Towards a modular version of Klyachko's theorem on Lie powers.

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(joint work with Roger Bryant)

Free associative algebra

Let V be a f.d. vector space over a field K with basis $\{x_1, \dots, x_n\}$.

Let $T(V)$ be the free associative K-algebra freely generated by $x_1, \dots, x_n$

Thus

$$\boxed{\begin{array}{l} T(V) = T^0(V) \oplus T^1(V) \oplus \cdots \oplus T^r(V) \oplus \cdots \\ \text{where } T^0(V) = K\cdot 1 \text{ and } T^r(V) \\ \text{is a f.d. subspace with basis} \\ \quad \{x_{i_1} \cdots x_{i_r} : i_1, \dots, i_r \in \{1, \dots, n\}\} \end{array}}$$

- $T^r(V)$ is spanned by all

$$v_1 \cdots v_r$$

with $v_1, \dots, v_r \in V$

- $T^r(V)$ may be identified with $V^{\otimes r}$ via

$$v_1 \cdots v_r \longleftrightarrow v_1 \otimes \cdots \otimes v_r.$$

Free Lie algebra

$T(V)$ is a Lie algebra under $[\cdot,\cdot]$, where

$$[a,b] := ab - ba$$

for all $a, b \in T(V)$

Let $L(V)$ denote the Lie subalgebra of $T(V)$ generated by $x_1, \dots, x_n$.

Then, by a theorem of Witt, $L(V)$ is a free Lie algebra freely generated by $x_1, \dots, x_n$.

$$\boxed{L(V) = L^1(V) \oplus L^2(V) \oplus \cdots \oplus L^r(V) \oplus \cdots \quad \text{where } L^r(V) = L(V) \cap T^r(V).}$$

- $L^1(V)$ is identified with V
- $L^r(V)$ is spanned by all left-normed commutators

$$[v_1, v_2, \dots, v_r]$$

with $v_1, v_2, \dots, v_r \in V$.

Tensor and Lie powers

Now let G be a group and suppose that V is a KG-module.

- $T^r(V)$ $(= V^{\otimes r})$ is a KG-module called the rth tensor power of V.

Action of $g \in G$:-

$$(v_1 v_2 \cdots v_r)g = (v_1 g)(v_2 g) \cdots (v_r g)$$

ie $$(v_1 \otimes v_2 \otimes \cdots \otimes v_r)g = (v_1 g) \otimes (v_2 g) \otimes \cdots \otimes (v_r g)$$

- $L^r(V)$ is a KG-submodule of $V^{\otimes r}$ called the rth Lie power of V.

Action of $g \in G$:-

$$[v_1, v_2, \ldots, v_r]g = [v_1 g, v_2 g, \ldots, v_r g]$$

Tensor and Lie powers

We started with a finite dimensional KG-module V and created two (infinite) families of finite dimensional KG-modules :-

$$
\begin{array}{lccccc}
\text{Tensor powers:} & V^{\otimes 0}, & V^{\otimes 1}, & V^{\otimes 2}, \ldots, & V^{\otimes r}, & \ldots \\
 & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \\
\text{Lie powers:} & & L^1(V), & L^2(V), \ldots, & L^r(V), & \ldots
\end{array}
$$

Research theme

Given a field K, a group G and a particular KG-module V, find $V^{\otimes r}$, $L^r(V)$ (and related modules) up to isomorphism.

Which indecomposable modules occur as direct summands?
With what multiplicity?

Polynomial $GL(n, K)$-modules

From now on we take K to be an alg. closed field of char $p \geq 0$, $G = GL(n, K)$

- W a f.d. KG-module with basis $\{w_1, \ldots, w_d\}$. For each $g = (g_{ij}) \in G$ let A_g denote the $d \times d$ matrix giving the action of g on W wrt the basis $\{w_1, \ldots, w_d\}$.

- We say that W is a polynomial module (of degree r) if there exist d^2 polynomials (homogeneous of degree r) $\varphi_{k,l}$ in n^2 variables such that the $(k,l)^{th}$ entry of A_g is $\varphi_{k,l}(g_{ij})$ for all $g \in G$.

Polynomial $GL(n,K)$-modules

Let $\mathrm{mod}_K(n,r)$ denote the category of finite dimensional polynomial KG-modules of degree r.

FACTS

1. If W is polynomial of degree r and U is polynomial of degree s then $W \otimes_K U$ is polynomial of degree $r+s$.

2. $\mathrm{mod}_K(n,r)$ is closed under direct sums, submodules and factor modules.

Examples

1. Let E denote the n-dimensional natural KG-module. Then E is a polynomial module of degree 1.

2. $E^{\otimes r}$ is a polynomial module of degree r, by FACT 1

3. $L^r(E)$ is a polynomial module of degree r, by FACT 2

The Schur functor

Let $\mathrm{Mod}(KS_r)$ denote the category of all f.d. KS_r-modules.

If $\underline{\underline{n \geq r}}$ then there is an exact functor

$$f_r : \mathrm{Mod}_K(n,r) \longrightarrow \mathrm{Mod}(KS_r)$$

called the Schur functor.

- $f_r(W) = W^{(1^r)}$

  a certain subspace (weight-space) of W, regarded as a KS_r-module via the embedding $S_r \subseteq GL(n,K)$

  $$\sigma \mapsto \left(\begin{array}{c|c} \text{perm matrix} & 0 \\ \hline 0 & \text{id} \end{array}\right)$$

- $\boxed{f_r(E^{\otimes r}) = KS_r}$

- $f_r(L^r(E)) = \mathrm{Lie}(r)$

  Lie module of the symmetric group.

Indecomposable summands of $E^{\otimes r}$

- For each partition λ of r into at most n parts there is a certain indecomposable "tilting" module $T(\lambda)$ in $\mathrm{Mod}_k(n,r)$.
- We say that a partition λ is row p-regular if no p parts of λ are equal.
- Let $n \geq r$. Then the set

$$\{ f_r(T(\lambda)) : \lambda \text{ is row } p\text{-regular} \}$$

is a full set of projective indecomposable kS_r - modules.

- Write $P^\lambda = f_r(T(\lambda))$.
  The head of P^λ is a simple kS_r-module denoted by D^λ.

Indecomposable summands of $E^{\otimes r}$

$$E^{\otimes r} \cong \bigoplus t_\lambda T(\lambda)$$

where the sum ranges over all row p-regular partitions of r into at most n parts

and $t_\lambda = \dim D^\lambda$.

<u>Remarks</u>

① When char $k = p = 0$, this result is classical.

The modules $E^{\otimes r}$ are known to be semisimple (Schur, 1927) and we have

- $T(\lambda)$ is simple
- $D^\lambda = S^\lambda$, the Specht module

so that $t_\lambda = \dfrac{n!}{\prod(\text{hook lengths})}$ $\left(= \#\text{ standard tableaux of shape } \lambda \right)$

② If char $k = p > 0$ then there is not, in general, an explicit formula for the multiplicities t_λ.

Indecomposable summands of $L^r(E)$

- Recall that $L^r(E)$ is a KG-submodule of $E^{\otimes r}$. We would like to describe $L^r(E)$ up to isomorphism.

> **Useful fact** If $\operatorname{char} k = p \nmid r$, then $L^r(E)$ is a direct summand of $E^{\otimes r}$

- Thus, in this case,

$$L^r(E) \cong \bigoplus l_\lambda T(\lambda)$$

where the sum ranges over all row p-regular partitions of r into at most n parts and

$$0 \leq l_\lambda \leq t_\lambda = \dim D^\lambda .$$

> **Question**: What is l_λ?

Indecomposable summands of $L^r(E)$

If char $k = p \nmid r$ then $L^r(E)$ is a direct summand of $E^{\otimes r}$ and we let l_λ denote the multiplicity of $T(\lambda)$ in $L^r(E)$.

Multiplicity formulae

- If char $k = 0$

$$l_\lambda = \frac{1}{r} \sum_{d|r} \mu(d)\, \chi^\lambda(\tau^{r/d})$$ — Wever, 1949

- If char $k > 0$

$$l_\lambda = \frac{1}{r} \sum_{d|r} \mu(d)\, \beta^\lambda(\tau^{r/d})$$ — Donkin + Erdmann, 1998

here μ denotes the Möbius function
χ^λ denotes the character of $D^\lambda = S^\lambda$
β^λ denotes the Brauer character of D^λ
and τ denotes a cycle of length r in S_r

Question : When is $l_\lambda > 0$?

Klyachko's theorem

Let char $k = 0$.

Recall that

$$E^{\otimes r} \cong \bigoplus t_\lambda T(\lambda)$$

where the sum ranges over all partitions of r into at most n parts, the modules $T(\lambda)$ are simple and $t_\lambda = \dim S^\lambda$, the Specht module.

Theorem, (Klyachko, 1974)

Let $r \geq 3$ and let λ be a partition of r into at most n parts.

$$T(\lambda) \mid L^r(E) \iff \begin{array}{l} \lambda \neq (1^r) \\ \lambda \neq (r) \\ \lambda \neq (2^2) \\ \lambda \neq (2^3) \end{array}$$

That is, almost all isomorphism types of indecomposable summands of $E^{\otimes r}$ occur as summands of $L^r(E)$.

Klyachko's theorem – Sketch of proof

- S_r acts on the left of $E^{\otimes r}$ by place permutations.
- Let ξ be a primitive rth root of unity in K and let

$$U_\xi = \{ u \in E^{\otimes r} : \tau u = \xi u \}$$

where τ is a cycle of length r in S_r.

[Note: U_ξ is the module from Roger's talk in the case where we take $G = GL(n, K)$ and $V = E$ the natural KG-module]

$$\boxed{\text{First show that } L^r(E) \cong U_\xi \text{ as } KG\text{-modules for \underline{any} primitive root } \xi}$$

- Thus we want to know when $T(\lambda) \mid U_\xi$.
- Since $T(\lambda)$ is <u>simple</u> we find that

$$f_r(T(\lambda)) = D^\lambda$$

- Also, $\quad f_r(U_\xi) = I \uparrow_{\langle\tau\rangle}^{S_r}$

where I is <u>any</u> 1-dimensional faithful $K\langle\tau\rangle$-module.

Klyachko's theorem - Sketch of proof

- Thus

$$T(\lambda) \mid L^r(E) \iff D^\lambda \mid I\uparrow_{\langle\tau\rangle}^{S_r}$$

$$\iff \operatorname{Hom}(I\uparrow_{\langle\tau\rangle}^{S_r}, D^\lambda) \neq 0$$

$$\iff \operatorname{Hom}(I, D^\lambda\downarrow_{\langle\tau\rangle}^{S_r}) \neq 0.$$

Now show that if $r > 6$ and D^λ is faithful then $D^\lambda\downarrow_{\langle\tau\rangle}^{S_r}$ contains a faithful 1-dimensional submodule.

- Remains to show that $T(1^r), T(r) \nmid L^r(E)$ and $T(2^2) \nmid L^4(E)$, $T(2^3) \nmid L^6(E)$.

Use multiplicity formulae, for example.

□

<u>Remark</u> If $\operatorname{char} k = p > 0$ and $p \nmid r$ then this argument works with little modification so that almost all isomorphism types of indecomposable summands of $E^{\otimes r}$ occur as summands of $L^r(E)$.

Klyachko's theorem in arbitrary degree?

Question Which indecomposable summands $T(\lambda)$ of $E^{\otimes r}$ also occur as direct summands of $L^r(E)$?

- If $\operatorname{char} k \nmid r$: almost all.
- What about when $\operatorname{char} k \mid r$?

Suppose that $r = p^m k$ where $p \nmid k$.
Then the Decomposition theorem gives

$$L^{p^m k}(E) \cong L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^m k})$$

where $B_{p^i k} \subseteq L^{p^i k}(E)$ and $B_{p^i k} \mid E^{\otimes p^i k}$.

- If $k=1$, this decomposition is trivial.
- For $k>1$, we find that $B_{p^m k} \mid L^{p^m k}(E)$

Idea: Try to show that almost all $T(\lambda)$ occur as summands of $B_{p^m k}$.

A modular version of Klyachko's theorem?

We will use information about the B's from Roger's talk :-

Let Δ be the set of k^{th} roots of unity in K.

① $B_{p^m k}$ is isomorphic to a direct sum of modules of the form

$$U_{\delta_1} \otimes \cdots \otimes U_{\delta_{p^m}}$$

where $\delta_1, \ldots, \delta_{p^m} \in \Delta$ and $\delta_1 \cdots \delta_{p^m}$ is a primitive k^{th} root of unity.

[Recall: $U_\delta = \{ u \in E^{\otimes k} : \tau u = \delta u \}$
where τ is a cycle of length r in S_r]

② Suppose that $k > 2$ and $p^m \neq 2$ if $k = 3$.
Then each such product

$$U_{\delta_1} \otimes \cdots \otimes U_{\delta_{p^m}}$$

occurs as a summand of $B_{p^m k}$.

Main result

Let $r = p^m k$ where $m \geqslant 0$, $k > 2$ and $p \nmid k$.

Let λ be a p-regular partition of r with at most n parts.

$T(\lambda) \mid B_{p^m k} \iff$ (i) $\lambda \neq \mu$, where $\mu' = (p-1, \ldots, p-1, b)$ with $1 \leq b \leq p-1$.

(ii) $\lambda \neq (r)$

(iii) $\lambda \neq (2^2)$

(iv) $\lambda \neq (2^3)$

(v) $\lambda \neq (4,2)$ if $p^m = 2$ and $k = 3$.

Thus almost all isomorphism types of indecomposable summands of $E^{\otimes r}$ and hence occur as summands of $B_{p^m k}$ and hence of $L^r(E)$

Remarks * Proof is an extended version of Klyachko's argument.

[We show that $T(\lambda) \mid U_{\delta_1} \otimes \cdots \otimes U_{\delta_{p^m}}$ for some choice of $\delta_1, \ldots, \delta_{p^m}$ with $\delta_1 \cdots \delta_{p^m}$ primitive]

* Need $k > 2$ so that we have "enough" choice.