. The eddy-viscosity formula for the $k - \varepsilon$ turbulence model is

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$$

where $C_\mu = 0.09$. Identify suitable velocity and length scales, u_0 and l_0.

Q3.
In the $k - \omega$ model of turbulence the kinematic eddy viscosity is given by

$$\nu_t = \frac{k}{\omega}$$

and transport equations are solved for turbulent kinetic energy k and specific dissipation rate ω. A modeled scalar-transport equation for ω is

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\omega}\frac{\partial \omega}{\partial x_i}\right) + \frac{\alpha}{\nu_t}P^{(k)} - \beta\omega^2$$

where $\mathrm{D}/\mathrm{D}t$ is a derivative following the flow, and summation is implied by the repeated index i. Here, $P^{(k)}$ is the rate of production of k, whilst σ_ω, α and β are constants.

In the log-law region of a turbulent boundary layer,

$$P^{(k)} = C_\mu k\omega = \frac{u_\tau^3}{\kappa y} \qquad \text{and} \qquad k = C_\mu^{-1/2} u_\tau^2$$

where κ is von Kármán's constant, C_μ is a model constant, u_τ is the friction velocity and y is the distance from the boundary. Show that this implies the following relationship between coefficients in the modeled scalar-transport equation for ω:

$$\left(\frac{\beta}{C_\mu} - \alpha\right)\sigma_\omega\sqrt{C_\mu} = \kappa^2$$

Q4.

In the analysis of turbulent flows it is common to decompose the velocity field into mean (U, V, W) and fluctuating (u, v, w) parts as part of the Reynolds-averaging process.

(a) The rate of production of the $\overline{uu}$ stress component per unit mass is given by

$$P_{11} = -2\left(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z}\right)$$

By inspection/pattern-matching, write down an analogous expression for P_{22}.

(b) Define the term *anisotropy* when applied to fluctuating quantities in turbulent flow and give *two* reasons why, for turbulent boundary layers along a plane wall $y = 0$, the wall-normal velocity variance is smaller than the streamwise variance.

(c) Describe the main principles of, and the main differences between
(i) eddy-viscosity
(ii) Reynolds-stress transport
models of turbulence, and give advantages and disadvantages of each type of closure.

Q5.

(a) By considering momentum transport by turbulent fluctuations show that $-\rho\overline{u_i' u_j'}$ can be interpreted as an additional effective stress in the mean momentum equation.

(b) For a linear eddy-viscosity turbulence model with strain-independent eddy viscosity μ_t, write expressions for the typical shear stress $-\rho\overline{u'v'}$ and normal stress $-\rho\overline{u'^2}$ in an arbitrary velocity field. Show that, for certain mean-velocity gradients, this type of model may predict physically unrealisable stresses.

(c) Explain why, for a zero-pressure-gradient, fully-developed, boundary-layer flow of the form $(\overline{u}(y),0,0)$, the mean shear stress τ is independent of distance y from the boundary. Find the mean-velocity profile if the total effective viscosity μ_{eff}:
(i) is constant;
(ii) varies linearly with wall distance: $\mu_{\text{eff}} = Cy$, where C is a constant.

(d) In the widely-used $k - \varepsilon$ model of turbulence:
(i) state the physical quantities represented by k and ε;
(ii) write down the expression for eddy viscosity μ_t in terms of ρ, k and ε in the standard $k - \varepsilon$ model;
(iii) explain briefly (and without detailed mathematics) how k and ε are determined.

(e) What special issues arise in the modeling and computation of near-wall turbulent flow? State the two main methods for dealing with the solid-wall boundary condition and give a brief summary of the major elements of each.

Q6.
In a general incompressible velocity field (U, V, W) the turbulent shear stress component τ_{12} is given, for an eddy-viscosity turbulence model, by

$$\tau_{12} = \mu_t \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)$$

where μ_t $(= \rho \nu_t)$ is the dynamic eddy viscosity and ρ is density. Use pattern-matching or index-permutation and the incompressibility condition to write expressions for the other independent stress components: τ_{23}, τ_{31}, τ_{11}, τ_{22}, τ_{33}.

Q7.
In the $k - T$ turbulence model, T is a turbulent time scale and the eddy-viscosity formulation takes the form

$$\mu_t = C \rho^{\alpha} k^{\beta} T^{\gamma}$$

where C is a dimensionless constant. Use dimensional analysis to find α, β and γ.

Q8. (Exam 2017 – part)
In turbulent flow, mean and fluctuating components are often denoted by an overbar ($\bar{\ }$) and prime ($'$) respectively. The fluctuating velocity components in the x, y, z or 1, 2, 3 coordinate directions are u', v' and w' respectively. At a particular point in a flow of air a hot-wire anemometer measures the following turbulent statistics:

$$\overline{u'^2} = 2.6 \ \text{m}^2 \ \text{s}^{-2}, \qquad \overline{v'^2} = 1.4 \ \text{m}^2 \ \text{s}^{-2}, \qquad \overline{w'^2} = 2.0 \ \text{m}^2 \ \text{s}^{-2},$$

$$\overline{u'v'} = -0.9 \ \text{m}^2 \ \text{s}^{-2}, \qquad \overline{v'w'} = \overline{w'u'} = 0.0$$

The density of air is $\rho = 1.2$ kg m^{-3}.

(a) At this point determine:
  (i) the turbulent kinetic energy (per unit mass), k;
  (ii) the dynamic shear stress τ_{12}.

(b) The only non-zero mean-velocity gradients are

$$\frac{\partial \bar{u}}{\partial y} = 4 \ \text{s}^{-1}, \qquad \frac{\partial \bar{u}}{\partial x} = -1.5 \ \text{s}^{-1}, \qquad \frac{\partial \bar{v}}{\partial y} = 1.5 \ \text{s}^{-1}$$

Assuming a linear eddy-viscosity model of turbulence, deduce the eddy viscosity μ_t on the basis of the shear stress found in part (a)(ii).

(c) Using the eddy viscosity calculated in part (b), what does the turbulence model *predict* for the fluctuating velocity variances $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$.

Q9. (Exam 2018)
In the fully-turbulent region of an equilibrium turbulent boundary layer the mean-velocity is given by the log-law profile:

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y u_\tau}{\nu}\right) + B \qquad (*)$$

where U is the wall-parallel component of mean velocity, u_τ is the friction velocity, ν is the kinematic viscosity, y is the distance from the boundary, κ $(= 0.41)$ is von Kármán's constant and B $(= 5.0)$ is a constant. In the same region the turbulent kinetic energy (per unit mass) k is related to the friction velocity by

$$u_\tau = C_\mu^{1/4} k^{1/2}$$

where C_μ $(= 0.09)$ is another constant.

(a) Define the friction velocity u_τ in terms of wall shear stress τ_w and fluid density ρ.

(b) Find the mean-velocity gradient $(\partial U/\partial y)$ implied by equation (*), and hence deduce an expression for the kinematic eddy viscosity ν_t in this flow.

(c) In a flow of water ($\nu = 1.0 \times 10^{-6}$ m^2 s^{-1}) the velocity at 5 mm from the boundary is 3 m s^{-1}. Assuming that this is within the log-law region, find, at this point:
(i) the friction velocity u_τ;
(ii) the eddy viscosity ν_t;
(iii) the turbulent kinetic energy k;
(iv) the turbulence intensity and turbulent viscosity ratio (stating definitions).
Correct units should be given.

(d) In a simple shear flow, the turbulent shear stress component τ_{12} is given by

$$\tau_{12} = \mu_t \frac{\partial U}{\partial y}$$

where $\mu_t = \rho\nu_t$ is the dynamic eddy viscosity, and 1,2 indices refer to x, y directions, respectively, with the x direction that of the mean flow velocity.

(i) Explain why this constitutive relationship can *not* hold true in a general flow, and give a generalisation that does.

(ii) Write down an expression for the τ_{11} turbulent stress component, using an eddy-viscosity model in an arbitrary incompressible flow.

Q10. (Exam 2019)
The logarithmic mean velocity profile for the atmospheric boundary layer is:

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{z}{z_0}\right) \qquad (*)$$

where U is mean velocity at height z above the ground, u_τ is the friction velocity, z_0 is the roughness length and κ $(= 0.41)$ is von Kármán's constant.

(a) Define the friction velocity u_τ in terms of boundary shear stress τ_w and air density ρ.

(b) Find the mean-velocity gradient $(\mathrm{d}U/\mathrm{d}z)$ from (*), and hence deduce an expression for the kinematic eddy viscosity ν_t in terms of u_τ and z.

(c) Define the turbulent kinetic energy (per unit mass) k in terms of velocity fluctuations.

(d) For flat countryside, a typical roughness length is $z_0 = 0.1$ m, whilst air density $\rho = 1.2$ kg m^{-3}. In an equilibrium boundary layer the turbulent kinetic energy and friction velocity are related by

$$u_\tau = C_\mu^{1/4} k^{1/2}$$

where $C_\mu = 0.09$ is a constant. If the wind speed at a height of 10 m is 15 m s^{-1}, find the values of:

(i) the friction velocity, u_τ;
(ii) the turbulent kinetic energy, k;
(iii) the kinematic eddy viscosity, ν_t.

(e) Assuming a linear eddy-viscosity model of turbulence, use the values in part (d) to determine values of the kinematic normal stresses, $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$.

(f) What is unphysical about the answers to part (e)? Suggest two classes of turbulence model that could be used to remedy this.

Q11. (Exam 2020 – part)

(a) In the standard $k-\varepsilon$ turbulence model, the kinematic eddy viscosity is

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

where $C_\mu = 0.09$. State the physical meaning of the variables k and ε and write expressions for the kinematic turbulent stresses $\overline{u'v'}$ and $\overline{u'^2}$ in terms of the mean velocity gradients in a general incompressible flow.

A real turbulent flow with velocity fluctuations (u', v', w') needs to satisfy the realisability conditions of positive normal stresses; e.g.

$$\overline{u'^2} \geq 0$$

and the Cauchy-Schwarz inequalities; e.g.

$$\left| \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}} \right| \leq 1$$

(b) For a simple shear flow (where $\partial U/\partial y$ is the only non-zero mean-velocity gradient) with shear-strain parameter

$$S_1^* = \frac{k}{\varepsilon}\frac{\partial U}{\partial y}$$

write expressions for $\overline{u'v'}$, $\overline{u'^2}$ and $\overline{v'^2}$ using the standard $k-\varepsilon$ model, and hence deduce the maximum value of S_1^* for realisability.

(c) For plane strain ($\partial U/\partial x = -\partial V/\partial y$ the only non-zero mean-velocity gradients) with normal-strain parameter

$$S_2^* = \frac{k}{\varepsilon}\frac{\partial U}{\partial x}$$

write expressions for $\overline{u'v'}$, $\overline{u'^2}$ and $\overline{v'^2}$ using the standard $k-\varepsilon$ model, and hence deduce the maximum value of S_2^* for realisability.

(d) In homogeneous turbulence the k and ε transport equations reduce to

$$\frac{\mathrm{D}k}{\mathrm{D}t} = P^{(k)} - \varepsilon$$

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = (C_{\varepsilon 1}P^{(k)} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k}$$

where $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.92$ are constants, and $P^{(k)}$ is the rate of production of k. By finding $\mathrm{D}\tau/\mathrm{D}t$ or otherwise, where $\tau = k/\varepsilon$ is the turbulent timescale, show that these equations admit a constant-τ solution, and find the numerical value of the ratio of production to dissipation, $P^{(k)}/\varepsilon$, in such a state.

Q12. (Exam 2021 – part)
In this question the (x, y, z) components of mean velocity are denoted (U, V, W), the turbulent fluctuations by (u, v, w) and Reynolds averaging is denoted by an overbar.

At one point in an incompressible 2-dimensional flow (some) mean-velocity gradients are:

$$\frac{\partial U}{\partial x} = -0.22\ \mathrm{s}^{-1}, \qquad \frac{\partial U}{\partial y} = 0.69\ \mathrm{s}^{-1}, \qquad \frac{\partial V}{\partial x} = 0.14\ \mathrm{s}^{-1}$$

The turbulent kinetic energy $k = 0.37\ (\mathrm{m\ s}^{-1})^2$ and, with a particular turbulence model, the kinematic eddy viscosity $\nu_t = 0.41\ \mathrm{m}^2\mathrm{s}^{-1}$.

(i) Calculate $\partial V/\partial y$.

(ii) Assuming an eddy-viscosity model of turbulence, find the values at this point of the following turbulent correlations:

$$\overline{u^2},\ \overline{v^2},\ \overline{w^2},\ \overline{uv}$$

(iii) Does the turbulence model generate "realisable" Reynolds stresses at this point? Justify your answer.

Q13. (Exam 2023 – part)
In turbulent flow, mean and fluctuating components are often denoted by an overbar ($^{-}$) and prime ($'$) respectively. The fluctuating velocity components in the x, y, z coordinate directions are u', v' and w' respectively. At a particular point in a flow of air a laser-doppler anemometer has been used to measure some turbulent statistics:

$$\overline{u'^2} = 2.0\ (\mathrm{m\ s}^{-1})^2, \quad \overline{v'^2} = 0.8\ (\mathrm{m\ s}^{-1})^2, \quad \overline{w'^2} = 1.2\ (\mathrm{m\ s}^{-1})^2, \quad \overline{u'v'} = -0.6\ (\mathrm{m\ s}^{-1})^2$$

and some mean-velocity gradients:

$$\frac{\partial \bar{u}}{\partial x} = -2.4\ \mathrm{s}^{-1}, \qquad \frac{\partial \bar{v}}{\partial y} = 0.7\ \mathrm{s}^{-1}, \qquad \frac{\partial \bar{u}}{\partial y} = 0.8\ \mathrm{s}^{-1}, \qquad \frac{\partial \bar{v}}{\partial x} = 0.6\ \mathrm{s}^{-1}$$

The density of air is $\rho = 1.2\ \mathrm{kg\ m}^{-3}$.

(a) Assuming incompressible flow, determine:
  (i) the mean-velocity gradient $\partial \bar{w}/\partial z$;
  (ii) the turbulent kinetic energy (per unit mass), k;
  (iii) the dynamic shear stress component τ_{xy}.

(b) Assuming a linear eddy-viscosity model of turbulence, deduce the eddy viscosity μ_t on the basis of the shear stress found in part (a)(iii).

(c) Using the eddy viscosity calculated in part (b), what does the turbulence model *predict* for the fluctuating velocity variances $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{w'^2}$? Are these possible physical values?