

# Tropical Linear Algebra (Achievements and Challenges)

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### III. Tropical permanent

# Tropical permanent - basics

$$A = (a_{ij}) \in \overline{\mathbb{R}}^n$$

(Conventional) permanent:

$$\text{per}(A) = \sum_{\pi \in P_n} \prod_{i \in N} a_{i, \pi(i)}$$

Tropical permanent:

$$\text{tper}(A) = \sum_{\pi \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} a_{i, \pi(i)}$$

where  $P_n$  ... the set of permutations of  $N = \{1, \dots, n\}$

In conventional notation:

$$\text{tper}(A) = \max_{\pi \in P_n} \sum_{i \in N} a_{i, \pi(i)}$$

The weight of  $\pi \in P_n$  :

$$w(\pi, A) = \prod_{i \in N}^{\otimes} a_{i, \pi(i)} = \sum_{i \in N} a_{i, \pi(i)}$$

$\pi \in P_n$  is *optimal* if

$$w(\pi, A) = tper(A)$$

The task of finding an optimal permutation ... *assignment problem*  
... *Hungarian method*  $O(n^3)$

Aspects and variants in: *Burkard RE, Dell'Amico M, Martello S (2009) Assignment problems. SIAM, Philadelphia*

Denote

$$ap(A) = \{\pi \in P_n; w(\pi, A) = tper(A)\}$$

**Lemma:** If  $A, C, D \in \overline{\mathbb{R}}^{n \times n}$  and  $C$  and  $D$  are diagonal matrices then

$$ap(C \otimes A \otimes D) = ap(A) \text{ and}$$

$$tper(C \otimes A \otimes D) = tper(C) \otimes tper(A) \otimes tper(D)$$

**Theorem (Hungarian Method)** For every  $A \in \overline{\mathbb{R}}^{n \times n}$  with  $tper(A) > \varepsilon$  there exist diagonal matrices  $C$  and  $D$  such that

$$B \stackrel{df}{=} C \otimes A \otimes D \leq 0$$

$tper(C \otimes A \otimes D) = 0$  and (hence)

$$ap(A) = \left\{ \pi \in P_n; b_{i,\pi(i)} = 0 \text{ for all } i \in N \right\}$$

$C$  and  $D$  can be found in  $O(n^3)$  time

Dequantisation:  $(a^k + b^k)^{1/k} \longrightarrow \max(a, b)$  for  $k \longrightarrow \infty$

$A$  ... a nonnegative matrix

$\{A^k\}_{k=1}^{\infty}$  ... sequence of *Hadamard (Schur)* powers

$$\left(\text{per}(A^k)\right)^{1/k} \longrightarrow \max_{\pi \in P_n} \prod_{i \in N} a_{i, \pi(i)} = \sum_{\pi \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} a_{i, \pi(i)} = \text{tper}(A)$$

in max-times

# Van der Waerden Conjecture

Among all doubly stochastic  $n \times n$  matrices the permanent obtains its minimum for the matrix  $A = (a_{ij})$ , where  $a_{ij} = \frac{1}{n}$  for all  $i, j \in N$ .

Among all doubly stochastic  $n \times n$  matrices the **tropical** permanent obtains its minimum for the matrix  $A = (a_{ij})$ , where  $a_{ij} = \frac{1}{n}$  for all  $i, j \in N$ .

# Linear (in)dependence

$v_1, \dots, v_n \in \overline{\mathbb{R}}^m$  are called  
(Weakly) *LD* iff for some  $k$  and  $\alpha_j \in \overline{\mathbb{R}}$

$$v_k = \sum_{j \neq k}^{\oplus} \alpha_j \otimes v_j$$

*Gondran-Minoux LD* iff for some  
 $U, V \subseteq \{1, \dots, n\}$ ,  $U \cap V = \emptyset$ ,  $U, V \neq \emptyset$  and  $\alpha_j \in \overline{\mathbb{R}}$

$$\sum_{j \in U}^{\oplus} \alpha_j \otimes v_j = \sum_{j \in V}^{\oplus} \alpha_j \otimes v_j$$

*Strongly LD* iff

$$\sum_{j=1, \dots, n}^{\oplus} v_j \otimes x_j = v$$

does not have a unique solution for any  $v \in \mathbb{R}^m$

$\omega \in \{\text{strong, GM, weak}\}$

$\omega$ -independent  $\equiv$  not  $\omega$ -dependent

SLI  $\implies$  GMLI  $\implies$  WLI

# Relations between independences

SLI  $\implies$  GMLI  $\implies$  WLI

SLI  $\implies$  GMLI  $\implies m \geq n$

WLI does not imply  $m \geq n$  in general:

$v_1, \dots, v_n \in \overline{\mathbb{R}}^m$  may be WLI for any  $n$  if  $m \geq 3$  ("dimensional anomaly")

WLI  $\implies m \geq n$  if  $m = 2$

# Checking independence

No polynomial method for checking SLI or GMLI is known (?)  
OPEN: Are SLI and GMLI polynomially verifiable or  $NP$ -complete?  
LI can be checked in  $O(mn^2)$  time

# Types of regularity

$\omega$ -regular matrix ... square matrix with  $\omega$ -independent columns

$A$  is strongly regular  $\implies A$  is GM regular  $\implies$  regular

$\omega$ -rank of an  $m \times n$  matrix  $A$  is the greatest  $r$  such that  $A$  contains an  $r \times r$   $\omega$ -regular submatrix

Independence of columns and full column rank for an  $m \times n$  matrix  $A$ :

$A$  has SLI columns  $\iff A$  has full strong rank

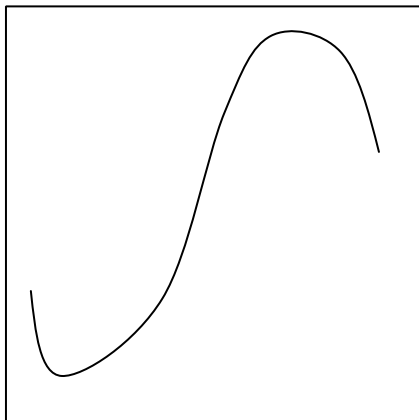
$A$  has GMLI columns  $\iff A$  has full GM rank

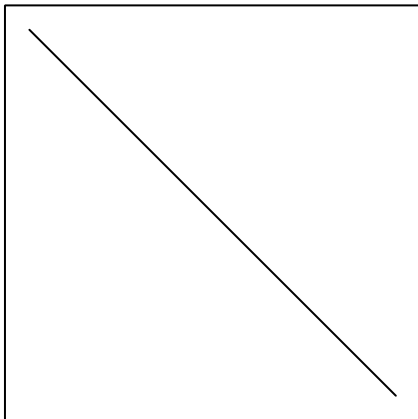
$\implies$  disproved by S.Gaubert (2003)

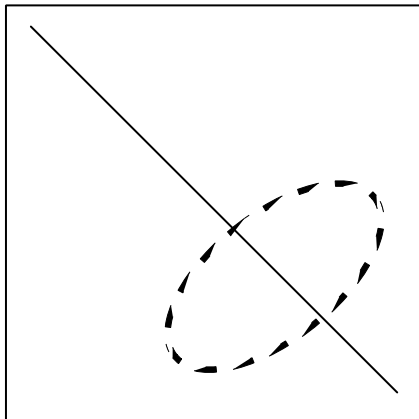
# Criteria for checking regularity

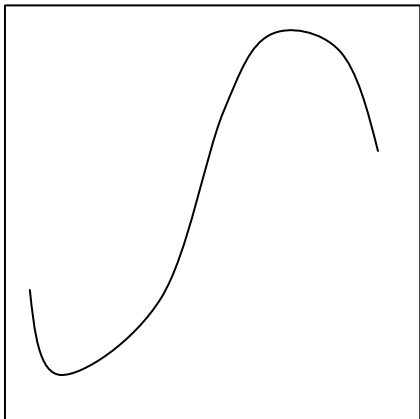
$A$  is strongly regular  $\iff |ap(A)| = 1$

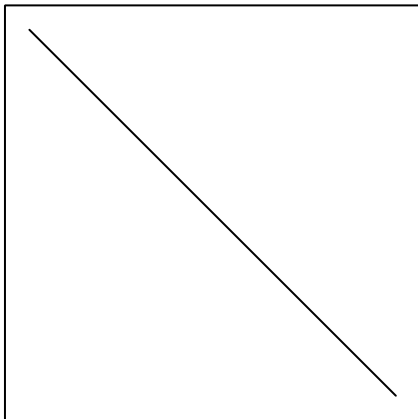
$A$  is GM regular  $\iff$  all permutations in  $ap(A)$  are of the same parity

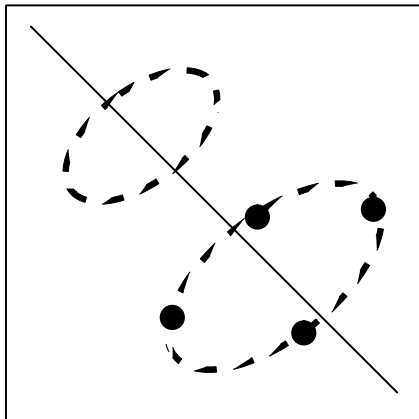












# Criteria for checking regularity

$A$  strongly regular  $\iff$

$$|ap(A)| = 1 \iff$$

$D_A$  does not contain a *cycle*

$A$  GM regular  $\iff$

$$ap(A) \subseteq P_n^+ \text{ or } ap(A) \subseteq P_n^- \iff$$

$D_A$  does not contain an *even cycle*

# Criteria for checking regularity

*Sign* of a cyclic permutation (cycle)  $\sigma = (i_1 i_2 \dots i_k) \dots$   
 $sgn(\sigma) = (-1)^{k-1}$

$\sigma$  has odd parity  $\iff$  sign of  $\sigma$  is  $-1 \iff k$  even

If  $\pi = \pi_1 \circ \dots \circ \pi_r$  then the *sign* of  $\pi$  is

$$sgn(\pi) = sgn(\pi_1) \cdot \dots \cdot sgn(\pi_r)$$

$sgn(\pi) = -1$  ( $+1$ ) ...  $\pi$  is *odd* (*even*)

$\pi$  odd  $\implies$  at least one of the constituent cycles is even

# Criteria for checking regularity

The set of odd (even) permutations ...  $P_n^-$  ( $P_n^+$ )

$$ap^+(A) = ap(A) \cap P_n^+$$

$$ap^-(A) = ap(A) \cap P_n^-$$

$$tper^+(A) = \max_{\pi \in P_n^+} \sum_{i \in N} a_{i, \pi(i)}$$

$$tper^-(A) = \max_{\pi \in P_n^-} \sum_{i \in N} a_{i, \pi(i)}$$

# Criteria for checking regularity

$A$  is Gondran-Minoux regular  $\iff$

$ap^+(A) = \emptyset$  or  $ap^-(A) = \emptyset$   $\iff$

$tper^+(A) \neq tper^-(A)$   $\iff$

$D_A$  does *not* contain an even cycle ... can be checked in  $O(n^3)$   
time

# Parity assignment problem

OPEN: Find **both**  $tper^+(A)$  and  $tper^-(A)$

Note:  $tper^+(A) \oplus tper^-(A) = tper(A)$

Solvable in linear time (after solving the assignment problem) for  
Monge matrices  
diagonally dominant symmetric matrices

Tropical polynomials:

$$p(z) = a_0 \oplus a_1 \otimes z \oplus \dots \oplus a_n \otimes z^n$$

But also

$$8.3 \otimes z^{-7.2} \oplus (-2.6) \otimes z^{3.7} \oplus 6.5 \otimes z^{12.3}$$

In general, a tropical polynomial is

$$p(z) = \sum_{r=0, \dots, p}^{\oplus} c_r \otimes z^{j_r}, \quad c_r, j_r \in \mathbb{R}$$

$j_p$  ... the *degree* and

$p + 1$  ... the *length* of  $p(z)$

# Tropical polynomials

$$p(z) = \sum_{r=0, \dots, p}^{\oplus} c_r \otimes z^{j_r}$$

$p(z)$  ... formal algebraic expressions with  $z$  as an indeterminate  
or

$p(z)$  ... tropical functions of  $z$

In conventional notation  $p(z)$ :

$$\max_{r=0, \dots, p} (c_r + j_r z)$$

**Theorem (Cunninghame-Green, 1983)** Every tropical polynomial can be factorised to linear terms in linear time in the degree  $p$ .

$$(a \oplus b)^k = a^k \oplus b^k$$

$$(1 \oplus z)^2 = 2 \oplus z^2$$

$$(1 \oplus z)^2 = 2 \oplus 1 \otimes z \oplus z^2$$

$$2 \oplus z^2 = 2 \oplus 1 \otimes z \oplus z^2 \text{ as functions}$$

$$1 \otimes z \leq 2 \oplus z^2$$

The term  $c_s \otimes z^{j_s}$  of a tropical polynomial  $\sum_{r=0, \dots, p}^{\oplus} c_r \otimes z^{j_r}$  is called *inessential* if

$$c_s \otimes z^{j_s} \leq \sum_{r \neq s}^{\oplus} c_r \otimes z^{j_r}$$

holds for every  $z \in \mathbb{R}$  and *essential* otherwise

Inessential terms can be omitted if tropical polynomials are considered as functions

# Tropical characteristic polynomial

Tropical characteristic polynomial for  $A \in \overline{\mathbb{R}}^{n \times n}$ :

$$\begin{aligned}\chi_A(\lambda) &= tper(A \oplus \lambda \otimes I) \\ &= tper \begin{pmatrix} a_{11} \oplus \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} \oplus \lambda & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \oplus \lambda \end{pmatrix} \\ &= \delta_n \oplus \delta_{n-1} \otimes \lambda \oplus \dots \oplus \delta_1 \otimes \lambda^{n-1} \oplus \lambda^n\end{aligned}$$

$$\chi_A(\lambda) = (\lambda \oplus c_1) \otimes (\lambda \oplus c_2) \otimes \dots \otimes (\lambda \oplus c_n)$$

$c_1, \dots, c_n$  ... corners of  $\chi_A(\lambda)$

All eigenvalues are corners (but not all corners are eigenvalues!)

$$\max_i c_i = \lambda(A)$$

# Calculating the coefficients of a tropical characteristic polynomial

If  $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$  then for  $k = 1, \dots, n$ :

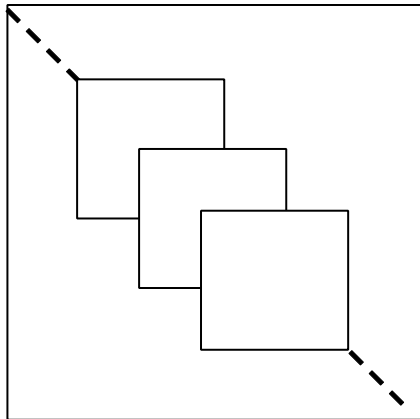
$$\delta_k = \max \{ tper(B); B \text{ is a } k \times k \text{ **principal** submatrix of } A \}$$

$$\delta_n = tper(A)$$

$$\delta_1 = \max(a_{11}, a_{22}, \dots, a_{nn})$$

$$\chi_A(\lambda) = \lambda^n \text{ if and only if } D_A \text{ is acyclic}$$

# Calculating the coefficients of a tropical characteristic polynomial



# Calculating the coefficients of a tropical characteristic polynomial

Checking all  $\binom{n}{k}$  principal submatrices  $\longrightarrow$  polynomial for a fixed  $k$

$$\sum_{k=1}^n \binom{n}{k} = 2^n - 1$$

No polynomial method for finding  $\delta_k$  for all  $k$  (?)

Modification obtained after removing the word "principal" is

$O(n^3)$  for every  $k$

# Calculating the coefficients of a tropical characteristic polynomial

All **essential terms** of a tropical characteristic polynomial can be found in  $< O(n^4)$  time

OPEN: Can all coefficients be found in polynomial time?

Related combinatorial problem:

*Given a digraph  $D$  with  $n$  nodes and a positive integer  $k$  ( $k < n$ ), is there a set of pairwise node-disjoint cycles covering exactly  $k$  nodes of  $D$ ?*

# An application: the job rotation problem

A company with  $n$  employees requires workers to swap their jobs  
To avoid disruption: only  $k$  ( $k < n$ ), should actually swap their jobs

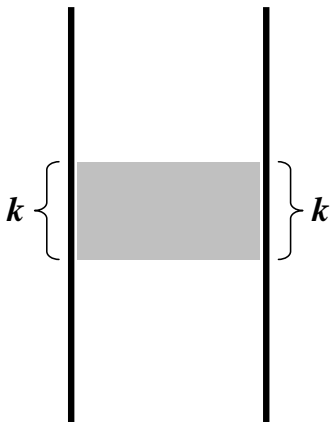
$a_{ij}$  ... costs associated with the swaps  $\longrightarrow$  min

$a_{ij}$  ... measures of preference of the swaps  $\longrightarrow$  max

The *best principal submatrix problem* (BPSM):

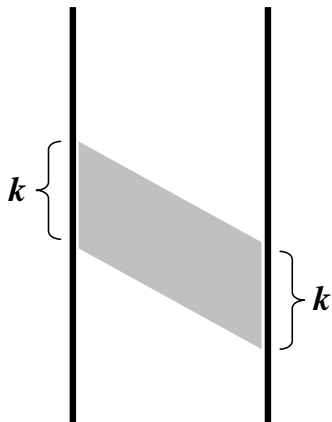
*Given a real  $n \times n$  matrix  $A$  and a  $k < n$ , find a  $k \times k$  principal submatrix of  $A$  whose optimal assignment problem value is maximal.*

# An application: the job rotation problem



$n$  positions

$n$  positions



$n$  positions

$n$  positions

# Tropical Cayley-Hamilton

$$A = (a_{ij}) \in \mathbb{R}^{n \times n}, v \in \mathbb{R}$$

$$p^+(A, v) = |\{\pi \in P_n^+; w(\pi, A) = v\}|$$

$$p^-(A, v) = |\{\pi \in P_n^-; w(\pi, A) = v\}|$$

$$P_k(A) = \{B; B \text{ is a } k \times k \text{ principal submatrix of } A\}$$

# Tropical Cayley-Hamilton

The *tropical characteristic equation* for  $A$ :

$$\lambda^n \oplus \sum_{k \in J}^{\oplus} c_{n-k} \otimes \lambda^k = c_1 \otimes \lambda^{n-1} \oplus \sum_{k \in \bar{J}}^{\oplus} c_{n-k} \otimes \lambda^k$$

where

$$c_k = \max \left\{ v; \sum_{B \in P_k(A)} p^+(B, v) \neq \sum_{B \in P_k(A)} p^-(B, v) \right\}, \quad k = 1, \dots, n$$

$$J = \{j; d_j > 0\}, \quad \bar{J} = \{j; d_j < 0\}$$

$$d_k = (-1)^k \left( \sum_{B \in P_k(A)} p^+(B, c_k) - \sum_{B \in P_k(A)} p^-(B, c_k) \right), \quad k = 1, \dots, n$$

## **Theorem (Tropical Cayley-Hamilton, Olsder and Roos, 1988)**

Every real square matrix  $A$  satisfies its tropical characteristic equation.

Finding the tropical characteristic equation is usually hard

A special case: If  $ap(A) = \{id\}$  then the tropical characteristic equation of  $A$  is:

$$\begin{aligned} & \lambda^n \oplus \delta_2 \otimes \lambda^{n-2} \oplus \delta_4 \otimes \lambda^{n-4} \oplus \dots \\ = & \delta_1 \otimes \lambda^{n-1} \oplus \delta_3 \otimes \lambda^{n-3} \oplus \delta_5 \otimes \lambda^{n-5} \oplus \dots \end{aligned}$$

where  $\delta_k =$  the sum of the  $k$  greatest diagonal values of  $A$

# An escape from max-plus: max-min ("bottleneck algebra")

$$a \oplus b = \max(a, b)$$

$$a \otimes b = \min(a, b)$$

$$a, b \in \mathbb{B}$$

$(\mathbb{B}, \leq)$  ... dense linearly ordered set without maximum and minimum

# Bottleneck assignment problem (bottleneck tropical permanent)

Assignment problem  $\longrightarrow$  *Bottleneck* assignment problem

Bottleneck permanent:

$$bper(A) = \sum_{\pi \in P_n}^{\oplus} \prod_{i \in N}^{\otimes} a_{i, \pi(i)} = \max_{\pi \in P_n} \min_{i \in N} a_{i, \pi(i)}$$

$$w(\pi, A) = \min_{i \in N} a_{i, \pi(i)}$$

$$bap(A) = \{\pi \in P_n; w(\pi, A) = bper(A)\}$$

Best computational complexity:  $O(n^2 \sqrt{n / \log n})$

# Trapezoidal matrices

$$A \oplus B = \dots \quad A \otimes B = \dots$$

$A \in \mathbb{B}^{m \times n}$  has SLI columns if  $A \otimes x = b$  has a unique solution for at least one  $b \in \mathbb{B}^m$

$A$  is strongly regular if it has SLI columns and  $m = n$

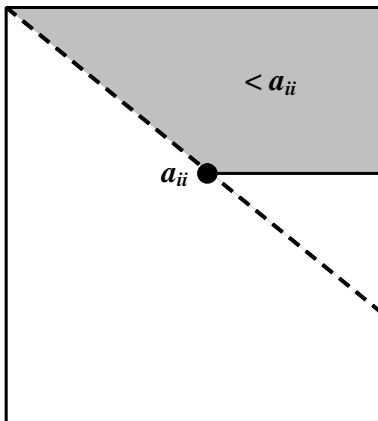
$A$  is said to be similar to  $B$  ( $A \sim B$ ) if  $A$  can be obtained from  $B$  by permuting the rows and/or columns (independently)

$A = (a_{ij})$  is called *trapezoidal* if

$$a_{ij} > a_{kj} \text{ for every } i, j, k, \quad k \leq i \text{ and } k < j$$

# Trapezoidal matrices

$a_{ii} > a_{kj}$  for every  $i, j, k$ ,  $k \leq i$  and  $k < j$



**Theorem (PB et al, 1987)** Let  $A \in \mathbb{B}^{m \times n}$ . Then

$A$  has SLI columns  $\iff A \sim B$ ,  $B$  trapezoidal

$m = n$ ,  $A$  has strong permanent  $\implies A \sim B$ ,  $B$  trapezoidal

**Corollary:**  $m = n$ ,  $A$  has strong permanent  $\implies A$  is strongly regular (but not  $\longleftarrow$ )

Strong bottleneck permanent can be checked in  $O(n^2 \log n)$  time

**Corollary:** Bottleneck assignment problem can be solved in  $O(n^2 \log n)$  time if there is exactly one optimal permutation.

Thank you!