Mathematical Logic - MATH33021

http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/teaching/MathematicalLogic/index.php

What is expected from students on this course. *The material from Logic that is covered in this course is technically not very difficult. However, it is very abstract and this normally causes a lot of problems. The only way one can understand the material is to engage with it. The course is designed under the assumption that students adhere to the following guidelines:*

- *Attend all lectures and **both** tutorials. You will have access to the lecture notes accompanying the lectures, but you are invited to produce handwritten notes during the lectures.*
- *After the lectures, revise the material, by using the lecture notes, your handwritten notes and the example sheets. There is a weekly learning plan on BlackBoard*$^{\copyright,\mathit{TM}}$ *detailing the expected timing.*
- *Write out your answers to questions from the example sheets on paper, in a way that another human can make sense of. This is particularly important for the tests and the exam, where you have to do exactly this. So you need to have build up a good routine for writing at assessment time.*
- *Work with the lecture notes when working through the example sheets. Most of the questions on the example sheets are easy, once you have actually revised the notes.*
- *Ask questions: To the lecturer, to fellow students, and to yourself.*
- ***In summary:*** *Learn actively and continuously. Due to its abstract nature, much of the material is hard to memorize if you don't get used to it.*

You are supposed to work 7–8 hours each week for this course in addition to attending the lectures and the tutorials.

Please turn over for the warmup questions.

Warmup Questions for the course

Please do the following questions in the first week as a revision of the language of Mathematics. If you have forgotten some of the terminology or if you have difficulties answering the questions, please revise the material as a matter of urgency and bring up any problems in the tutorials in week 1. There will be no solutions posted to these questions as they cover level 1 material, however we will talk about them in both tutorials of week 1.

1. Give an example of a set that is not contained in any $\mathbb{C}^n$ (where $\mathbb{C}$ is the set of complex numbers and $n \in \mathbb{N}$).

2. Calculate $\{\mathbb{Z}\} \cap \{\mathbb{R}\}$.

3. Let X be a set. Recall that a binary relation R of X is a subset of the cartesian product $X \times X$; also recall that we write xRy instead of $(x, y) \in R$. For example the set $\{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$ is a binary relation of $\mathbb{R}$. Give three examples of a binary relation of $\mathbb{N}$.

4. *The definition of a function.*

(i) Write down the definition of a **function** (also known as **map**, the words are used interchangeably). If you are not totally sure about the definition, please find the definition in some source (you could use lecture notes from previous years or a maths book).

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$ and let $g : \mathbb{R} \longrightarrow [0, +\infty)$ be the function defined by $g(x) = x^2$. Use your definition of a function in (i) to prove or disprove that $f = g$.

(iii) Use your definition of a function in (i) to calculate the number of functions from the set $\{0, 1\}$ to the set $\{\emptyset, -2, \mathbb{R}\}$.

(iv) Use your definition of a function in (i) to calculate the number of functions from the empty set to $\mathbb{N}$.

(v) Let X, Y be sets and let $R \subseteq X \times Y$. Write down a property P of R such that P is true if and only if R is the **graph** of a function $X \longrightarrow Y$. If you have forgotten what the graph of a function is: revisit the definition of a function in (i).

(vi) Revisit your definition of a function in (i) and double check if it needs an amendment after having done the other items above.

5. *The power set.* For a set X we denote the power set of X by $\mathcal{P}(X)$.

(i) Write down an injective (also known as "1-1") function $X \longrightarrow \mathcal{P}(X)$ and a surjective (aka "onto") function $\mathcal{P}(X) \longrightarrow X$.

(ii) Find a bijection of $\mathcal{P}(X)$ with the set of all functions $X \longrightarrow \{0, 1\}$.

(iii) Find a function $f : \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{N}$ such that $f(A) \in A$ for all $A \in \mathcal{P}(A)$ with $A \neq \emptyset$.

(iv) Let X, Y be sets and let Z be the set of all maps $X \longrightarrow Y$. Show that the map $G : Z \longrightarrow \mathcal{P}(X \times Y)$ defined by $G(f) = \mathit{Graph}(f)$ is injective, but it is in general not surjective.

6. *Arbitrary unions and intersections.* Let I be a set and for each $i \in I$ let X_i be a set. We define

$$\bigcup_{i \in I} X_i = \{x \mid \exists i \in I : x \in X_i\}$$

and call it the **union of the X_i with $i \in I$**, and

$$\bigcap_{i \in I} X_i = \{x \mid \forall i \in I : x \in X_i\}$$

and call it the **intersection of the X_i with $i \in I$**.

(i) Now let $I = \{a, b\}$ be a set with exactly two elements. Let $X_a = \{\mathbb{Z}\}$ and let $X_b = \{\mathbb{R}\}$. Compute $\bigcup_{i \in I} X_i$ and $\bigcap_{i \in I} X_i$.

(ii) Compute $\bigcup_{n \in \mathbb{N}} [-n, n]$, where $[-n, n]$ is the closed interval of $\mathbb{R}$ between $-n$ and n.

(iii) Compute $\bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus [-n, n])$.

7. *Images and preimages of functions.* Let X be a set and let $f : X \longrightarrow Y$ be a map. Recall that for $A \subseteq X$ and $B \subseteq Y$ the set

$f(A) = \{f(a) \mid a \in A\}$ is called the image of A under f, and
$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ is called the preimage of B under f.

(i) For $A \subseteq X$ show that $A \subseteq f^{-1}(f(A))$ and give an example showing that $f^{-1}(f(A)) \neq A$ in general.

(ii) For $B \subseteq Y$ show that $f(f^{-1}(B)) \subseteq B$ and give an example showing that $f(f^{-1}(B)) \neq B$ in general

(iii) Show that a function $f : X \longrightarrow Y$ is bijective if and only if for every $y \in Y$ the set $f^{-1}(\{y\})$ consist of a single element.

(iv) Show that for a bijective function $f : X \longrightarrow Y$ and every $y \in Y$ we have $f^{-1}(\{y\}) = \{g(y)\}$, where g is the compositional inverse of f, hence $g : Y \longrightarrow X$ and $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$. This indicates the notation f^{-1} for the function g and this is how the compositional inverse of a bijective function is normally denoted.

Show that the map

$$F : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$$

defined by $F(B) := f^{-1}(B)$, has the following properties.

(v) If $B \in \mathcal{P}(Y)$, then $F(Y \setminus B) = X \setminus F(B)$.

(vi) If I is a set and $B_i \in \mathcal{P}(Y)$ for all i, then $F(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} F(B_i)$.

(vii) Now let $H : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ be defined by $H(A) =$ the image of A under f. Which of the following properties are true?

(a) If $A \in \mathcal{P}(X)$, then $H(X \setminus A) = Y \setminus H(A)$

(b) If I is a set and $A_i \in \mathcal{P}(X)$ for all i, then $H(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} H(A_i)$.

8. *Equivalence relations.*

(i) Let R be a binary relation of a set X such that

(a) $\forall x \in X : xRx$

(b) $\forall x, y, z \in X(xRy \;\&\; zRy \;\Rightarrow xRz)$

Show that R is an equivalence relation of X.

(ii) Explain why every equivalence relation of a set X is of the form

$$\{(x, y) \in X \mid f(x) = f(y)\}$$

for some surjective function f with domain X (you need to create f and its codomain).

9. In this question we first write out a somewhat ambiguous definition of a group:

A group is a set G together with a binary operation $*$ of G such that

(a) G is closed under $*$.

(b) G is associative, i.e. for all a, b, c we have $(a * b) * c = a * (b * c)$.

(c) There is some element $e \in G$ such that for all $a \in G$ we have $a * e = e * a = a$

(d) For every $a \in G$ there is some $b \in G$ with $a * b = e$ and $b * a = e$.

Questions:

(i) What is a binary operation of a set?

(ii) What does it mean that G is "closed under $*$" and how does this relate to the question in (i)?

(iii) Why does the requirement in (d) make sense?