

# APPLICATION OF THE MODIFIED PHYSICAL PROGRAMMING METHOD TO GENERATING THE ENTIRE PARETO FRONTIER IN MULTIOBJECTIVE OPTIMIZATION

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**Abstract.** *In an industrial design setting, the decision-maker (DM) has to trade-off many conflicting criteria such as minimum weight, acquisition and lifecycle costs, and desirable performance characteristics. In this process it is natural to exclude from the consideration any design solution which can be improved without any trade-off and violation of the constraints. It leads to the Pareto optimal solutions. The DM selects the ultimate solution among the Pareto set on the basis of additional subjective (qualitative) requirements.*

*Unfortunately, wider industrial application of vector optimization methods seems to be constrained by the high computational resources required and the limited time scales available to perform extensive optimisation search. Thus, when considering the design of complex products such as aircraft, the DM is able to consider only a few possible (Pareto) solutions. Under such conditions, it is necessary to minimize the computational time required to find a Pareto optimal solution. The choice of an appropriate numerical optimization method is therefore important, in particular, with respect to the ability to generate an even distribution of the complete Pareto frontier in order to gain maximum information on the Pareto surface with minimum computational time.*

*The Physical Programming (PP) Method recently suggested by Messac appears to match many of the above requirements. The PP has been modified by the authors to make it simpler and more efficient for practical applications. The modification is based on shrinking the search domain to make its location in the objective space easier. The algorithm to obtain an even distribution of the Pareto set is outlined. It is shown that the method is able to find the Pareto surface for different test cases including the optimization of the wing having a Zhukowskii profile. It is also shown in this paper that the method generates the entire Pareto frontier for both the convex and concave surfaces in the cases of bi-objective and multiobjective optimisation.*

## 1 INTRODUCTION

In real industrial design, the decision-maker (DM) has to take into account many different criteria such as low initial cost, manufacturability, long life and good performance which cannot be satisfied simultaneously. In fact, it is possible only to consider a trade-off

among all (or almost all) criteria. The task becomes even more complicated because of additional constraints which always exist in practice.

Mathematically, the trade-off analysis can be formulated as a vector nonlinear optimization problem under constraints [1]. In general, the solution of such a problem is not unique. It is naturally to exclude from the consideration any design solution which can be improved without deterioration of any discipline and violation of the constraints; in other words, a solution which can be improved without any trade-off. This leads to the Pareto optimal solutions. Mathematically, each Pareto point is a solution of the multidisciplinary optimization problem. A designer selects the ultimate solution among the Pareto set on the basis of additional requirements (which may be subjective). In order to ensure an informed decision it is desirable to have a sufficient number of Pareto points to represent the whole Pareto set. Furthermore, it is important that the Pareto set is evenly distributed, otherwise the representation of the Pareto surface becomes inefficient.

The Physical Programming (PP) Method was suggested by Messac in [2, 3]. This method generates Pareto points on both convex and non-convex Pareto frontiers as it was shown in [5]. The method does not use any weight coefficients and allows one to take into account the DM experience immediately. In this sense, it looks to be the most interesting method for practical applications under the above stated conditions. In the PP, the designer assigns each objective to one of the four categories (class-functions). The optimization is based on minimization of an aggregate preference function determined by the preference functions (class-functions) with preferences set *a priori*. The notion of the generalized Pareto optimal solution was introduced in [4] on the basis of the class-functions.

The PP has been recently modified by the authors [6] to make it simpler and more efficient for practical applications. The class-functions are generalized to shrink the search domain and make its location in space more optimal. This is critical when generating an even set of the Pareto frontier. The proposed modification allows us to combine the advantages of the PP, NBI and NC methods. The algorithm to obtain an even distribution of the Pareto set is described in this paper. One of its main advantages is that it does not provide non-Pareto solutions while local Pareto solutions may be easily recognized and removed. A comparison against the standard approach shows the modified approach is able to generate a much more even Pareto set. The suggested method is then generalized for the search of the generalized Pareto solutions. A simpler structure of the class-functions is derived below.

Numerical solutions of the tested multi-objective optimization problems under constraints are obtained using the MATLAB optimization software.

## 2 MULTIOBJECTIVE OPTIMIZATION PROBLEM. PARETO OPTIMIZATION

It is assumed that an optimization problem is described in terms of a design variable vector  $\mathbf{x} = (x_1, x_2, \dots, x_N)^T$  in the design space  $\mathbf{X} \in R^N$ . A function  $\mathbf{f} \in R^M$  evaluates the quality of a solution by assigning it to an objective vector  $\mathbf{y} = (y_1, y_2, \dots, y_M)^T$  ( $y_i = f_i(\mathbf{x})$ ,  $f_i: R^N \rightarrow R^1$ ,  $i = 1, 2, \dots, M$ ) in the objective space  $\mathbf{Y} \in R^M$ . Thus,  $\mathbf{X}$  is mapped by  $\mathbf{f}$  onto  $\mathbf{Y}$ :  $\mathbf{X} \mapsto \mathbf{Y}$ . A multiobjective optimization problem may be formulated in the following form:

$$\text{Minimize } [\mathbf{y}(\mathbf{x})] \tag{1}$$

subject to  $K$  inequality constraints

$$g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, K \quad (2)$$

and  $P$  equality constraints

$$h_j(\mathbf{x}) = 0, \quad j = 1, 2, \dots, P \quad (3)$$

The feasible design space  $\mathbf{X}^*$  is defined as the set  $\{\mathbf{x} | g_j(\mathbf{x}) \leq 0, j = 1, 2, \dots, K; \text{ and } h_i(\mathbf{x}) = 0, i = 1, 2, \dots, P\}$ . The feasible criterion (objective) space  $\mathbf{Y}^*$  is defined as the set  $\{\mathbf{Y}(\mathbf{x}) | \mathbf{x} \in \mathbf{X}^*\}$ . The feasibility means no constraint is violated.

A design vector  $\mathbf{a}$  ( $\mathbf{a} \in \mathbf{X}^*$ ) is called a Pareto optimum iff it does not exist any  $\mathbf{b} \in \mathbf{X}^*$  such that:  $\mathbf{y}(\mathbf{b}) \leq \mathbf{y}(\mathbf{a})$  and exist  $l \leq M$ :  $y_l(\mathbf{b}) < y_l(\mathbf{a})$ .

A design vector is called a local Pareto optimum if it is a Pareto optimum within its some neighbourhood.

### 3 PHYSICAL PROGRAMMING METHOD

In the PP, both objective and constraint functions are treated as design metrics. Each design metric  $F_i(\mathbf{x})$  is referred to one of possible (determined in advance) utility dimensionless functions  $\bar{F}_i(F_i(\mathbf{x}))$  called *class functions*. The determination of the class-functions reflects the qualitative classifications of possible preferences. The preferences are split into two major groups: hard class-functions  $H$  and soft ones  $S$ . The former reflect the presence of constraints (2) and (3), while the latter ones reflect the preferences settled for objective functions which may include subjective preferences of the DM based, say, on his/her experience and intuition.

As it is easy to see from Figure 1, the more desirable the range is, the smaller the value of the class-function. The preference between the ranges is enforced, that means: the highly undesirable region significantly worse than the undesirable, the undesirable region significantly worse than the tolerable region, and so on. This demand is reflected by the following conditions [3]:

$$\Delta_k \bar{F}_i > n_{sc} \Delta_{k-1} \bar{F}_i, \quad 1 \leq k \leq 4, \quad (4)$$

where  $\Delta_k \bar{F}_i = \bar{F}_i(F_{i,k+1}) - \bar{F}_i(F_{ik})$ ,  $\Delta_0 \bar{F}_i = \bar{F}_i(F_{i1})$ ,  $F_{ik}$  ( $k = 1, 2, \dots, 4$ ) are the boundary points of the preference ranges for a metric  $F_i$ ;  $i$  is a number of a soft-class function,  $n_{sc}$  is the number of soft design metrics.

Ultimately, the problem is reduced to the following optimization problem under constraints:

$$\min G(\mathbf{x}) = \frac{1}{n_{sc}} \log_{10} \left[ \sum_{i=1}^{n_{sc}} \bar{F}_i(F_i) \right], \quad (5)$$

subject to:

$$\begin{aligned}
F_i(\mathbf{x}) &\leq F_{i5} && (\text{for } 1S) \\
F_i(\mathbf{x}) &\geq F_{i5} && (\text{for } 2S) \\
F_{i5L} &\leq F_i(\mathbf{x}) \leq F_{i5R} && (\text{for } 3S) \\
F_{i5L} &\leq F_i(\mathbf{x}) \leq F_{i5R} && (\text{for } 4S)
\end{aligned} \tag{6}$$

and constraints related with the hard classes

$$\begin{aligned}
F_i(\mathbf{x}) &\leq F_{iM} && (\text{for } 1H) \\
F_i(\mathbf{x}) &\geq F_{im} && (\text{for } 2H) \\
F_{im} &\leq F_i(\mathbf{x}) \leq F_{iM} && (\text{for } 3H)
\end{aligned} \tag{7}$$

Any constraints on the design variables  $\mathbf{x}$  are considered as ordinary constraints. The logarithm in (5) is only used to diminish the difference between the maximal and minimal values that may effect on the convergence of an iteration algorithm. It is easy to see that the more “undesirable” the range is, the more it effects on the value of the AOF. Thus, preferences play somewhat the same role as weights in the weighted-sum method [3]. In (5) and henceforth, we assume that in the formulation  $\bar{F}(F_i)$  the appropriate category of the class-function is used. This means that if an objective  $F_i$  is to be minimized then  $1S$  class-function is utilised, if  $F_i$  is to be maximized then  $2S$  class-function is considered and so on.

A robust implementation of the PP necessitates that problem (5) - (7) has a unique solution, which means absence of local minima. To guarantee this, each class-function must be a strictly convex function [3], that is:

$$\frac{d\bar{F}}{dF_i} \in C^1, \quad \frac{d^2\bar{F}}{dF_i^2} > 0 \tag{8}$$

It is easy to see that both high-order polynomial and cubic spline functions are not acceptable. Messac [3] derived the class-functions that satisfy all requirements mentioned above. They have been implemented into the PP and successfully used for solving many MDO problems. We suggest another approximation of the class-functions which also satisfy all the requirements, but look simpler and more compact.

Let us assume that the derivative of the class-function  $1S$  has the following form:

$$\begin{aligned}
\frac{d\bar{F}}{dF} &= Ae^{\alpha(F)}, \\
A &> 0, \\
\alpha'(F) &> 0,
\end{aligned} \tag{9}$$

where index  $i$  is omitted for the sake of simplicity. Then, function  $\bar{F}$  is strictly convex because both the first and the second derivatives are positive.

Further,  $\alpha$  must be a smooth function which satisfies condition (4). Since  $\Delta_k \bar{F} = A \int_{F_k}^{F_{k+1}} e^{\alpha(F)} dF$ , the latter condition is equivalent to

$$A\Delta_k F \int_0^1 e^{\alpha^{(k)}(\xi)} d\xi = \Delta_k \bar{F}, \quad (10)$$

where  $\alpha^{(k)} = \alpha^{(k)}(\xi^{(k)})$ ,  $\xi^{(k)} = \frac{F - F_k}{F_{k+1} - F_k}$  ( $F_k \leq F \leq F_{k+1}$ ).

Additional condition on  $\alpha$  follows from (8) , (9) and means:

$$\alpha^{(k)}(0) = \alpha^{(k-1)}(1) \quad (11)$$

It follows that the linear function

$$\alpha^{(k)}(\xi) = a_k \xi + b_k \quad (12)$$

is able to satisfy conditions (10) and (11) if

$$a_k = A\Delta_k F e^{b_k} (e^{a_k} - 1) / \Delta_k \bar{F}, \quad (k = 1, 2, \dots, 4) \quad (13)$$

$$b_k = a_{k-1} + b_{k-1}$$

Conditions (13) give a recurrent relation for the calculation of  $a_k$  and  $b_k$ , where the former condition assumes solution of a nonlinear equation.

The values of  $\bar{F}$  at the boundaries of the ranges do not depend on  $F_{ik}$  and are fixed using relation

$$\Delta_k \bar{F} = \beta n_{sc} \Delta_{k-1} \bar{F}, \quad (k = 1, 2, 3, 4) \quad (14)$$

$$\beta > 1, \Delta_0 \bar{F} = 1$$

Having integrated (9), the ultimate solution is obtained as follows

$$\bar{F}^{(k)} = \bar{F}_k + \Delta_k \bar{F} \frac{e^{a_k \xi^{(k)}} - 1}{e^{a_k} - 1} \quad (F_k \leq F \leq F_{k+1}), \quad (15)$$

where  $\xi^{(k)} = \frac{F - F_k}{F_{k+1} - F_k}$ ,  $\bar{F}_k = \bar{F}(F_k)$ ,  $k = 1, 2, 3, 4$ .

In the rest region,  $F < F_1$ , it is possible to choose the following exponential function

$$\bar{F}^{(0)} = e^{A(F - F_1)} \quad (16)$$

then the initial conditions in (13) may be chosen as uniform:

$$a_0 = 0, \quad b_0 = 0. \quad (17)$$

As we see,  $\bar{F}^{(0)} = 1$  always.

The parameter  $A$  must have the dimensionality of  $F^{-1}$ . It is natural to set it as

$$A = \frac{1}{F_5 - F_1}. \quad (18)$$

Thus, the class-function  $\bar{F}$  is given by (15) and (16) where the coefficients are determined by (13), (17) and (18). Obviously, the function  $\bar{F}$  is positive. From (16), it follows that  $\bar{F}(F_1) = 1$ .

Finally, we will give a brief comment on solving the nonlinear equation with respect to  $a_k$  in (13). The equation can be rewritten as follows (index  $i$  is omitted):

$$a_k = \varphi(a_k), \quad (19)$$

where  $\varphi(a_k) = \ln(1 + a_k / \omega_k)$ ,  $\omega_k = A\Delta_k F e^{b_k} / \Delta_k \bar{F}$ .

*Proposition 1:* A positive solution of (13) always exist and the method of simple iteration

$$a_k^{(n+1)} = \varphi(a_k^{(n)}) \quad (n = 0, 1, 2, \dots) \quad (20)$$

monotonically converges to this solution at any initial  $a_k^{(0)} > -\ln \omega_k$ .

The class-function  $2S$  is obtained as the mirror-function of  $1S$ . The class-functions  $3S$  and  $4S$  can be created in regions  $F < F_{1L}$  and  $F > F_{1R}$  using the class-functions  $2S$  and  $1S$ , accordingly. In the rest region  $F_{1L} < F < F_{1R}$ , the class functions are defined as follows:

$$\bar{F}^{(0)}(F) = A_0 \xi^{(0)}(F)^{2m} + \varepsilon_0, \quad (21)$$

where

$$\xi^{(0)}(F) = \frac{2F - F_{1L} - F_{1R}}{F_{1R} - F_{1L}},$$

$$A_0 = \frac{F_{1R} - F_{1L}}{4m} A, \quad \varepsilon_0 = 1 - A_0.$$

Integer parameter  $m$  defines the “flatness” of the function in the vicinity of the point of the minimum. It is naturally to set  $m \gg 1$  for  $4S$  class-function. In the case of the  $3S$  class-function,  $F_{1L}$  should be close to  $F_{1R}$ , and  $F_l = (F_{1R} + F_{1L})/2$ . Formula (21) then represents a local smoothness at point  $F_l$ . Then,  $\bar{F}(F_l)$  approximately equals 1. Yet, it is not difficult to reach the exact equality for  $3S$  class-functions at  $F_l$  using the following function in the interval  $F_{1R} \leq F \leq F_{2R}$ :

$$\bar{F}^{(1)} = A_1 (e^{a_1 [\xi^{(1)}]^2} - 1) + 1,$$

$$A_1 = \frac{\bar{F}_2 - 1}{e^{a_1} - 1}, \quad a_1 = \frac{(F_{2R} - F_{1R})A}{2A_1}.$$

In this case, in (13)  $b_2 = a_l$ . The similar solution can be obtained in the interval  $F_{2L} \leq F \leq F_{1L}$ .

#### 4 GENERATION OF EVEN DISTRIBUTION OF A PARETO SET

The modified PP method for generating a well-distributed Pareto set has been suggested in [6]. It is important to emphasise that the method is able to generate the entire representation of the Pareto frontier and does not generate either non-Pareto or local Pareto solutions. The method is described in [5] and is based on shrinking of the search domain. In this paper the method is testified on different test cases.

The approach [6] can be extended on the  $2S$ - $4S$  class-functions. First, let us introduce the notion of reference class-functions. A class-function  $\bar{F}(F_i)$  is said to be a reference class-function if its preferences include the entire interval of the variation of the objective  $F_i$  in the space  $\mathbf{Y}^*$ . We map the objective space  $\mathbf{Y}^*$  onto the class-function space  $\mathbf{Z}^*$  by the reference class-functions  $\bar{F}(F_i)$  ( $i = 1, \dots, n_{sc}$ ):  $\mathbf{Y} \mapsto \mathbf{Z}$  ( $\mathbf{Z} \in R^{n_{sc}}$ ). The solution belonging to a Pareto set in the space  $\mathbf{Z}^*$  can be interpreted as a generalized Pareto optimal (GPO) solution [4].

Vector  $\mathbf{a} \in \mathbf{X}^*$  is said to be GPO solution iff it does not exist any vector  $\mathbf{b} \in \mathbf{X}^*$  such that  $\overline{F}(F_i(\mathbf{b})) \leq \overline{F}(F_i(\mathbf{a}))$  ( $i = 1, \dots, n_{sc}$ ) and there is  $l \leq M$ :  $\overline{F}(F_l(\mathbf{b})) < \overline{F}(F_l(\mathbf{a}))$  where  $\overline{F}(F_i)$  are the reference class-functions. The generalization is naturally related with the PP formulation of the optimization problem based on the introduction of such nonmonotonic classes as 3S and 4S. In [4], it was proved that any solution of optimization problem (5)-(7) is a generalized Pareto solution. If the problem is considered in the standard form for minimization (1)-(3), then it is easy to prove the following statement.

*Proposition 2:* Assume that the minimization problem (1)-(3) is solved. If the vector  $\mathbf{a}$  is a Pareto optimal solution then it is a GPO solution and vice a versa.

Thus, a classical Pareto solution is always a GPO solution. The problem of seeking GPO solutions is associated only with minimization. The appropriate class-functions can be considered as new objectives and then the problem can be solved by the algorithm described above. Following this approach, only the 1S class-function is to be introduced for the new objectives which are the original class-functions.

## 5 TEST CASES

The method described above is validated using a few test cases. The test cases include examples with both Pareto convex and concave frontiers. It is shown that the location of the Pareto points in the standard realization of the PP may become very sensitive to the displacement of box  $D$  (or  $D^*$ ) in the case of a concave Pareto frontier.

### Test case 1:

A 3D minimization problem:

$$\min (x, y, z)^T \quad (22)$$

under constraints

$$\begin{aligned} x^2 + y^2 + z^2 &\geq 1 \\ x > 0, y > 0, z > 0 \end{aligned} \quad (23)$$

The standard definition of the anchor points again leads to non-uniqueness. It is easy to see that in our formulation [6] of the anchor points we have only three anchor points: (0,0,1), (1,0,0) and (0,1,0). Not the entire orthogonal projection of the Pareto surface onto the utopia surface appears to be in the triangle created by the anchor points (see Figure 2). For this reason, the known methods encounter problems when attempting to capture the entire Pareto frontier. Our approach [6] is able to provide the complete representation of the Pareto frontier. The utopia plane is shown in Figure 2 by the dashed line. The solution providing the complete Pareto frontier is given in Figure 3.

### Test case 2:

This test case is related with the optimization of an aircraft wing with Zhukowskii profile. It is assumed that the wing has a finite thickness. The problem allows one to obtain analytical solutions for both the gas dynamic and stress analysis parts.

Let us introduce the complex plane  $\zeta = x + iy$ . We consider the family of wing profiles which are obtained by conformal mapping  $z = z(\zeta)$  of the exterior part of the circle with radius  $a$  and centre coordinates at point  $\zeta_c = x_c + iy_c$ , where

$$z = \zeta + \frac{\lambda^2}{\zeta}, \quad (24)$$

$$\lambda = -x_c + \sqrt{a^2 - y_c^2}$$

Such wing profiles suggested by Zhukovskii [1] have one remarkable property. Under assumption on incompressible flow, there is the exact analytical gas dynamic solution. In particular, the lift force  $F_y$  is as follows:

$$F_y = 4\pi\rho U_\infty^2 (y_c \cos \alpha + \sqrt{a^2 - y_c^2} \sin \alpha), \quad (25)$$

where  $\rho$  is the density of the flow,  $U_\infty$  is the velocity of the flow at infinity,  $\alpha$  is the angle of attack.

Assume that the wing has the thickness  $h$ . The critical pressure of the gas flow above which small displacements become irreversible [2] is:

$$p_{cr} \approx \frac{Eh^2}{R^2}, \quad (26)$$

where  $E$  is Young's modulus and  $R$  is the local radius of the profile taken at the middle of the chord.

The weight of the wing with length  $L$  can be approximated to:

$$W = 2\rho_w ch, \quad (27)$$

where  $\rho_w$  is the density of the wing,  $c$  is the chord of the wing.

Now, we can formulate the following optimization problem.

$$\begin{aligned} \min \quad & W \\ \max \quad & (F_y, p_{cr}) \end{aligned} \quad (28)$$

The vector of design values is  $(x_c, y_c, a, h)^T$ . The input parameters are the following:  $\rho$ ,  $U_\infty$ ,  $\alpha$ , and the material of the wing which uniquely defines  $\rho_w$ ,  $\sigma$  and  $E$ . In (26), an effective radius  $R$  is only considered taken at the middle of the chord. It is also assumed that only the range of small angles of attack  $\alpha$  is considered.

In this example, the anchor points are not unique if the standard definition is used. In our definition we will use the following priority in the optimization:  $W$ ,  $F_y$ ,  $p_{cr}$ . Then, the requirements for all anchor points are formulated in the Table 1. In the Table 1,  $c = c(x_c, y_c, a)$  is the chord of the profile. As can be seen from the Table, when determining each of the anchor points in the standard formulation, one of the variables  $h$ ,  $c$  and  $R$  does not affect the objective function to which the anchor point is related, thus causing the non-uniqueness. In our modified formulation, the non-uniqueness is removed by introducing second-priority objective function and by imposing non-violation of the higher priority objective function.



	1	2	3
Standard formulation	$\min h, \min c, R - \text{var.}$	$\max c, \max R, h - \text{var.}$	$\max h, \min R, c - \text{var.}$
Modified formulation	$\min h, \min c, \max R$	$\max c, \max R, \max h$	$\max h, \min R, \min c$

Table 1: Determination of anchor points

The distribution of the Pareto solutions in the objective space is shown in Figure 4. All the objectives representing the lift force, critical pressure and weights are given in SI units. The wing profile corresponding to the highlighted Pareto point and the velocity field around the profile are shown in Fig. 5.

## 6 CONCLUSIONS

The Physical Programming method has been modified to generate an even distribution of the entire Pareto set. The modification is based on the generalization of the class-functions which leads to shrinking of the search domain. The orientation of the search domain in space can be easily conducted. It allows the method to provide an even distribution of the entire Pareto surface. The generation is performed for both convex and non-convex Pareto frontiers. The method does not generate non-Pareto solutions. A simple algorithm has been proposed to remove local Pareto solutions which are not the global Pareto solutions. A new, more compact representation of the class-functions has been suggested.

The algorithm has been also extended to the case of the generalized Pareto solutions. The suggested approach has been verified by different test cases, including the generation of both convex and concave Pareto frontiers.

Given the encouraging results obtained so far, the next step will be to evaluate the effectiveness of the approach on industrial optimisation problems.

## 7 ACKNOWLEDGEMENTS

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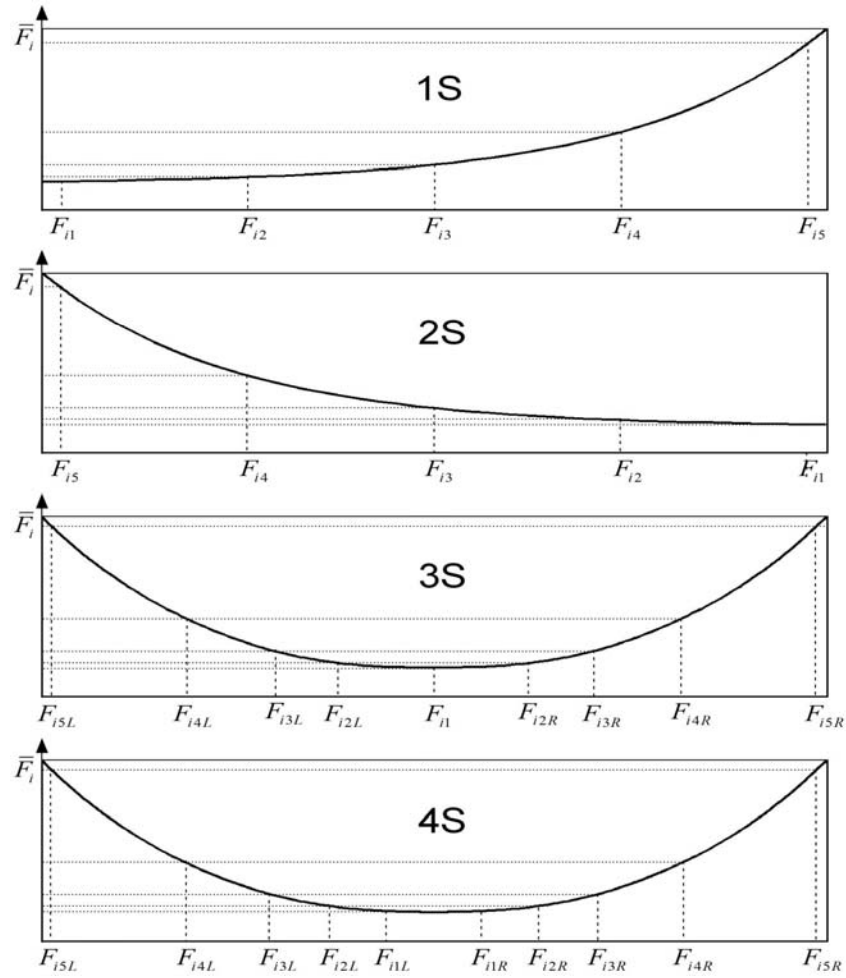


Fig. 1: The class-functions.

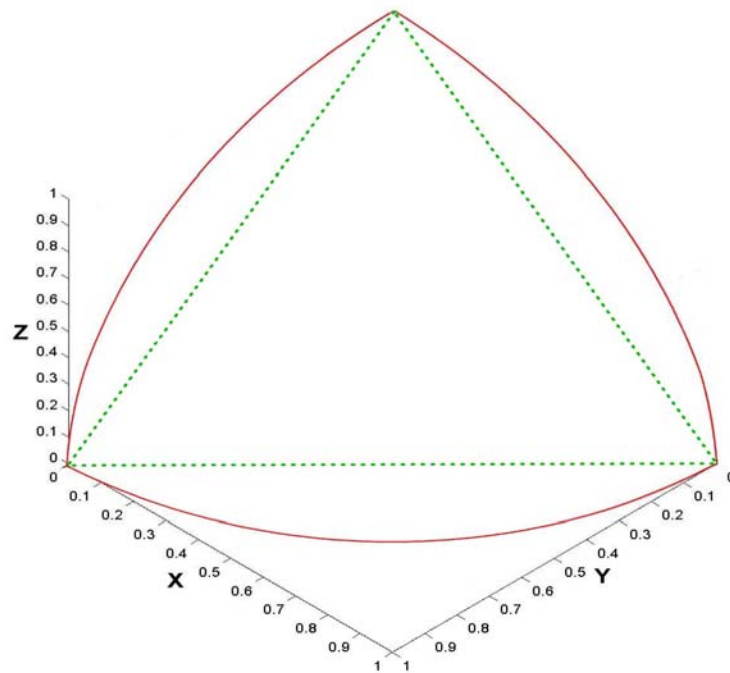


Fig. 2: 3D algebraic test case. Projection of the Pareto surface onto the utopia plane.

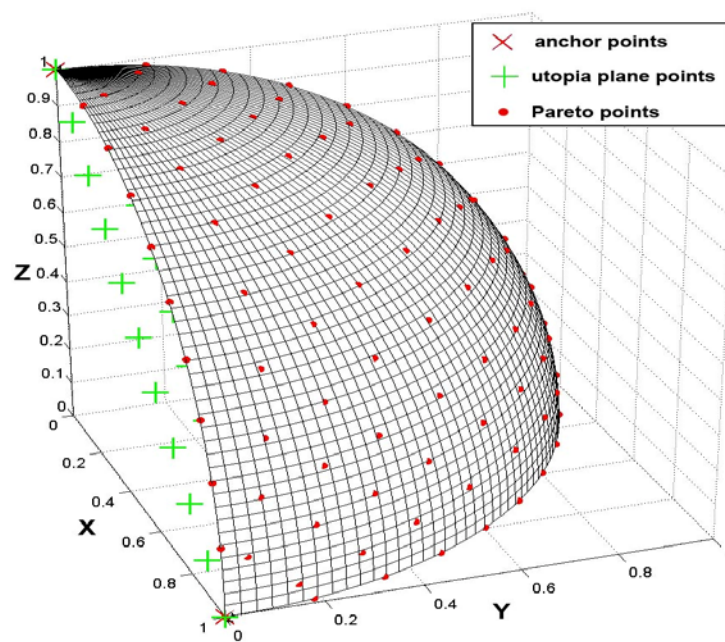


Fig. 3: 3D algebraic test case. Pareto surface.

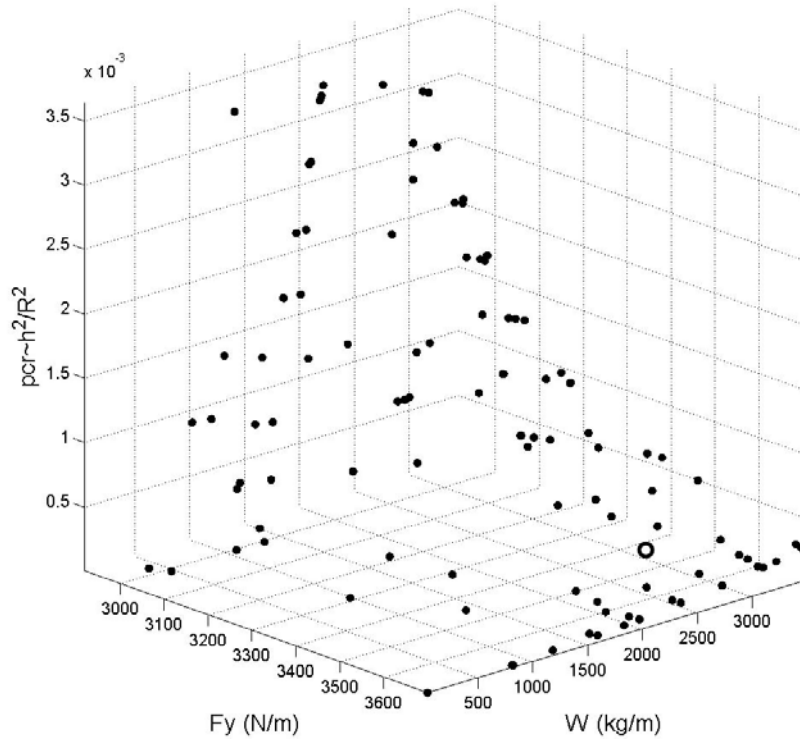


Fig. 4: Zhukowskii profile test case. Pareto frontier.

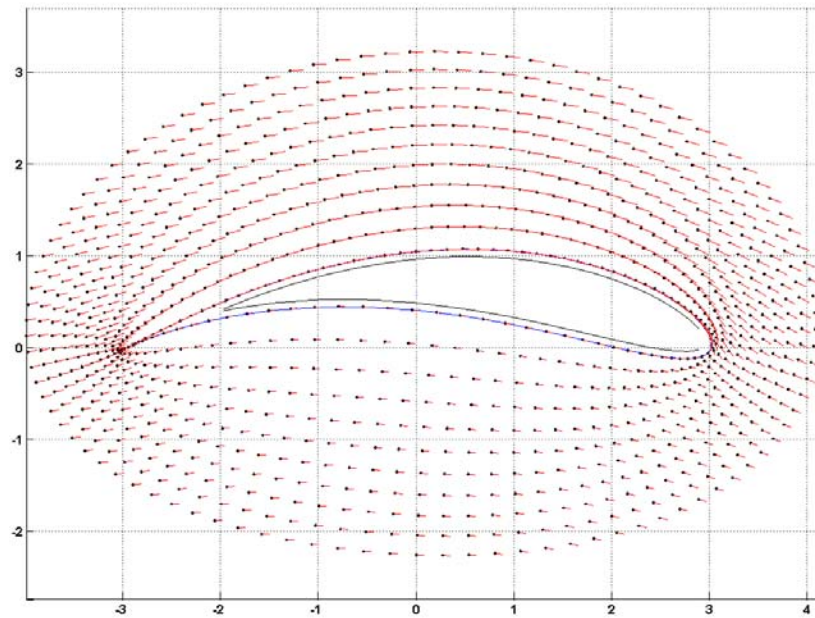


Fig. 5: Wing profile corresponding to a Pareto solution and flow velocity field.