PREDICATE LOGIC 2009/10

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http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/teaching/PredicateLogic/index.php

Revision of propositional logic

Contents

1. Languages

Propositional logic (sometimes called sentential logic or propositional calculus) attempts to formalize the reasoning that can be done with connectives like **not**, **and**, **or**, and **if . . . then**. We will define the formal language of propositional logic, $\mathcal{L}_P$, by specifying its symbols and rules for assembling these symbols into the formulas of the language.

Definition 1.1. The **symbols** (or **letters**) of $\mathcal{L}_P$ are:

(i) Brackets: (and).
(ii) Connectives: $\neg$ and $\to$.
(iii) Atomic formulas: A_0, A_1, A_2, . . . , A_n, . . .

We still need to specify the ways in which the symbols of $\mathcal{L}_P$ can be put together.

Definition 1.2. The **formulas** of $\mathcal{L}_P$ are those finite sequences or strings of the symbols given in Definition 1.1 which satisfy the following rules:

(i) Every atomic formula is a formula.
(ii) If α is a formula, then $(\neg\alpha)$ is a formula.
(iii) If α and β are formulas, then $(\alpha \to \beta)$ is a formula.
(iv) No other sequence of symbols is a formula.

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We will often use lower-case Greek characters to represent $\mathcal{L}_P$-formulas, as we did in the definition above, and upper-case Greek characters to represent sets of formulas.

The atomic formulas, A_0, A_1, ..., are meant to represent statements that cannot be broken down any further using our connectives, such as "The moon is made of cheese." Thus, one might translate the the English sentence "If the moon is red, it is not made of cheese" into the formula $(A_0 \to (\neg A_1))$ of $\mathcal{L}_P$ by using A_0 to represent "The moon is red" and A_1 to represent "The moon is made of cheese." Note that the truth of the formula depends on the interpretation of the atomic sentences which appear in it. Using the interpretations just given of A_0 and A_1, the formula $(A_0 \to (\neg A_1))$ is true, but if we instead use A_0 and A_1 to interpret "My telephone is ringing" and "Someone is calling me", respectively, $(A_0 \to (\neg A_1))$ is false.

Definition 1.2 says that that every atomic formula is a formula and every other formula is built from shorter formulas using the connectives and brackets in particular ways.

Proposition 1.3. *The set of formulas of $\mathcal{L}_P$ is countable.*

Informal Conventions

At first glance, $\mathcal{L}_P$ may not seem capable of breaking down English sentences with connectives other than **not** and **if ... then**. However, the sense of many other connectives can be captured by these two by using suitable circumlocutions. We will use the symbols $\wedge$, $\vee$, and $\leftrightarrow$ to represent **and**, **or**,[1] and **if and only if** respectively. Since they are not among the symbols of $\mathcal{L}_P$, we will use them as abbreviations for certain constructions involving only $\neg$ and $\to$. Namely,

- $(\alpha \wedge \beta)$ is short for $(\neg(\alpha \to (\neg\beta)))$,
- $(\alpha \vee \beta)$ is short for $((\neg\alpha) \to \beta)$, and
- $(\alpha \leftrightarrow \beta)$ is short for $((\alpha \to \beta) \wedge (\beta \to \alpha))$.

Interpreting A_0 and A_1 as before, for example, one could translate the English sentence "The moon is red and made of cheese" as $(A_0 \wedge A_1)$. (Of course this is really $(\neg(A_0 \to (\neg A_1)))$, **i.e.** "It is not the case that if the moon is green, it is not made of cheese.") $\wedge$, $\vee$, and $\leftrightarrow$ were not included among the official symbols of $\mathcal{L}_P$ partly because we can get by without them and partly because leaving them out makes it easier to prove things about $\mathcal{L}_P$.

For the sake of readability, we will occasionally use some informal conventions that let us get away with writing fewer brackets:

- We will usually drop the outermost brackets in a formula, writing $\alpha \to \beta$ instead of $(\alpha \to \beta)$ and $\neg\alpha$ instead of $(\neg\alpha)$.
- We will let $\neg$ take precedence over $\to$ when brackets are missing, so $\neg\alpha \to \beta$ is short for $((\neg\alpha) \to \beta)$, and fit the informal connectives into this scheme by letting the order of precedence be $\neg$, $\wedge$, $\vee$, $\to$, and $\leftrightarrow$.
- Finally, we will group repetitions of $\to$, $\vee$, $\wedge$, or $\leftrightarrow$ to the right when brackets are missing, so $\alpha \to \beta \to \gamma$ is short for $(\alpha \to (\beta \to \gamma))$.

Just like formulas using $\vee$, $\wedge$, or $\neg$, formulas in which brackets have been omitted as above are not official formulas of $\mathcal{L}_P$, they are convenient abbreviations for official

[1] We will use **or** inclusively, so that "A or B" is still true if both of A and B are true.

formulas of $\mathcal{L}_P$. Note that a precedent for the precedence convention can be found in the way that $\cdot$ commonly takes precedence over $+$ in writing arithmetic formulas.

The following notion will be needed later on.

Definition 1.4. Suppose φ is a formula of $\mathcal{L}_P$. The set of **subformulas** of φ, $\mathcal{S}(\varphi)$, is defined as follows.

(i) If φ is an atomic formula, then $\mathcal{S}(\varphi) = \{\varphi\}$.
(ii) If φ is $(\neg\alpha)$, then $\mathcal{S}(\varphi) = \mathcal{S}(\alpha) \cup \{(\neg\alpha)\}$.
(iii) If φ is $(\alpha \to \beta)$, then $\mathcal{S}(\varphi) = \mathcal{S}(\alpha) \cup \mathcal{S}(\beta) \cup \{(\alpha \to \beta)\}$.

For example, if φ is $(((\neg A_1) \to A_7) \to (A_8 \to A_1))$, then $\mathcal{S}(\varphi)$ includes A_1, A_7, A_8, $(\neg A_1)$, $(A_8 \to A_1)$, $((\neg A_1) \to A_7)$, and $(((\neg A_1) \to A_7) \to (A_8 \to A_1))$ itself.

Note that if you write out a formula with all the official brackets, then the subformulas are just the parts of the formula enclosed by matching brackets, plus the atomic formulas. In particular, every formula is a subformula of itself. Note that some subformulas of formulas involving our informal abbreviations $\vee$, $\wedge$, or $\leftrightarrow$ will be most conveniently written using these abbreviations. For example, if ψ is $A_4 \to A_1 \vee A_4$, then

$$\mathcal{S}(\psi) = \{\, A_1,\, A_4,\, (\neg A_1),\, (A_1 \vee A_4),\, (A_4 \to (A_1 \vee A_4)) \,\}\,.$$

(As an exercise, where did $(\neg A_1)$ come from?)

Unique Readability

The slightly paranoid — er, truly rigorous — might ask whether Definitions 1.1 and 1.2 actually ensure that the formulas of $\mathcal{L}_P$ are unambiguous, i.e. can be read in only one way according to the rules given in Definition 1.2. To actually prove this one must add to Definition 1.1 the requirement that all the symbols of $\mathcal{L}_P$ are distinct and that no symbol is a subsequence of any other symbol. With this addition, one can prove the following:

Theorem 1.5 (Unique Readability Theorem). *A formula of $\mathcal{L}_P$ must satisfy exactly one of conditions 1–3 in Definition 1.2.*

2. Truth Assignments

Whether a given formula φ of $\mathcal{L}_P$ is true or false usually depends on how we interpret the atomic formulas which appear in φ. For example, if φ is the atomic formula A_2 and we interpret it as "$2 + 2 = 4$", it is true, but if we interpret it as "The moon is made of cheese", it is false. Since we don't want to commit ourselves to a single interpretation — after all, we're really interested in general logical relationships — we will define how any assignment of **truth values** T ("true") and F ("false") to atomic formulas of $\mathcal{L}_P$ can be extended to all other formulas. We will also get a reasonable definition of what it means for a formula of $\mathcal{L}_P$ to follow logically from other formulas.

Definition 2.1. A **truth assignment** or a **valuation** is a function v whose domain is the set of all formulas of $\mathcal{L}_P$ and whose range is the set $\{T, F\}$ of truth values, such that:

(i) $v(A_n)$ is defined for every atomic formula A_n.

(ii) For any formula α,

$$v(\,(\neg\alpha)\,) = \begin{cases} T & \text{if } v(\alpha) = F \\ F & \text{if } v(\alpha) = T. \end{cases}$$

(iii) For any formulas α and β,

$$v(\,(\alpha \to \beta)\,) = \begin{cases} F & \text{if } v(\alpha) = T \text{ and } v(\beta) = F \\ T & \text{otherwise.} \end{cases}$$

Given interpretations of all the atomic formulas of $\mathcal{L}_P$, the corresponding truth assignment would give each atomic formula representing a true statement the value T and every atomic formula representing a false statement the value F. Note that we have not defined how to handle any truth values besides T and F in $\mathcal{L}_P$. Logics with other truth values have applications, e.g. in computer science.

For an example of how non-atomic formulas are given truth values on the basis of the truth values given to their components, suppose v is a truth assignment such that $v(A_0) = T$ and $v(A_1) = F$. Then $v(\,((\neg A_1) \to (A_0 \to A_1))\,)$ is determined from $v(\,(\neg A_1)\,)$ and $v(\,(A_0 \to A_1)\,)$ according to clause 3 of Definition 2.1. In turn, $v(\,(\neg A_1)\,)$ is determined from of $v(A_1)$ according to clause 2 and $v(\,(A_0 \to A_1)\,)$ is determined from $v(A_1)$ and $v(A_0)$ according to clause 3. Finally, by clause 1, our truth assignment must be defined for all atomic formulas to begin with; in this case, $v(A_0) = T$ and $v(A_1) = F$. Thus $v(\,(\neg A_1)\,) = T$ and $v(\,(A_0 \to A_1)\,) = F$, so $v(\,((\neg A_1) \to (A_0 \to A_1))\,) = F$.

A convenient way to write out the determination of the truth value of a formula on a given truth assignment is to use a **truth table**: list all the subformulas of the given formula across the top in order of length and then fill in their truth values on the bottom from left to right. Except for the atomic formulas at the extreme left, the truth value of each subformula will depend on the truth values of the subformulas to its left. For the example above, one gets something like:

A_0	A_1	$(\neg A_1)$	$(A_0 \to A_1)$	$(\neg A_1) \to (A_0 \to A_1))$
T	F	T	F	F

The use of finite truth tables to determine what truth value a particular truth assignment gives a particular formula is justified by the following proposition, which asserts that only the truth values of the atomic sentences in the formula matter.

Proposition 2.2. *Suppose δ is any formula and u and v are truth assignments such that $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ. Then $u(\delta) = v(\delta)$.*

Corollary 2.3. *Suppose u and v are truth assignments such that $u(A_n) = v(A_n)$ for every atomic formula A_n. Then $u = v$,* **i.e.** *$u(\varphi) = v(\varphi)$ for every formula φ.*

Proposition 2.4. *If α and β are formulas and v is a truth assignment, then:*

(i) *$v(\neg\alpha) = T$ if and only if $v(\alpha) = F$.*

(ii) *$v(\alpha \to \beta) = T$ if and only if $v(\beta) = T$ whenever $v(\alpha) = T$;*

(iii) *$v(\alpha \wedge \beta) = T$ if and only if $v(\alpha) = T$ and $v(\beta) = T$;*

(iv) *$v(\alpha \vee \beta) = T$ if and only if $v(\alpha) = T$ or $v(\beta) = T$; and*

(v) *$v(\alpha \leftrightarrow \beta) = T$ if and only if $v(\alpha) = v(\beta)$.*

Truth tables are often used even when the formula in question is not broken down all the way into atomic formulas. For example, if α and β are any formulas and we know that α is true but β is false, then the truth of $(\alpha \to (\neg\beta))$ can be determined by means of the following table:

α	β	$(\neg\beta)$	$(\alpha \to (\neg\beta))$
T	F	T	T

Definition 2.5. If v is a truth assignment and φ is a formula, we will often say that v **satisfies** φ if $v(\varphi) = T$. Similarly, if Σ is a set of formulas, we will often say that v satisfies Σ if $v(\sigma) = T$ for every $\sigma \in \Sigma$. We will say that φ (respectively, Σ) is **satisfiable** if there is some truth assignment which satisfies it.

Definition 2.6. A formula φ is a **tautology** if it is satisfied by every truth assignment. A formula ψ is a **contradiction** if there is no truth assignment which satisfies it.

For example, $(A_4 \to A_4)$ is a tautology while $(\neg(A_4 \to A_4))$ is a contradiction, and A_4 is a formula which is neither. One can check whether a given formula is a tautology, contradiction, or neither, by grinding out a complete truth table for it, with a separate line for each possible assignment of truth values to the atomic subformulas of the formula. For $A_3 \to (A_4 \to A_3)$ this gives

A_3	A_4	$A_4 \to A_3$	$A_3 \to (A_4 \to A_3)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

so $A_3 \to (A_4 \to A_3)$ is a tautology. Note that, by Proposition 2.2, we need only consider the possible truth values of the atomic sentences which actually occur in a given formula.

One can often use truth tables to determine whether a given formula is a tautology or a contradiction even when it is not broken down all the way into atomic formulas. For example, if α is any formula, then the table

α	$(\alpha \to \alpha)$	$(\neg(\alpha \to \alpha))$
T	T	F
F	T	F

demonstrates that $(\neg(\alpha \to \alpha))$ is a contradiction, no matter which formula of $\mathcal{L}_P$ α actually is.

Proposition 2.7. *If α is any formula, then $((\neg\alpha)\vee\alpha)$ is a tautology and $((\neg\alpha)\wedge\alpha)$ is a contradiction.*

Proposition 2.8. *A formula β is a tautology if and only if $\neg\beta$ is a contradiction.*

After all this warmup, we are finally in a position to define what it means for one formula to follow logically from other formulas.

Definition 2.9. A set of formulas Σ **implies** a formula φ, written as $\Sigma \models \varphi$, if every truth assignment v which satisfies Σ also satisfies φ. We will often write $\Sigma \not\models \varphi$ if it is not the case that $\Sigma \models \varphi$. In the case where Σ is empty, we will usually write $\models \varphi$ instead of $\emptyset \models \varphi$.

Similarly, if Δ and Γ are sets of formulas, then Δ **implies** Γ, written as $\Delta \models \Gamma$, if every truth assignment v which satisfies Δ also satisfies Γ.

For example, $\{ A_3, (A_3 \to \neg A_7) \} \models \neg A_7$, but $\{ A_8, (A_5 \to A_8) \} \not\models A_5$. (There is a truth assignment which makes A_8 and $A_5 \to A_8$ true, but A_5 false.) Note that a formula φ is a tautology if and only if $\models \varphi$, and a contradiction if and only if $\models (\neg\varphi)$.

Proposition 2.10. *If Γ and Σ are sets of formulas such that $\Gamma \subseteq \Sigma$, then $\Sigma \models \Gamma$.*

Problem 2.11. *How can one check whether or not $\Sigma \models \varphi$ for a formula φ and a finite set of formulas Σ?*

Proposition 2.12. *Suppose Σ is a set of formulas and ψ and ρ are formulas. Then $\Sigma \cup \{\psi\} \models \rho$ if and only if $\Sigma \models \psi \to \rho$.*

Proposition 2.13. *A set of formulas Σ is satisfiable if and only if there is no contradiction χ such that $\Sigma \models \chi$.*

3. Deductions

In this section we develop a way of defining logical implication that does not rely on any notion of truth, but only on manipulating sequences of formulas, namely formal proofs or deductions. (Of course, any way of defining logical implication had better be compatible with that given in section 2.) To define these, we first specify a suitable set of formulas which we can use freely as premisses in deductions.

Definition 3.1. The three **axiom schema** of $\mathcal{L}_P$ are:
A1: $(\alpha \to (\beta \to \alpha))$
A2: $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
A3: $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$.
Replacing α, β, and γ by particular formulas of $\mathcal{L}_P$ in any one of the schemas A1, A2, or A3 gives an **axiom** of $\mathcal{L}_P$.

For example, $(A_1 \to (A_4 \to A_1))$ is an axiom, being an instance of axiom schema A1, but $(A_9 \to (\neg A_0))$ is not an axiom as it is not the instance of any of the schema. As had better be the case, every axiom is always true:

Proposition 3.2. *Every axiom of $\mathcal{L}_P$ is a tautology.*

Second, we specify our one (and only!) rule of inference.[2]

Definition 3.3 (Modus Ponens)**.** Given the formulas φ and $(\varphi \to \psi)$, one may infer ψ.

We will usually refer to Modus Ponens by its initials, MP. Of course, MP preserves truth:

Proposition 3.4. *Suppose φ and ψ are formulas. Then $\{ \varphi, (\varphi \to \psi) \} \models \psi$.*

[2]Natural deductive systems, which are usually more convenient to actually execute deductions in than the system being developed here, compensate for having few or no axioms by having many rules of inference.

With axioms and a rule of inference in hand, we can execute formal proofs in $\mathcal{L}_P$.

Definition 3.5. Let Σ be a set of formulas. A **deduction** or **proof** from Σ in $\mathcal{L}_P$ is a finite sequence $\varphi_1\varphi_2\ldots\varphi_n$ of formulas such that for each $k \leq n$,

(i) φ_k is an axiom, or
(ii) $\varphi_k \in \Sigma$, or
(iii) there are $i, j < k$ such that φ_k follows from φ_i and φ_j by MP.

A formula of Σ appearing in the deduction is called a **premiss**. Σ **proves** a formula α, written as $\Sigma \vdash \alpha$, if α is the last formula of a deduction from Σ. We'll usually write $\vdash \alpha$ for $\emptyset \vdash \alpha$, and take $\Sigma \vdash \Delta$ to mean that $\Sigma \vdash \delta$ for every formula $\delta \in \Delta$.

In order to make it easier to verify that an alleged deduction really is one, we will number the formulas in a deduction, write them out in order on separate lines, and give a justification for each formula. Like the additional connectives and conventions for dropping brackets in section 1, this is not officially a part of the definition of a deduction.

Example 3.6. Let us show that $\vdash \varphi \to \varphi$.

1. $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$ — A2
2. $\varphi \to ((\varphi \to \varphi) \to \varphi)$ — A1
3. $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$ — 1,2 MP
4. $\varphi \to (\varphi \to \varphi)$ — A1
5. $\varphi \to \varphi$ — 3,4 MP

Hence $\vdash \varphi \to \varphi$, as desired. Note that indication of the formulas from which formulas 3 and 5 beside the mentions of MP.

Example 3.7. Let us show that $\{\, \alpha \to \beta,\ \beta \to \gamma \,\} \vdash \alpha \to \gamma$.

1. $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$ — A1
2. $\beta \to \gamma$ — Premiss
3. $\alpha \to (\beta \to \gamma)$ — 1,2 MP
4. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ — A2
5. $(\alpha \to \beta) \to (\alpha \to \gamma)$ — 4,3 MP
6. $\alpha \to \beta$ — Premiss
7. $\alpha \to \gamma$ — 5,6 MP

Hence $\{\, \alpha \to \beta,\ \beta \to \gamma \,\} \vdash \alpha \to \gamma$, as desired.

It is frequently convenient to save time and effort by simply referring to a deduction one has already done instead of writing it again as part of another deduction. If you do so, please make sure you appeal only to deductions that have already been carried out.

Example 3.8. Let us show that $\vdash (\neg\alpha \to \alpha) \to \alpha$.

1. $(\neg\alpha \to \neg\alpha) \to ((\neg\alpha \to \alpha) \to \alpha)$ — A3
2. $\neg\alpha \to \neg\alpha$ — Example 3.6
3. $(\neg\alpha \to \alpha) \to \alpha$ — 1,2 MP

Hence $\vdash (\neg\alpha \to \alpha) \to \alpha$, as desired. To be completely formal, one would have to insert the deduction given in Example 3.6 (with φ replaced by $\neg\alpha$ throughout) in place of line 2 above and renumber the old line 3.

Certain general facts are sometimes handy:

Proposition 3.9. *If $\varphi_1\varphi_2\ldots\varphi_n$ is a deduction of $\mathcal{L}_P$, then $\varphi_1\ldots\varphi_\ell$ is also a deduction of $\mathcal{L}_P$ for any ℓ such that $1 \leq \ell \leq n$.*

Proposition 3.10. *If $\Gamma \vdash \delta$ and $\Gamma \vdash \delta \to \beta$, then $\Gamma \vdash \beta$.*

Proposition 3.11. *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$.*

Proposition 3.12. *If $\Gamma \vdash \Delta$ and $\Delta \vdash \sigma$, then $\Gamma \vdash \sigma$.*

The following theorem often lets one take substantial shortcuts when trying to show that certain deductions exist in $\mathcal{L}_P$, even though it doesn't give us the deductions explicitly.

Theorem 3.13 (Deduction Theorem)**.** *If Σ is any set of formulas and α and β are any formulas, then $\Sigma \vdash \alpha \to \beta$ if and only if $\Sigma \cup \{\alpha\} \vdash \beta$.*

Example 3.14. Let us show that $\vdash \varphi \to \varphi$. By the Deduction Theorem it is enough to show that $\{\varphi\} \vdash \varphi$, which is trivial:

(i) φ Premiss

Compare this to the deduction in Example 3.6.

4. Soundness and Completeness

How are deduction and implication related, given that they were defined in completely different ways? We have some evidence that they behave alike; compare, for example, Proposition 2.12 and the Deduction Theorem. It had better be the case that if there is a deduction of a formula φ from a set of premisses Σ, then φ is implied by Σ. (Otherwise, what's the point of defining deductions?) It would also be nice for the converse to hold: whenever φ is implied by Σ, there is a deduction of φ from Σ. (So anything which is true can be proved.) The Soundness and Completeness Theorems say that both ways do hold, so $\Sigma \vdash \varphi$ if and only if $\Sigma \models \varphi$, i.e. $\vdash$ and $\models$ are equivalent for propositional logic. One direction is relatively straightforward to prove...

Theorem 4.1 (Soundness Theorem)**.** *If Δ is a set of formulas and α is a formula such that $\Delta \vdash \alpha$, then $\Delta \models \alpha$.*

...but for the other direction we need some additional concepts.

Definition 4.2. A set of formulas Γ is **inconsistent** if $\Gamma \vdash \neg(\alpha \to \alpha)$ for some formula α, and **consistent** if it is not inconsistent.

For example, $\{A_{41}\}$ is consistent by Proposition 4.3, but $\{A_{13}, \neg A_{13}\}$ is inconsistent.

Proposition 4.3. *If a set of formulas is satisfiable, then it is consistent.*

Proposition 4.4. *Suppose Δ is an inconsistent set of formulas. Then $\Delta \vdash \psi$ for any formula ψ.*

Proposition 4.5. *Suppose Σ is an inconsistent set of formulas. Then there is a finite subset Δ of Σ such that Δ is inconsistent.*

Corollary 4.6. *A set of formulas* Γ *is consistent if and only if every finite subset of* Γ *is consistent.*

To obtain the Completeness Theorem requires one more definition.

Definition 4.7. A set of formulas Σ is **maximally consistent** if Σ is consistent but $\Sigma \cup \{\varphi\}$ is inconsistent for any $\varphi \notin \Sigma$.

That is, a set of formulas is maximally consistent if it is consistent, but there is no way to add any other formula to it and keep it consistent.

Proposition 4.8. *Suppose* v *is a truth assignment. Then* $\Sigma = \{\, \varphi \mid v(\varphi) = T \,\}$ *is maximally consistent.*

We will need some facts concerning maximally consistent theories.

Proposition 4.9. *If* Σ *is a maximally consistent set of formulas,* φ *is a formula, and* $\Sigma \vdash \varphi$*, then* $\varphi \in \Sigma$*.*

Proposition 4.10. *Suppose* Σ *is a maximally consistent set of formulas and* φ *is a formula. Then* $\neg\varphi \in \Sigma$ *if and only if* $\varphi \notin \Sigma$*.*

Proposition 4.11. *Suppose* Σ *is a maximally consistent set of formulas and* φ *and* ψ *are formulas. Then* $\varphi \to \psi \in \Sigma$ *if and only if* $\varphi \notin \Sigma$ *or* $\psi \in \Sigma$*.*

It is important to know that any consistent set of formulas can be expanded to a maximally consistent set.

Theorem 4.12. *Suppose* Γ *is a consistent set of formulas. Then there is a maximally consistent set of formulas* Σ *such that* $\Gamma \subseteq \Sigma$*.*

Now for the main event!

Theorem 4.13. *A set of formulas is consistent if and only if it is satisfiable.*

Theorem 4.13 gives the equivalence between $\vdash$ and $\models$ in slightly disguised form.

Theorem 4.14 (Completeness Theorem)**.** *If* Δ *is a set of formulas and* α *is a formula such that* $\Delta \models \alpha$*, then* $\Delta \vdash \alpha$*.*

It follows that anything provable from a given set of premisses must be true if the premisses are, and *vice versa*. The fact that $\vdash$ and $\models$ are actually equivalent can be very convenient in situations where one is easier to use than the other. For example, most parts of Examples 3.6-3.8 are much easier to do with truth tables instead of deductions, even if one makes use of the Deduction Theorem.

Finally, one more consequence of Theorem 4.13.

Theorem 4.15 (Compactness Theorem)**.** *A set of formulas* Γ *is satisfiable if and only if every finite subset of* Γ *is satisfiable.*

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