The real spectrum of continuous definable functions in o-minimal structures

Marcus Tressl

Abstract. If M is an o-minimal expansion of a real closed field there is a natural, injective map from the n-types of M into the real spectrum of the ring of continuous, definable maps $M^n \longrightarrow M$. We determine the image of this map and deduce a Nullstellensatz and a division property as in the case of semialgebraic functions.

1. Introduction. In real algebraic geometry there are two theorems which are sometimes called the fundamental theorems of real algebraic geometry. They are

(I) Quantifier elimination: if $X \subseteq R^{n+1}$ is semialgebraic and $\pi : R^{n+1} \longrightarrow R^n$ is a projection then $\pi(X)$ is again semialgebraic.

(II) The finiteness theorem: if $X \subseteq R^n$ is closed and semialgebraic then X is a finite union of finite intersections of sets of the form $\{a \in R^n \mid f(a) \geq 0\}$ with $f \in R[X_1, ..., X_n]$.

Let $S_n(R)$ denote the set of n-types of R and let $\mathcal{C}(R^n)$ denote the ring of continuous, semialgebraic maps $R^n \longrightarrow R$. Let ι be the map from $S_n(R)$ into the real spectrum (c.f.Definition 2.3) of $\mathcal{C}(R^n)$ which sends a type p to $\iota(p) := \{s \in \mathcal{C}(R^n) \mid s \geq 0 \in p\}$. Furthermore let $\psi : \operatorname{Sper}\mathcal{C}(R^n) \longrightarrow \operatorname{Sper} R[X_1, ..., X_n]$ be the map induced from the inclusion. Then, by general model theory, (I) is equivalent to the injectivity of $\psi \circ \iota$; by basic real algebra $\psi \circ \iota$ is even bijective. On the other hand a reformulation of (II) says that ψ is an homeomorphism. So (I) and (II) are expressed in the following diagram:

$$\begin{tikzcd} & S_n(R) \arrow[dl, "\iota"'] \arrow[dr, "\psi\circ\iota", "1-1"'] & \\ \operatorname{Sper}\mathcal{C}(R^n) \arrow[rr, "\sim", "\psi"'] & & \operatorname{Sper} R[X_1, ..., X_n] \end{tikzcd}$$

It follows that ι is bijective. A geometric reformulation of this fact is the Division Property for continuous semialgebraic maps, which says the following: let $F, G : R^n \longrightarrow R$ be continuous, semialgebraic maps. If each zero of F is a zero of G, then there are some continuous, semialgebraic $H : R^n \longrightarrow R$ and some $k \in \mathbb{N}$ such that $G^k = F \cdot H$. Again this property can easily be reformulated into the Łojasiewicz-inequality for semialgebraic maps: let $F, G : R^n \longrightarrow R$ be continuous, semialgebraic maps. If each zero of F is a zero of G, then there are $k, m \in \mathbb{N}$ and $c \in R$ such that $|G(\bar{a})|^k \leq |F(\bar{a})| \cdot c \cdot (1 + a_1^2 + ... + a_n^2)^m$ for all $\bar{a} = (a_1, ..., a_n) \in X$.

In this paper we examine the map ι for o-minimal expansions M of real closed fields, where now $\mathcal{C}(M^n)$ denotes the ring of continuous, definable maps $M^n \longrightarrow M$. More precisely we determine the image of ι (Proposition 6.2) and get a Nullstellensatz (Theorem 7.1) as well as the general division property (Theorem 7.3) and the general Łojasiewicz inequality (Theorem 7.4) as stated in [vdD-Mi]. Let me explain the situation in the real exponential field $M = (\mathbb{R}, \exp)$. M is o-minimal by [Wi]. By Lemma 2.4 the support map $\operatorname{Sper}\mathcal{C}(M^n) \longrightarrow \operatorname{Spec}\mathcal{C}(M^n)$ is an homeomorphism, so we deal with prime ideals rather than with orderings. Let $E : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $E(x) := \exp(-\frac{1}{x^2})$, where we take $E(0) = 0$. Then $E(x)$ vanishes exactly at 0 but no power of $E(x)$ can be estimated by the product of x with a continuous map $\mathbb{R} \longrightarrow \mathbb{R}$. Hence there is a prime ideal $\mathfrak{p}$ of $\mathcal{C}(M)$, which contains $E(x)$ and excludes x. In particular $\mathfrak{p}$ is not in the image of ι. It turns out that this is the only obstruction for a prime ideal to be induced from a type.

So $\mathfrak{p} \in \operatorname{Spec} \mathcal{C}(M^n)$ is of the form $\{s \mid s = 0 \in p\}$ for some type $p \in S_n(M)$ if for all $F \in \mathcal{C}(M^n)$ with $E(F) \in \mathfrak{p}$ we have $F \in \mathfrak{p}$ (Theorem 8.2).

In a sequel to this paper we'll give a generalization of (II) for arbitrary o-minimal expansions M of real closed fields in the following way: let $\mathcal{C}^\infty(M^n)$ be the ring of definable C^∞-maps $M^n \longrightarrow M$. If the map $S_n(M) \longrightarrow \operatorname{Sper} \mathcal{C}(M^n) \longrightarrow \operatorname{Sper} \mathcal{C}^\infty(M^n)$ is injective (in other words: each definable subset of M^n is a boolean combination of sets of the form $\{a \in M^n \mid \varphi(a) \geq 0\}$, $\varphi \in \mathcal{C}^\infty(M^n)$), then each closed definable subset of M^n is a finite union of finite intersections of sets of the form $\{a \in M^n \mid \varphi(a) \geq 0\}$ with $\varphi \in \mathcal{C}^\infty(M^n)$. So if in addition M is an expansion of $(\mathbb{R}, \exp)$ then we get easily that each closed definable set is the zero set of a definable C^∞-function.

2. Preliminaries on continuous and definable maps.

The reader is assumed to be familiar with the elementary notions from model theory, such as formulas, theories and elementary extensions (see [Ho]) and with the basic theory of o-minimal structures as developed in zitat [vdD], [P-S] and [K-P-S]. Since types are in the focus of our proofs, we recall the definition. Throughout this paper we work in a language $\mathscr{L}$ expanding the language $\{+, -, \cdot, 0, 1\}$ of rings. If M is an $\mathscr{L}$-structure and $A \subseteq M$, then $\mathscr{L}(A)$ denotes the language obtained by adding a new constant c_a to $\mathscr{L}$ for each $a \in A$. Furthermore $\operatorname{Fml} \mathscr{L}_n(A)$ denotes the set of all $\mathscr{L}(A)$-formulas with at most n free, fixed variables. The A-definable subsets of M^n are precisely the subsets of M^n defined by formulas from $\operatorname{Fml} \mathscr{L}_n(A)$, where c_a in M means a. A function $F : X \longrightarrow M^m$, where $X \subseteq M^n$ is called A-definable, if the graph of F is an A-definable subset of $M^n \times M^m$. A maximal set of $\mathscr{L}(A)$-formulas which can be finitely realized in M is called an n-type over A. Thus an n-type over A is nothing but an ultrafilter of A-definable subsets of M^n. We write $S_n(M, A)$ for the set of n-types over A. The topology on $S_n(M, A)$ is given by the open-closed basis $\langle \varphi \rangle := \{p \in S_n(M, A) \mid \varphi \in p\}$ and coincides with the Stone-topology, if we regard types as ultrafilters. If $X \subseteq M^n$ is the set defined by $\varphi \in \operatorname{Fml} \mathscr{L}_n(A)$, then we write $\langle X \rangle_A$ for $\langle \varphi \rangle$. We use a further description of types, where "Spec" is the spectrum of a ring as a topological space:

PROPOSITION 2.1. *Let M be an expansion of a field, $A \subseteq M$, $n \in \mathbb{N}$ and let $\bar{x}$ be an n-tuple of variables. Let $X \subseteq M^n$ be A-definable and let $\mathcal{D}$ be the ring of all A-definable maps $X \longrightarrow M$. Then*

$$\begin{aligned} \langle X \rangle_A &\longrightarrow \operatorname{Spec} \mathcal{D} \\ p &\longmapsto \{F \mid F(\bar{x}) = 0 \in p\} \end{aligned}$$

is an homeomorphism (hence the Zariski topology of $\operatorname{Spec} \mathcal{D}$ *is equal to the boolean topology on the space $\langle X \rangle_A$).*

PROOF. Obviously $\mathfrak{p} := \{F \in \mathcal{D} \mid F(\bar{x}) = 0 \in p\}$ is a prime ideal of $\mathcal{D}$ if $p \in \langle X \rangle_A$. If $q \in \langle X \rangle_A$ is a type different from p and $\mathfrak{q} := \{F \in \mathcal{D} \mid F(\bar{x}) = 0 \in q\}$ then $\mathfrak{q}$ is not a specialization of $\mathfrak{p}$: take an A-definable subset $Z \subseteq X$ such that $p \notin \langle Z \rangle_A$ and $q \in \langle Z \rangle_A$. Then the characteristic function χ_Z of Z is in $\mathfrak{p}$ but not in $\mathfrak{q}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{D}$. It remains to show that there is a type $p \in \langle X \rangle_A$ such that $\mathfrak{p} = \{F \mid F(\bar{x}) = 0 \in p\}$. We have to show that

$$Th(M, A) \cup \{\bar{x} \in X\} \cup \{F(\bar{x}) = 0 \mid F \in \mathfrak{p}\} \cup \{G(\bar{x}) \neq 0 \mid G \in \mathcal{D} \setminus \mathfrak{p}\}$$

is consistent. Let $F_1, ..., F_k \in \mathfrak{p}$ and $G_1, ..., G_n \in \mathcal{D} \setminus \mathfrak{p}$. Let $\chi_i \in \mathcal{D}$ be the characteristic function of the set of zeros of F_i. Then

$$F := F_1 + \chi_1 \cdot F_2 + \chi_1 \cdot \chi_2 \cdot F_3 + ... + (\chi_1 \cdot ... \cdot \chi_{k-1}) \cdot F_k \in \mathfrak{p}$$

and for all $\bar{a} \in X$ with $F(\bar{a}) = 0$ we have $F_1(\bar{a}) = ... = F_k(\bar{a}) = 0$. Now if we take $G := G_1 \cdot ... \cdot G_n \notin \mathfrak{p}$, it remains to find some $\bar{a} \in X$ such that $F(\bar{a}) = 0$ and $G(\bar{a}) \neq 0$. If $F(\bar{a}) \neq 0$ for all $\bar{a} \in X$ with $G(\bar{a}) \neq 0$ then obviously there is an A-definable map $H : X \longrightarrow M$ such that $G = H \cdot F \in \mathfrak{p}$, a contradiction. □

If X is any set and $f, g : X \longrightarrow K$ are maps into an ordered field K, then we write $f \wedge g$ and $f \vee g$ for the pointwise infimum and supremum of maps $f, g : X \longrightarrow K$. $f^+ := f \vee 0$ and $f^- := -(f \wedge 0)$ denotes the positive and the negative part of f. Furthermore we write $\{f = g\}, \{f \leq g\}, ...$ for the sets $\{x \in X \mid f(x) = g(x)\}$, $\{x \in X \mid g(x) \leq f(x)\}, ...$. We collect some easy properties of rings of continuous functions now, which can be found in [Gil-Jer] and which remains valid in rings of continuous, definable functions.

LEMMA 2.2. *Let K be an ordered field, let X be a set and let $f, g : X \longrightarrow K$ be maps. Then*

(i) $|f| = f^+ + f^-$, $f \vee g = \frac{1}{2}(f + g + |f - g|)$.

(ii) $(fg)^+ = f^+g^+ + f^-g^-$ *and* $(fg)^- = f^-g^+ + f^+g^-$

(iii) $(f + g)^+ \leq f^+ + g^+$ *and* $(f + g)^- \leq (f^- + g^-)$

(iv) Let R be a ring of maps $X \longrightarrow K$ which contains f^+ and f^- for all $f \in R$.
If $\mathfrak{a}$ is a radical ideal of R, then $f \in \mathfrak{a} \Leftrightarrow f^+, f^- \in \mathfrak{a} \Leftrightarrow |f| \in \mathfrak{a}$.
If $\mathfrak{b}$ is an ideal of R such that $f^2 \in \mathfrak{b}$ implies $f \in \mathfrak{b}$ for all $f \in R$ with $f \geq 0$, then $\mathfrak{b}$ is a radical ideal.

PROOF. The statements (i), (ii) and (iii) are obvious.

(iv). The assertion on $\mathfrak{a}$ follows from $(f^+)^2 = f \cdot f^+ = |f| \cdot f^+$ and $(f^-)^2 = -f \cdot f^- = |f| \cdot f^-$. Let $f \in R$ such that $f^k \in \mathfrak{b}$. Then $|f|^{2^k} = f^{2^k} \in \mathfrak{b}$ and, by assumption, $|f| \in \mathfrak{b}$. Since $f^+, f^- \geq 0$ we get $f \in \mathfrak{b}$ from the equations above. □

DEFINITION 2.3. If R is a commutative, unitary ring, then the real spectrum $\operatorname{Sper} R$ of R is the following. The elements, the orderings of R, are subsets $P \subseteq R$ such that $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = R$ and $P \cap -P$ is a prime ideal (hence a proper ideal) of R. Another description of orderings is given by a tuple $(\mathfrak{p}, T)$, where $\mathfrak{p} \in \operatorname{Spec} R$ and T is an ordering of the field $\kappa(\mathfrak{p})$ =quotient field of $R/\mathfrak{p}$ (T is the set of positive elements of $\kappa(\mathfrak{p})$). The connection is given by $P = \pi^{-1}(T)$, where $\pi : R \longrightarrow \kappa(\mathfrak{p})$ is the residue map. The prime ideal $P \cap -P = \mathfrak{p}$ is called the support of P. The topology on $\operatorname{Sper} R$ is given by the subbasis of open sets $\{P \in \operatorname{Sper} R \mid a \notin -P\}$ for $a \in R$ (see [B-C-R] for the use of the real spectrum in real geometry).

LEMMA 2.4. *Let K be a real closed field, let X be a set and let R be a ring of maps $X \longrightarrow K$, such that the following conditions hold:*

(a) If $f \in R$ then $f^+, f^-, \sqrt{|f|} \in R$.

(b) If $f, g \in R$ and $0 \leq g \leq f$, then there are $k \in \mathbb{N}$ and $h \in R$ such that $g^k = f \cdot h$.

Then the support map $\operatorname{supp} : \operatorname{Sper} R \longrightarrow \operatorname{Spec} R$ *is an homeomorphism with inverse* $\mathfrak{p} \mapsto \{f \in R \mid f^- \in \mathfrak{p}\}$.

PROOF. If $P \in \operatorname{Sper} R$, then by (a), P contains all elements $f \in R$ with $f^- \in \operatorname{supp} P$. Hence the lemma follows if we know that $P := \{f \in R \mid f^- \in \mathfrak{p}\} \in \operatorname{Sper} R$ for each $\mathfrak{p} \in \operatorname{Spec} R$. If $f^-, g^- \in \mathfrak{p}$, then $(f \cdot g)^- = f^-g^+ + f^+g^- \in \mathfrak{p}$. If $f \in R$ then $f^- \cdot (-f)^- = 0 \in \mathfrak{p}$, hence $f \in P$ or $-f \in P$. If $f \in \mathfrak{p}$, then $(f^-)^2 = -f^- \cdot f \in \mathfrak{p}$, hence $f \in P$ and similar $f \in -P$. If $f \in P \cap -P$, then $f^+, f^- \in \mathfrak{p}$, hence $f = f^+ - f^- \in \mathfrak{p}$. Thus we know that $P \cdot P \subseteq P$, $P \cup -P = R$ and $P \cap -P = \mathfrak{p}$ without the help of condition (b).

$P+P\subseteq P$: if $f,g\in P$, then $0\leq (f+g)^-\leq f^-+g^-\in\mathfrak{p}$. By condition (b) we get $(f+g)^-\in\sqrt{(f^-+g^-)\cdot R}\subseteq\mathfrak{p}$. $\square$

Let $f,g: X\longrightarrow K$ be continuous maps with $\{f=0\}\subseteq\{g=0\}$. The division problem asks in what way a suitable product $f\cdot h$, with continuous $h: X\longrightarrow K$, can be obtained from g. The following lemma says how this property depends on the behaviour of g in a neighbourhood of the zeros of f in general.

LEMMA 2.5.*(basic division lemma)*
Let X be a topological space, let K be an ordered field and let $f,g: X\longrightarrow K$ be continuous maps. Let U be an open neighbourhood of $\{f=0\}$ and let $h: U\longrightarrow K$ be continuous.
(i) If $g=f\cdot h$ on U, then the map $q: X\longrightarrow K$ defined by

$$q(x):=\begin{cases}\frac{g(x)}{f(x)} & \text{if } f(x)\neq 0\\ h(x) & \text{if } f(x)=0\end{cases}$$

is continuous and $g=f\cdot q$.
(ii) If $|g|\leq|h\cdot f|$ on U and $s: K\longrightarrow K$ is a continuous map, such that $\lim\limits_{x\to 0}\frac{s(x)}{x}=0$ then the map $q: X\longrightarrow K$ defined by

$$q(x):=\begin{cases}\frac{s(g(x))}{f(x)} & \text{if } f(x)\neq 0\\ 0 & \text{if } f(x)=0\end{cases}$$

is continuous and $s(g)=f\cdot q$.
(iii) If $|g|\leq|h\cdot f|$ on U and $f_1: X\longrightarrow K$ is a continuous map with $\{f=0\}\subseteq\{f_1=0\}$, then the map

$$q(x):=\begin{cases}\frac{f_1(x)\cdot g(x)}{f(x)} & \text{if } f(x)\neq 0\\ 0 & \text{if } f(x)=0\end{cases}$$

is continuous and $f_1\cdot g=f\cdot q$.

PROOF. (i) is obvious.

(ii). If $x\in U$ and $g(x)\neq 0$, then $|q(x)|\leq|\frac{s(g(x))}{g(x)}\cdot h(x)|$. From the assumption on s we get for each $a\in X$ with $f(a)=0$, that $\lim\limits_{x\to a}q(x)=0=q(a)$.

(iii). The map

$$f_2(x):=\begin{cases}\frac{g(x)}{f(x)} & \text{if } f(x)\neq 0\\ 0 & \text{if } f(x)=0\end{cases}$$

is bounded in a neighbourhood of each zero of f. It follows that $f_1\cdot f_2$ is continuous everywhere. $\square$

If M is an expansion of a real closed field, then the explicit formulas in Lemma 2.5(ii) says that conditions (a) and (b) from Lemma 2.4 are fulfilled for

(1) The ring of continuous maps $X\longrightarrow M$ for a topological space X.
(2) The ring of A-definable maps $F: X\longrightarrow M$, where $A\subseteq M$ and $X\subseteq M^n$ is A-definable.
(3) The ring of A-definable, continuous maps $F: X\longrightarrow M$, where $A\subseteq M$ and $X\subseteq M^n$ is A-definable.

Example (2) together with Proposition 2.1 yields

COROLLARY 2.6. *Let M be an expansion of a real closed field, $A\subseteq M$ and let $X\subseteq M^n$*

be A-definable. Let $\mathcal{D}$ be the ring of A-definable maps $X \longrightarrow M$. Then

$$\begin{aligned} \langle X\rangle_A &\longrightarrow \operatorname{Sper}\mathcal{D} \\ p &\longmapsto \{F \mid F(\bar{x}) \geq 0 \in p\} \end{aligned}$$

is an homeomorphism (hence the Harrison topology of Sper $\mathcal{D}$ *is equal to the constructible topology).* □

COROLLARY 2.7. *If $\mathfrak{a}$ is a proper radical ideal of the ring $\mathcal{C}$ of A-definable, continuous maps $X \longrightarrow M$ which contains a prime ideal, then $\mathfrak{a}$ is a prime ideal itself.*

PROOF. Because $\operatorname{Sper}\mathcal{C} \cong \operatorname{Spec}\mathcal{C}$ and the closure of a point in the real spectrum of any ring is a chain with respect to inclusion ([B-C-R], 7.1.22). □

We keep the notation and write $\iota : \operatorname{Spec}\mathcal{D} \longrightarrow \operatorname{Spec}\mathcal{C}$ for the map induced from the inclusion $\mathcal{C} \longrightarrow \mathcal{D}$. We write Z for the image of ι in the sequel.

LEMMA 2.8.
(i) ι is injective if and only if each A-definable subset of X is the boolean combination of zero sets of elements of $\mathcal{C}$.
(ii) If $\mathfrak{K}$ is a chain in Z, then $\bigcap \mathfrak{K}, \bigcup \mathfrak{K} \in Z$.
(iii) If $\mathfrak{p} \in \operatorname{Spec}\mathcal{C}$, then there are $\mathfrak{q}_1, \mathfrak{q}_2 \in Z$ such that $\mathfrak{q}_1 \subseteq \mathfrak{p} \subseteq \mathfrak{q}_2$. Furthermore there is a least element in Z containing $\mathfrak{p}$.

PROOF. (i). The "if" part follows easily. The "only if" part follows with a compactness argument; we omit the proof, since we do not need this later on.

(ii). A prime ideal $\mathfrak{p}$ lies in Z if and only if for all $F_1, ..., F_k \in \mathfrak{p}$ and all $G \in \mathcal{C} \setminus \mathfrak{p}$ there is some $\bar{a} \in X$ such that $F_1(\bar{a}) = ... = F_k(\bar{a}) = 0$ and $G(\bar{a}) \neq 0$. This is a finite condition, thus (ii) follows from Zorn's Lemma.

(iii). If $G \in \mathcal{C} \setminus \mathfrak{p}$, then G is not identical zero, hence there is some $\bar{a} \in X$ such that $G(\bar{a}) \neq 0$. Since $\mathfrak{p}$ is prime, this means that $Th(M, A) \cup \{\bar{x} \in X\} \cup \{G(\bar{x}) \neq 0 \mid G \in \mathcal{C} \setminus \mathfrak{p}\}$ is consistent. A type $q_1 \in S_n(M, A)$ containing the latter set generates a prime ideal $\mathfrak{q}_1 \in Z$ with $\mathfrak{q}_1 \subseteq \mathfrak{p}$.

Let $F_1, ..., F_k \in \mathfrak{p}$. Then $F := F_1^2 + ...F_k^2 \in \mathfrak{p}$. Since $\mathfrak{p}$ is a proper ideal, F must have a zero in X, hence $F_1, ..., F_k$ must have a common zero in X. As above we get $\mathfrak{q}_2 \in Z$ such that $\mathfrak{p} \subseteq \mathfrak{q}_2$.

Since $\operatorname{Spec}\mathcal{C} \cong \operatorname{Sper}\mathcal{C}$, the elements in Z containing $\mathfrak{p}$ are totally ordered by inclusion. By (ii) the intersection of these points is again in Z. □

3. o-minimal expansions of real closed fields. For the rest of this paper, M denotes an o-minimal expansion of a real closed field in the language $\mathscr{L}$. We fix an integer n and an M-definable subset $X \subseteq M^n$. We write

$$\mathcal{D}(X) := \mathcal{D}(X, M) := \{F : X \longrightarrow M \mid F \text{ is } M\text{-definable}\}$$

$$\mathcal{C}(X) := \mathcal{C}(X, M) := \{F : X \longrightarrow M \mid F \text{ is } M\text{-definable and continuous}\}$$

If $A \subseteq M$ then the definable closure cl(A) is an elementary restriction of M, since M expands a group ([P-S]). Therefore all our results remain valid if we replace the clause "M-definable" by "A-definable" and we proceed with M-definable sets in the sequel. We write $S_n(M)$ instead of $S_n(M, M)$ and $\langle X\rangle$ instead of $\langle X\rangle_M$. By Corollary 2.6 we may identify the topological space of n-types $\langle X\rangle$ with $\operatorname{Sper}\mathcal{D}(X)$. By Lemma 2.5 we may identify the topological space $\operatorname{Sper}\mathcal{D}(X)$ with $\operatorname{Spec}\mathcal{D}(X)$ and $\operatorname{Sper}\mathcal{C}(X)$ with $\operatorname{Spec}\mathcal{C}(X)$. We want to study the inclusion map $\mathcal{C}(X) \longrightarrow \mathcal{D}(X)$. Again, let $\iota : \operatorname{Spec}\mathcal{D}(X) \longrightarrow \operatorname{Spec}\mathcal{C}(X)$ denote the map induced by this inclusion and let Z be the image of ι.

If $\bar{a} \in M^n$ we take $||\bar{a}|| := \sqrt{a_1^2 + ... + a_n^2}$. If $Y \subseteq M^n$ is M-definable we define the distance function $M^n \longrightarrow M$ to be $\operatorname{dist}_Y(\bar{a}) := \inf_{\bar{y}\in Y} ||\bar{a} - \bar{y}||$. By o-minimality the map dist_Y is well defined.

PROPOSITION 3.1. *If M is o-minimal, then $\iota : \langle X\rangle = \operatorname{Spec}\mathcal{D}(X) \longrightarrow \operatorname{Spec}\mathcal{C}(X)$ is injective.*

PROOF. It follows from cell decomposition ([P-S] or [vdD-Lew], §1), that each M-definable subset of M^n is a boolean combination of closed M-definable subsets of M^n. Since each closed, M-definable set $Y \subseteq M^n$ is the zero set of dist_Y and $\operatorname{dist}_Y|_X \in \mathcal{C}(X)$, we get the proposition from Lemma 2.8(i). □

If $\alpha_1, ..., \alpha_k$ are elements from an elementary extension N of M we write $M\langle\alpha_1, ..., \alpha_k\rangle$ for the definable closure of $M \cup \{\alpha_1, ..., \alpha_k\}$. As mentioned above $M\langle\alpha_1, ..., \alpha_k\rangle$ is an elementary extension of M.

PROPOSITION 3.2. *If $p \in \langle X\rangle$ is realized by $\bar{\alpha}$, then $M\langle\bar{\alpha}\rangle$ is the quotient field of*

$$\{F(\bar{\alpha}) \mid F \in \mathcal{C}(X)\}$$

PROOF. We may assume that $X = M^n$ and $\dim p = n$, where dim denotes the dimension of $M\langle\bar{\alpha}\rangle$ over M in the sense of the theory of M ([P-S]). Let $\beta \in M\langle\bar{\alpha}\rangle$, and suppose β is M-bounded. Let $H : M^n \longrightarrow M$ be M-definable, such that $\beta = H(\bar{\alpha})$. By cell decomposition there is some open, M-definable subset of M^n such that $p \in U$, H is continuous on U and $|H(\bar{a})| < b$ $(\bar{a} \in U)$ for some $b \in M$. Let $F = \operatorname{dist}_{M^n\setminus U}$ be the distance function. F is M-definable and continuous. We define $G : M^n \longrightarrow M$ by $G(\bar{x}) = F(\bar{x})\cdot H(\bar{x})$. Since $H(U)$ is an M-bounded subset of M and $F(M^n \setminus U) = 0$ we get that G is continuous everywhere in M^n. Since $F(\bar{a}) > 0$ for all $\bar{a} \in U$ we have $F(\bar{\alpha}) \neq 0$. Thus $\beta = \frac{G(\bar{\alpha})}{F(\bar{\alpha})}$ with M-definable, continuous maps $G, F : M^n \longrightarrow M$. □

Proposition 3.2 shows that in the case $\mathfrak{p} \in Z$, the residue field $\kappa(\mathfrak{p})$ can be canonically expanded to an $\mathscr{L}$-structure which is an elementary extension of M in the following way:

COROLLARY 3.3. *Let $p \in \langle X\rangle$ and let $\bar{\alpha}$ be a realization of p. Let $N = M\langle\bar{\alpha}\rangle$. If $F \in \mathcal{C}(X)$ then we write F_N for the map $X_N \longrightarrow N$ which is defined by the same formula as F. Let $\mathfrak{p} = \iota(p)$. The map $\mathcal{C}(X) \longrightarrow N$ which sends F to $F_N(\bar{\alpha})$ induces an embedding*

$$\begin{aligned} \mathcal{C}(X)/\mathfrak{p} &\longrightarrow N \\ F/\mathfrak{p} &\longmapsto F_N(\bar{\alpha}) \end{aligned}$$

and $\kappa(\mathfrak{p}) \cong N$ as fields. If we expand $\kappa(\mathfrak{p})$ to an $\mathscr{L}$-structure with this isomorphism, then the expansion does not depend on the choice of the realization $\bar{\alpha}$ of p. In particular if $\alpha_i = x_i \bmod \mathfrak{p}$, then $N = \kappa(\mathfrak{p})$ and $F \bmod \mathfrak{p} = F_N(x_1 \bmod \mathfrak{p}, ..., x_n \bmod \mathfrak{p})$. □

In [Pi] a description of $\kappa(\mathfrak{p})$ as stalk of the sheaf of continuous, M-definable function on M^n at p says: $\kappa(\mathfrak{p}) = \varinjlim \{\mathcal{C}(U) \mid U \subseteq M^n$ open and M-definable, $p \in \langle U\rangle\}$

4. T-convex valuation rings. In this section T denotes a complete o-minimal expansion of the theory of real closed fields in the language $\mathscr{L}$. We collect some results from the paper [vdD-Lew].

DEFINITION 4.1. Let M be a model of T. A convex valuation ring V of M is called *T-convex*, if V is closed under all $\emptyset$-definable, continuous maps $M \longrightarrow M$.

For example, if $M_0 \prec M$ and if V is the convex hull of M_0 in M, then V is T-convex. If $V \subseteq M$ is T-convex, then $\operatorname{cl}\emptyset \subseteq V$.

PROPOSITION 4.2. *For all convex valuation rings $V \subseteq M \models T$ the following conditions are equivalent:*

(i) V is T-convex.
(ii) If $M_0 \prec M$, $M_0 \subseteq V \subseteq M$ and $F : M^n \longrightarrow M$ is a continuous, M_0-definable map, then $F(V^n) \subseteq V$.

PROOF. [vdD-Lew], (2.9) and (2.10). □

LEMMA 4.3. *Let $M \prec N$ be models of T and let V be a T-convex subring of N, which contains M. If $\alpha \in V$ and $\alpha > M$, then $M\langle\alpha\rangle \subseteq V$.*

PROOF. [vdD-Lew], (2.13). □

THEOREM 4.4. *Let $V \subseteq M \models T$ be T-convex and $M_0 \prec M$ with $M_0 \subseteq V$. Then M_0 is maximal definably closed in V if and only if the composed map $M_0 \longrightarrow V \longrightarrow \kappa_V := V/\mathfrak{m}_V$ is onto. In this case we have:*

(i) V is the convex hull of M_0 in M and M_0 is tame in M, i.e. each M_0-bounded element in M is infinitely close to an element of M_0.
(ii) If $M_0' \subseteq V$ is maximal definably closed, then there is a unique $\mathscr{L}$-isomorphism $M_0 \longrightarrow M_0'$, which commutes with the residue maps.

PROOF. [vdD-Lew], (2.12) and (2.15). □

Thus $M_0 \longrightarrow \kappa_V$ induces an $\mathscr{L}$-structure on κ_V, which is independent of the choice of M_0. κ_V is again a model of T. We always consider κ_V to be this expansion of the field κ_V. Theorem 4.4 implies

COROLLARY 4.5. *Let $V \subseteq M \models T$ be T-convex and $M \prec N$. If $W \subseteq N$ is a T-convex ring of N lying over V, then the canonical morphism $\kappa_V \longrightarrow \kappa_W$ is an elementary map of models of T.* □

We are going to change the condition "T-convex" for valuation rings into a condition for convex subgroups of $(M, +, <)$. This gives a characterization of the maximal ideals of T-convex rings.

REMARK If $X \subseteq M$ is convex, $0 \in X$ and $M_0 \prec M$ is maximal definably closed with the property that $M_0 \cap X = 0$, then M_0 is tame in M. Such an M_0 exists iff X is infinitesimal over $\operatorname{cl}\emptyset$.

DEFINITION 4.6. Let G be a convex subgroup of $(M, +)$, such that G is infinitesimal over $\operatorname{cl}\emptyset$. We define $B^T(G) =$ the convex hull of all $M_0 \prec M$ with $G \cap M_0 = 0$.

We have $G \neq \{0\} \;\Rightarrow\; B^T(G) \neq M$, because for each $g \in G,\ g \neq 0$ the element $\frac{1}{g}$ is not in $B^T(G)$. $B^T(G)$ is the union of T-convex rings, hence $B^T(G)$ is T-convex, too. $B^T(G)$ is the largest T-convex ring of M such that G is contained in the maximal ideal $\mathfrak{m}_{B^T(G)}$ of $B^T(G)$. If $M_0 \prec M$ is maximal with $M_0 \cap G = 0$, then M_0 is maximal in $B^T(G)$ and vice versa.

If M is a pure real closed field and G is a convex subgroup of M with $1 \notin G$, then

$$\mathfrak{m}_{B^T(G)} = \sqrt{G} := \{a \in M \mid a^n \in G \text{ for some } n \in \mathbb{N}\}$$
$$B^T(G) = \{a \in M \mid a \cdot \sqrt{G} \subseteq \sqrt{G}\}$$

We write $\mathfrak{M}$ for the set of all $\emptyset$-definable, continuous maps s with $s(0) = 0$.

DEFINITION 4.7. If M is a model of T and G is a convex subgroup of $(M, +)$ with $G \cap \operatorname{cl}\emptyset = 0$ we define the *T-radical of G*: $\sqrt[T]{G} := \{s(g) \mid g \in G,\ s \in \mathfrak{M}\}$. If $\sqrt[T]{G} = G$ we call G a *T-radical group.*

LEMMA 4.8. *If $G \cap \operatorname{cl}\emptyset = 0$, then $\sqrt[T]{G} = \mathfrak{m}_{B^T(G)}$. G is T-radical if and only if for all $a \in M \setminus G$, which are infinitesimal over $\operatorname{cl}\emptyset$ we have $G \cap \operatorname{cl} a = 0$.*

PROOF. Suppose the condition in the supplement holds for G. Then we have $G = \mathfrak{m}_{B^T(G)}$, since each $a \in \mathfrak{m}_{B^T(G)}$ is infinitesimal over $\operatorname{cl}\emptyset$; because $\operatorname{cl} a$ is an elementary restriction of M we have $G \cap \operatorname{cl} a \neq 0$ thus $a \in G$.

By the mean value theorem $\sqrt[T]{G}$ is a convex subgroup of $(M, +)$ which contains G. If $M_0 \prec M$ with $M_0 \cap G = 0$, then $M_0 \cap \sqrt[T]{G} = 0$, too. Therefore we have $B^T(\sqrt[T]{G}) = B^T(G)$. It follows $\sqrt[T]{G} \subseteq \mathfrak{m}_{B^T(\sqrt[T]{G})} = \mathfrak{m}_{B^T(G)}$.

We are going to prove the condition in the supplement for $\sqrt[T]{G}$. Let $a \in M$, $a > 0$ be infinitesimal over $\operatorname{cl}\emptyset$ and $\operatorname{cl} a \cap \sqrt[T]{G} \neq 0$. We have to show that $a \in \sqrt[T]{G}$. Let $f : M \longrightarrow M$ be $\emptyset$-definable, $g \in G$ and $s \in \mathfrak{M}$ such that $f(a) = s(g)$. We may suppose that $g, s(g) > 0$. The elements a, g and $s(g)$ are positive and infinitesimal over $\operatorname{cl}\emptyset$. Therefore $f(x) \to 0$ as $x \to 0$, $x > 0$. There is some $b \in \operatorname{cl}\emptyset$, > 0, such that $f|_{(0,b]} : (0, b] \longrightarrow (0, f(b)]$ is an increasing homeomorphism. The map $f_1 : M \longrightarrow M$ defined by

$$f_1(x) = \begin{cases} x & \text{if } x \leq 0 \\ f(x) & \text{if } x \in (0, b] \\ f(b) + (x - b) & \text{if } x > b \end{cases}$$

is $\emptyset$-definable, continuous and bijective with $f_1(a) = f(a)$. The map f_1^{-1} is continuous too and we have $f_1^{-1}(0) = 0$. Thus $a = f_1^{-1}(s(g)) \in \sqrt[T]{G}$. This shows $\sqrt[T]{G} = \mathfrak{m}_{B^T(G)}$.

Finally suppose $G = \mathfrak{m}_{B^T(G)}$. If $a \in M \setminus G$ is infinitesimal over $\operatorname{cl}\emptyset$, then $\frac{1}{a} \in B^T(G)$ is infinite over $\operatorname{cl}\emptyset$. By Lemma 4.3 $\operatorname{cl} a \subseteq V$, hence $G \cap \operatorname{cl} a = 0$. $\square$

COROLLARY 4.9. *If G is a convex subgroup of $(M, +)$ with $G \cap \operatorname{cl}\emptyset = 0$ then G is T-radical if and only if G is the maximal ideal of a T-convex valuation ring of M. In this case $V = \{a \in M \mid a \cdot G \subseteq G\}$ is the valuation ring with maximal ideal G. Furthermore, if $M_0 \subseteq V$ is an elementary restriction of M and $f : M^n \longrightarrow M$ is M_0-definable, continuous with $f(0) = 0$, then $f(G^n) \subseteq G$.*

PROOF. Lemma 4.8 gives the equivalence. If G is the maximal ideal of a T-convex valuation ring V and $M_0 \subseteq V$ is an elementary restriction of M, then $g(G) \subseteq G$ for each M_0-definable, continuous map $g : M \longrightarrow M$ with $f(0) = 0$, since G is infinitesimal over M_0. So if $f : M^n \longrightarrow M$ is continuous, M_0-definable with $f(0) = 0$, then $g(y) := \max\{|f(\bar{x})| \mid \bar{x} \in M_0^n, ||\bar{x}|| \leq y\}$ defines an M_0-definable, continuous map $[0, \infty) \longrightarrow M_0$ with $g(0) = 0$ and $f(G^n) \subseteq \operatorname{co}_N g(G) \subseteq G$. $\square$

5. T-radical ideals. Again we fix an integer n, a model M of T and some M-definable set $X \subseteq M^n$. Furthermore let $\mathcal{C}(X) = \mathcal{C}(X, M)$ and let $\mathfrak{M}$ be the set of all $\emptyset$-definable, continuous maps s with $s(0) = 0$.

DEFINITION 5.1. An ideal $\mathfrak{a}$ of $\mathcal{C}(X)$ is called *T-radical* if $\mathfrak{a}$ is closed under $\mathfrak{M}$, that is $s \circ F \in \mathfrak{a}$, for all $F \in \mathfrak{a}$ and $s \in \mathfrak{M}$.

Note that if $\mathfrak{a}$ is T-radical and $F^3 \in \mathfrak{a}$, then $F = \sqrt[3]{x} \circ F^3 \in \mathfrak{a}$, thus $\mathfrak{a}$ is a radical ideal. If $\mathfrak{p} \in Z$, then $\mathfrak{p}$ is T-radical, since there is a type $p \in \langle X \rangle$ such that $\mathfrak{p} = \{F \in \mathcal{C}(X) \mid F = 0 \in p\}$. If $F \in \mathfrak{p}$ and $s \in \mathfrak{M}$, then $\{F = 0\} \subseteq \{s \circ F = 0\}$, hence $s \circ F \in \mathfrak{p}$. Proposition 6.2 below says, that if X is locally closed, then each T-radical prime ideal comes from a type in this way.

DEFINITION 5.2. Obviously the intersection of T-radical ideals is again T-radical and we may define for each ideal $\mathfrak{a}$ of $\mathcal{C}(X)$:

$$\sqrt[T]{\mathfrak{a}} := \text{ the } T\text{-radical ideal generated by } \mathfrak{a}$$

Furthermore we define

$$\mathfrak{a}^\diamond := \{F \in \mathfrak{a} \mid s \circ F \in \mathfrak{a} \text{ for all } s \in \mathfrak{M}\}$$

In order to give a more explicit description of $\sqrt[T]{\mathfrak{a}}$ we first prove, that $\mathfrak{a}^\diamond$ is a T-radical ideal. Note that $\mathfrak{a}^\diamond$ is closed under all $s \in \mathfrak{M}$.

LEMMA 5.3. *Let $f, g, h : M \longrightarrow M$ be continuous, $\emptyset$-definable with $f(0) = 0 = g(0)$. Then we have*

(i) If $\{f = 0\}$ is in the interior of $\{|g| \le |f \cdot h|\}$ and the isolated zeros of f are in the interior of $\{f = g\}$, then there is some $\emptyset$-definable, continuous map $q : M \longrightarrow M$ such that $g = f \cdot q$.

(ii) If $F \in \mathcal{C}(X)$, $\varepsilon \in M$, $\varepsilon > 0$ and $|g| \le |f \cdot h|$ on $(-\varepsilon, \varepsilon)$, then $(g \circ F)^5$ is in the ideal generated by F^k and $(f \circ F)^2$ for all $k \in \mathbb{N}$.

PROOF. (i). Let $\{a_1, ..., a_k\} \subseteq \mathrm{cl}\,\emptyset$ be the set of boundary points of $\{f = 0\}$. Let $i \in \{1, ..., k\}$ and assume first that a_i is a nonisolated zero of f. Since a_i is in the interior of $\{|g| \le |f \cdot h|\}$, o-minimality implies that

$$q(a_i) := \lim_{x \to a_i, f(x) \neq 0} \frac{g(x)}{f(x)}$$

exist. If a_i is an isolated zero, then the limit exists by assumption (and is equal to 1). Furthermore we define $q(a) = \frac{g(a)}{f(a)}$ for $a \in M$, $f(a) \neq 0$. Thus q is defined on the closure of $\{f \neq 0\}$. Obviously, q can be extended to a continuous, $\emptyset$-definable function $M \longrightarrow M$ and this extension fulfills $g = f \cdot q$.

(ii). Let $\mathfrak{a}$ be the ideal generated by F^k and $(f \circ F)^2$. By assumption $h_1 := g^2 \vee (fh)^2 - (fh)^2$ vanishes in $(-\varepsilon, \varepsilon)$. Since $F^{2k} \in \mathfrak{a}$, we have $\frac{1}{\varepsilon^{2k}} F^{2k} \cdot h_1(F) \in \mathfrak{a}$. From $0 \le h_1(F) \le \frac{1}{\varepsilon^{2k}} F^{2k} \cdot h_1(F)$ we get $h_1^2(F) \in \mathfrak{a}$ (Lemma 2.5(iii)). By assumption $(g^2 \vee (fh)^2)^2(F) = h_1^2(F) + (fh)^4(F) + 2h_1(F)(fh)^2(F) \in \mathfrak{a}$. Since $0 \le g^4 \le (g^2 \vee (fh)^2)^2$ we get $g^5(F) \in \mathfrak{a}$ from Lemma 2.5(iii). □

DEFINITION 5.4. A subset $\mathfrak{M}_0$ of $\mathfrak{M}$ is called a *full set* if for all $s \in \mathfrak{M}$, there is some $s_0 \in \mathfrak{M}_0$ and some $\varepsilon \in M$, $\varepsilon > 0$ such that $|s| \le |s_0|$ on $(0, \varepsilon)$.

PROPOSITION 5.5. *If $\mathfrak{a}$ is an ideal of $\mathcal{C}(X)$, then $\mathfrak{a}^\diamond$ is the largest T-radical ideal contained in $\mathfrak{a}$. Furthermore $\mathfrak{a}^\diamond$ is closed under all M-definable, continuous maps $f : M \longrightarrow M$ with $f(0) = 0$. If $\mathfrak{M}_0$ is a full set and $\mathfrak{a}$ is an ideal of $\mathcal{C}(X)$, then*

$$\mathfrak{a}^\diamond = \{F \in \mathfrak{a} \mid s_0(F), s_0(-F) \in \mathfrak{a} \textit{ for all } s_0 \in \mathfrak{M}_0\}$$

PROOF.

<u>Claim 1:</u> $\mathfrak{a}^\diamond$ is closed under all M-definable, continuous maps $f : M \longrightarrow M$ with $f(0) = 0$.

Let $\mathcal{P}$ be the definable closure of $\emptyset$, let $\mu > M$ and let V be the convex hull of $\mathcal{P}\langle\mu\rangle$ in $M\langle\mu\rangle$. If $V \subsetneqq M\langle\mu\rangle$, then $\dim M\langle\mu\rangle/M \geq 2$ (by Theorem 4.4, since V is T-convex), a contradiction. It follows, that for each M-definable, continuous map $f : M \longrightarrow M$ with $f(0) = 0$, there is some $s \in \mathfrak{M}$ and $\varepsilon \in M, \varepsilon > 0$ such that $|f| \leq |s|$ in $(-\varepsilon, \varepsilon)$. By Lemma 5.3(ii), $f^5(F) \in \mathfrak{a}$ for all $F \in \mathfrak{a}^\diamond$. The same is true for $\sqrt[5]{f}$, which shows that $f(F) \in \mathfrak{a}$ for all $F \in \mathfrak{a}^\diamond$. This implies that even $\mathfrak{a}^\diamond$ is closed under f.

Claim 2: If $\mathfrak{M}_0$ is a full set and $\mathfrak{a}$ is an ideal of $\mathcal{C}(X)$, then

$$\mathfrak{a}^\diamond = \{F \in \mathfrak{a} \mid s_0(F), s_0(-F) \in \mathfrak{a} \text{ for all } s_0 \in \mathfrak{M}_0\}.$$

Let $F \in \mathfrak{a}$ such that $s_0(F), s_0(-F) \in \mathfrak{a}$ for all $s_0 \in \mathfrak{M}_0$ and let $s \in \mathfrak{M}$. Choose $s_0, s_1 \in \mathfrak{M}_0$ and some $\varepsilon > 0$ such that $|s| \leq s_0^2(x)$ and $|s| \leq s_1^2(-x)$ on $(0, \varepsilon)$. Then $s_2(F) \in \mathfrak{a}$ if $s_2(x) := s_0^2(x) + s_1^2(-x)$ and $|s| \leq s_2$ on $(-\varepsilon, \varepsilon)$. By Lemma 5.3(ii) we have $s^5(F) \in \mathfrak{a}$. Again, if we replace s by $\sqrt[5]{s}$ we get $s(F) \in \mathfrak{a}$.

For the rest of the proof we work with the full set $\mathfrak{M}_0$ of all $s \in \mathfrak{M}$, which are increasing, > 0 on $(0, +\infty)$ and symmetric, i.e. $s(-x) = s(x)$.

Claim 3: If $F \in \mathfrak{a}^\diamond$ and $G \in \mathcal{C}(X)$ with $|G| \leq |F|$, then $G \in \mathfrak{a}^\diamond$.

We have $G^3 = F{\cdot}H$ for some $H \in \mathcal{C}(X)$ (Lemma 2.5(ii)). Thus $G = \sqrt[3]{F{\cdot}H} = \sqrt[3]{F}{\cdot}\sqrt[3]{H} \in \mathfrak{a}$. If $s \in \mathfrak{M}_0$, then s is symmetric and increasing, hence $0 \leq s(G) \leq s(F)$. Since $s(F) \in \mathfrak{a}^\diamond$ we get $s(G) \in \mathfrak{a}$ as we have just proved for G. By Claim 2 we have $G \in \mathfrak{a}^\diamond$.

Claim 4: $\mathfrak{a}^\diamond$ is a subgroup of $\mathcal{C}(X)$.

If $F, G \in \mathfrak{a}^\diamond$, then $|F - G| \leq |F| + |G| \leq 2(|F| \vee |G|)$ and by Claim 3 it remains to show that $|F| \vee |G| \in \mathfrak{a}^\diamond$ for all $F, G \in \mathfrak{a}^\diamond$. By definition we have $|F|, |G| \in \mathfrak{a}^\diamond$. Take $s \in \mathfrak{M}_0$. Then $s(|F| \vee |G|) \leq s(|F|) + s(|G|) \in \mathfrak{a}$. Thus $s^2(|F| \vee |G|) \in \mathfrak{a}$ for all $s \in \mathfrak{M}_0$ and consequently $s(|F| \vee |G|) \in \mathfrak{a}$ for all $s \in \mathfrak{M}_0$.

Claim 5: If $F \in \mathfrak{a}^\diamond$, $H \in \mathcal{C}(X)$ with $F, H \geq 0$ and $s_0 \in \mathfrak{M}_0$, then $s_0^2(F{\cdot}H) \in \mathfrak{a}$.

We have $s_0(F{\cdot}H) \leq s_0(F{\cdot}(1 + H)) \leq s_0(\sqrt{F})$ on $\{\sqrt{F} < \frac{1}{1+H}\}$, since s_0 is increasing on $(0, +\infty)$. Since s_0 does not vanish for $F \neq 0$ we know that the zeros of $s_0(\sqrt{F})$ are in the interior of $\{s_0(F{\cdot}H) \leq s_0(\sqrt{F})\}$. Hence $s_0^2(F{\cdot}H) \in \mathfrak{a}$.

In order to prove that $\mathfrak{a}^\diamond$ is an ideal, it remains to show that $F{\cdot}H \in \mathfrak{a}^\diamond$ for all $F \in \mathfrak{a}^\diamond$ and all $H \in \mathcal{C}(X)$. By Claim 3 we may assume that $F, H \geq 0$. Since $\{s_0^2 \mid s_0 \in \mathfrak{M}_0\}$ is a full set as well, we get $F{\cdot}H \in \mathfrak{a}^\diamond$ from Claim 5 and Claim 2.

Obviously $\mathfrak{a}^\diamond$ is the largest subset of $\mathfrak{a}$, which is closed under all $s \in \mathfrak{M}$. $\square$

If R is a commutative unitary ring and G is a subgroup of $(R, +)$, then the set $\{a \in R \mid a{\cdot}R \subseteq G\}$ is the largest ideal of R contained in G. Therefore, in our situation, the construction $\mathfrak{a} \mapsto \mathfrak{a}^\diamond$ can be easily extended to subgroups of $(\mathcal{C}(X), +)$:

COROLLARY 5.6. *If G is a subgroup of $(\mathcal{C}(X), +)$, then*

$$\{F \in G \mid \textit{for all } s \in \mathfrak{M} \textit{ and all } H, H_1 \in \mathcal{C}(X) \textit{ we have } H_1{\cdot}s(F{\cdot}H) \in G\}$$

is the largest T-radical ideal contained in G. $\square$

COROLLARY 5.7. *If $\mathfrak{M}_0 \subseteq \mathfrak{M}$ is a full set and the ideal $\mathfrak{a}$ is closed under all $s_0 \in \mathfrak{M}_0$, then $\mathfrak{a}$ is T-radical.* $\square$

COROLLARY 5.8. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{C}(X)$, then $\sqrt[T]{\mathfrak{a}} + \sqrt[T]{\mathfrak{b}} = \sqrt[T]{\mathfrak{a} + \mathfrak{b}}$.*

PROOF. Since $\sqrt[T]{\mathfrak{a}} \cup \sqrt[T]{\mathfrak{b}} \subseteq (\sqrt[T]{\mathfrak{a}} + \sqrt[T]{\mathfrak{b}})^\diamond$ we know that $\sqrt[T]{\mathfrak{a}} + \sqrt[T]{\mathfrak{b}}$ is T-radical. $\square$

COROLLARY 5.9. *If* $\mathfrak{p} \in \operatorname{Spec} \mathcal{C}(X)$, *then* $\mathfrak{p}^\diamond$ *is the largest* T*-radical prime ideal contained in* $\mathfrak{p}$.

PROOF. Because there is a type $q \in \operatorname{Spec} \mathcal{D}$ such that $\mathfrak{q} \cap \mathcal{C}(X) \subseteq \mathfrak{p}$, we have $\mathfrak{q} \subseteq \mathfrak{p}^\diamond$. By Corollary 2.7 we know that $\mathfrak{p}^\diamond$ is prime. □

COROLLARY 5.10. *If* $\mathfrak{a}$ *is a* T*-radical ideal, then each minimal prime ideal over* $\mathfrak{a}$ *is* T*-radical. Consequently*

$$\sqrt[T]{\mathfrak{b}} = \bigcap \{\mathfrak{q} \mid \mathfrak{q} \textit{ is } T - \textit{radical and prime with } \mathfrak{b} \subseteq \mathfrak{q}\}$$

for all ideals $\mathfrak{b}$ *of* $\mathcal{C}(X)$. □

PROPOSITION 5.11. *Let* $\mathfrak{a}$ *be an ideal of* $\mathcal{C}(X)$.
(i) If $\mathfrak{M}_0$ *is a full set of elements, which does not vanish in* $(0, +\infty)$ *and* $k \in \mathbb{N}$, *then*

$$\sqrt[T]{\mathfrak{a}} = \{H \cdot s_0(F^k) \mid s_0 \in \mathfrak{M}_0, H \in \mathcal{C}(X) \textit{ and } F \in \mathfrak{a},\ F \geq 0\}$$

(ii) Let $\mathfrak{N} \subseteq \mathfrak{M}$ *such that*
- *(a) If* $s \in \mathfrak{M}$, $s > 0$ *on* $(0, +\infty)$ *then there are some* $t \in \mathfrak{N}$ *and some* $\varepsilon > 0$ *such that* $|t| < s$ *on* $(0, \varepsilon)$.
- *(b) If* $t \in \mathfrak{N}$ *then* $|t| \neq 0$ *on some interval* $(0, \varepsilon)$.

Then

$$\sqrt[T]{\mathfrak{a}} = \{G \in \mathcal{C}(X) \mid t(|G|) \in \mathfrak{a} \textit{ for some } t \in \mathfrak{N}\}$$

PROOF. (i). Let $\mathfrak{b}$ be the ideal generated by $\{s(F) \mid s \in \mathfrak{M},\ F \in \mathfrak{a}\}$. By definition we have $\mathfrak{a} \subseteq \mathfrak{b}^\diamond$. Since $\mathfrak{b}^\diamond$ is T-radical we must have $\sqrt[T]{\mathfrak{a}} = \mathfrak{b}$. Let $H_1, ..., H_k \in \mathcal{C}(X)$, $F_1, ..., F_k \in \mathfrak{a}$, $s_1, ..., s_k \in \mathfrak{M}$ and $G := H_1 \cdot s_1(F_1) + ... + H_k \cdot s_k(F_k)$. Take $s_i^* \in \mathfrak{M}$, $s_i \geq 0$ such that $|s_i(F_i)| \leq s_i^*(F_i^2)$ and such that s_i^* is increasing on M. Let $H^* := |H_1| \vee ... \vee |H_k|$, $s = s_1^* \vee ... \vee s_k^*$ and $F = F_1^2 + ... + F_k^2 \in \mathfrak{a}$. Then $|G| \leq H^* \cdot s(F)$ and there is some $H^{**} \in \mathcal{C}(X)$ such that $G^3 = H^* \cdot H^{**} \cdot s(F)$. Thus $G = H \cdot \sqrt[3]{s}(F)$ if $H = \sqrt[3]{H^* \cdot H^{**}}$.

Replacing s by $\sqrt[3]{s}(\sqrt[k]{|x|})$ we have proved that for all $G \in \sqrt[T]{\mathfrak{a}}$ there are some $H \in \mathcal{C}(X)$, $s \in \mathfrak{M}$, s increasing and $F \in \mathfrak{a}$, $F \geq 0$ such that $G = H \cdot s^3(F^k)$. Take $s_0 \in \mathfrak{M}_0$ such that $s \leq s_0$ on some $(0, \varepsilon)$. Then $|G| = |H \cdot s^3(F^k)| \leq |H \cdot s_0^3(F^k)|$ on $\{F^k < \varepsilon\}$. Since $F \geq 0$ and s_0 does not vanish in $(0, +\infty)$, the zeros of $s_0^3(F^k)$ are in $\{F^k < \varepsilon\}$. Thus $G^3 = H_1 \cdot H \cdot s_0^3(F^k)$ for some $H_1 \in \mathcal{C}(X)$ and finally $G = H_2 \cdot s_0(F^k)$ with $H_2 = \sqrt[3]{H_1 \cdot H}$.

(ii). First take $G \in \mathcal{C}(X)$ and $t \in \mathfrak{N}$ such that $t(|G|) \in \mathfrak{a}$. Since $|t| \neq 0$ for small $x > 0$, there is an homeomorphism $s \in \mathfrak{M}$ and some $\varepsilon > 0$ such that $s = t$ on $(0, \varepsilon)$. Hence $|G| = s^{-1}(t(|G|))$ on $\{|G| < \varepsilon\}$ and consequently $G \in \sqrt[T]{\mathfrak{a}}$.

Conversely let $G \in \sqrt[T]{\mathfrak{a}}$. Let $s \in \mathfrak{M}$ be an increasing homeomorphism, $H \in \mathcal{C}(X)$ and $F \in \mathfrak{a}$, $F \geq 0$ such that $G = H \cdot s(F^3)$. Then $|G| \leq (1 + |H|)s(F^3) \leq \sqrt{s(F^3)}$ on $\{\sqrt{s(F^3)} < \frac{1}{1+|H|}\}$. Let $\varepsilon > 0$ such that $|t| \leq s^{-1}(x^2)$ on $(0, \varepsilon)$ and such that $|t|$ is increasing on $(0, \varepsilon)$. Then $|t(|G|)| \leq |t|(\sqrt{s(F^3)}) \leq F^3$ on $\{\sqrt{s(F^3)} < \min\{\frac{1}{1+|H|}, \varepsilon\}\}$. It follows $t(|G|)^3 = H_1 \cdot F^3$ with some $H_1 \in \mathcal{C}(X)$, consequently $t(|G|) = \sqrt[3]{H_1} \cdot F \in \mathfrak{a}$. □

6. T-radical prime ideals.

LEMMA 6.1. *Let* $\mathfrak{q} \in \operatorname{im} \iota$ *and let* $f : M \longrightarrow M$ *be* M*-definable and continuous. Let* $\mathfrak{p} \in \operatorname{Spec} \mathcal{C}(X)$ *such that* $\mathfrak{q} \subseteq \mathfrak{p}$ *and let* U *be the convex hull of* $\mathfrak{p}/\mathfrak{q}$ *in* $\kappa(\mathfrak{q})$. *Furthermore let* $f_\mathfrak{q} : \kappa(\mathfrak{q}) \longrightarrow \kappa(\mathfrak{q})$ *be the map defined by the same formula as* f. *Then* $f_\mathfrak{q}(U) \subseteq U$ *if and only if* $f \circ F \in \mathfrak{p}$ *for all* $F \in \mathfrak{p}$.

PROOF. By Corollary 3.3 we have for $F \in \mathcal{C}(X)$: $f_{\mathfrak{q}}(F \bmod \mathfrak{q}) = (f \circ F) \bmod \mathfrak{q}$. Suppose $f_{\mathfrak{q}}(U) \subseteq U$ and let $F \in \mathfrak{p}$. From $f_{\mathfrak{q}}(F \bmod \mathfrak{q}) \in U$ we get some $G \in \mathfrak{p}$ such that $|f \circ F \bmod \mathfrak{q}| \leq G \bmod \mathfrak{q}$ (in $\kappa(\mathfrak{q})$). This means $(G \pm f \circ F)^- \in \mathfrak{q} \subseteq \mathfrak{p}$ (by Lemma 2.4), hence $|f \circ F \bmod \mathfrak{p}| \leq G \bmod \mathfrak{p}$ (in $\kappa(\mathfrak{p})$). Since $G \in \mathfrak{p}$ we get $f \circ F \in \mathfrak{p}$.

Conversely suppose that $f \circ F \in \mathfrak{p}$ for all $F \in \mathfrak{p}$. Each element in U is infinitesimal over M, because for $0 < m \in M$ and $F \in \mathfrak{p}$ with $m < F \bmod \mathfrak{q}$ we have $(F - m) \bmod \mathfrak{p} \geq 0$ and $-F \bmod \mathfrak{p} \geq 0$, thus $-m \geq 0$, a contradiction. From $f(0) \in \mathfrak{p}$ we get $f(0) = 0$. Let $0 < \alpha \leq F \bmod \mathfrak{q}$ for some $F \in \mathfrak{p}$. Since f is M-definable and continuous, U is infinitesimal over M and $f(0) = 0$ we get that $|f_{\mathfrak{q}}(\alpha)| \leq |f_{\mathfrak{q}}(F \bmod \mathfrak{q})|$. By assumption $f_{\mathfrak{q}}(F \bmod \mathfrak{q}) \in \mathfrak{p}/\mathfrak{q}$, which gives the lemma. □

Up to now we have not needed any assumption on the M-definable set $X \subseteq M^n$. In the following key Proposition we need that X is locally closed. If $X = \{(a,b) \in M^2 \mid a, b > 0\} \cup \{(0,0)\}$, then $\mathfrak{p} := \{F \in \mathcal{C}(X) \mid \exists \varepsilon > 0 \forall t \in [0, \varepsilon) \lim_{x \in X,\ x \to (t,0)} F(x) = 0\}$ is a T-radical prime ideal of $\mathcal{C}(X)$ which is not induced by a type from $\langle X \rangle$.

PROPOSITION 6.2. *If X is locally closed, then each T-radical prime ideal of $\mathcal{C}(X)$ is in Z.*

PROOF. We fix an arbitrary $\mathfrak{q} \in Z$ with $\mathfrak{q} \subseteq \mathfrak{p}$ (by Lemma 2.8(iii) such a prime ideal exists). By Lemma 6.1 we know that the convex hull $\mathfrak{m}$ of $\mathfrak{p}/\mathfrak{q}$ in $\kappa(\mathfrak{q})$ is T-radical. Hence $\mathfrak{m}$ is the maximal ideal of a T-convex valuation ring V of $\kappa(\mathfrak{q})$ (Corollary 4.9). Note that $\mathcal{C}(X)/\mathfrak{q} \subseteq V$, since $(F \bmod \mathfrak{q}) \cdot \mathfrak{m} \subseteq \mathfrak{m}$ for all $F \in \mathcal{C}(X)$. By Theorem 4.4, there is an elementary restriction $N \prec \kappa(\mathfrak{q})$, which contains M, such that V is the convex hull of N and such that N is tame in $\kappa(\mathfrak{q})$. Let λ be the standard part map $V \longrightarrow N$ and let $\beta_i = \lambda(x_i \bmod \mathfrak{q})$ (note that $x_i \bmod \mathfrak{q} \in \mathcal{C}(X)/\mathfrak{q} \subseteq V$). We claim that $\iota(t(\beta_1, ..., \beta_n/M)) = \mathfrak{p}$. First we show that $p_0 := t(\beta_1, ..., \beta_n/M) \in \langle X \rangle$.

By Lemma 2.8(iii) there is a prime ideal $\mathfrak{q}' \in Z$ with $\mathfrak{p} \subseteq \mathfrak{q}'$. Let $\mathfrak{m}'$ be the convex hull of $\mathfrak{q}'/\mathfrak{q}$ in $\kappa(\mathfrak{q})$, let V' be the T-convex valuation ring according to $\mathfrak{m}'$ and let $N' \prec N$ be tame in $\kappa(\mathfrak{q})$ such that V' is the convex hull of N'. Finally let $\lambda' : V' \longrightarrow N'$ be the standard part map according to V'. By definition we have $\lambda'(x_i \bmod \mathfrak{q}) = x_i \bmod \mathfrak{q}'$ (again we have $\kappa(\mathfrak{q}') \prec N'$), hence $\lambda'(\beta_i) = x_i \bmod \mathfrak{q}'$ as well. We have the following situation. We have types q, p_0 and q' such that $q, q' \in \langle X \rangle$ and $M \prec N' \prec N \prec \kappa(\mathfrak{q})$, such that N' and N are tame in $\kappa(\mathfrak{q})$. Furthermore there are realization $\bar{\alpha} \models q$, $\bar{\beta} \models p_0$ and $\bar{\alpha}' \models \mathfrak{q}'$ such that $\bar{\alpha} \in \kappa(\mathfrak{q})$, $\bar{\beta} \in N$ is infinitely close to $\bar{\alpha}$ and $\bar{\alpha}' \in N'$ is infinitely close to $\bar{\beta}$. Therefore if $G \in \mathcal{C}(M^n)$, then $G(\bar{\alpha}) = 0 \Rightarrow G(\bar{\beta}) = 0 \Rightarrow G(\bar{\alpha}') = 0$. This means that $q \cap \mathcal{C}(M^n) \subseteq p_0 \cap \mathcal{C}(M^n) \subseteq q_1 \cap \mathcal{C}(M^n)$. Since X is locally closed and $q, q_1 \in X$ we get that $p_0 \in X$.

Let $F \in \mathcal{C}(X)$, thus $F : X \longrightarrow M$ is M-definable and continuous. Since $t(\beta_1, ..., \beta_n/M) \in \langle X \rangle$, the continuity of F implies

$$F_N(\beta_1, ...\beta_n) = F_N(\lambda(x_1 \bmod \mathfrak{q}), ..., \lambda(x_n \bmod \mathfrak{q})) = \lambda(F_{\kappa(\mathfrak{q})}(x_1 \bmod \mathfrak{q}, ..., x_n \bmod \mathfrak{q}))$$

Thus $F_N(\beta_1, ...\beta_n) = 0$ if and only if $F_{\kappa(\mathfrak{q})}(x_1 \bmod \mathfrak{q}, ..., x_n \bmod \mathfrak{q}) \in \mathfrak{m}$. Since $F_{\kappa(\mathfrak{q})}(x_1 \bmod \mathfrak{q}, ..., x_n \bmod \mathfrak{q}) = F \bmod \mathfrak{q}$ we get that $F_N(\beta_1, ...\beta_n) = 0$ if and only if $F \in \mathfrak{p}$, i.e. $\iota(t(\beta_1, ..., \beta_n/M)) = \mathfrak{p}$. □

7. Consequences. If $S \subseteq \mathcal{C}(X)$, we write $Z(S)$ for the set of all common zeros of elements of S. Conversely if $Y \subseteq X$ we write $I(Y)$ for the T-radical ideal of all $F \in \mathcal{C}(X)$, which vanishes on Y. Certainly we have $\overline{Y} \subseteq Z(I(Y))$, where $\overline{Y}$ denotes the topological closure of Y in X and $\sqrt[T]{(S)} \subseteq I(Z(S))$. If Y is M-definable, then $Z(\mathrm{dist}_Y |_X) = \overline{Y}$ and $\mathrm{dist}_Y |_X \in \mathcal{C}(X)$, hence $\overline{Y} = Z(I(Y))$. For finitely generated ideals we have:

THEOREM 7.1.*(Nullstellensatz)*
If X is locally closed and $\mathfrak{a}$ is a finitely generated ideal of $\mathcal{C}(X)$, then
(i) $\sqrt[T]{\mathfrak{a}} = I(Z(\mathfrak{a}))$
(ii) $\mathfrak{a}^\diamond = \{G \in \mathcal{C}(X) \mid G$ *vanishes in an open neighbourhood of* $Z(\mathfrak{a})\}$

PROOF. Let $F_1, ..., F_k$ be generators of $\mathfrak{a}$.
(i). If $G \in \mathcal{C}(X)$ vanishes on $\{F_1 = ... = F_k = 0\}$, then $G \in \mathfrak{q} \cap \mathcal{C}(X)$ for all $\mathfrak{q} \in \operatorname{Spec} \mathcal{D}$, which contains $F_1, ..., F_k$. By Corollary 5.10 and Proposition 6.2 we have $G \in \sqrt[T]{\mathfrak{a}}$.
(ii). If G vanishes in a neighbourhood of the zeros of $\sum F_i^2 \in \mathfrak{a}$ then $G \in \mathfrak{a}$ by Lemma 2.5(i). Since the set on the right hand side is T-radical we have $G \in \mathfrak{a}^\diamond$.
For the converse we first replace $\mathfrak{a}$ by the ideal generated by $F := \sqrt[3]{|F_1| + ... + |F_k|}$. This is possible, since for each $H \in \mathfrak{a}$ we have $|H| \leq H_1 \cdot F^3$ for some $H_1 \in \mathcal{C}(X)$, hence $\mathfrak{a} \subseteq F \cdot \mathcal{C}(X)$. Suppose $G \in \mathfrak{a}^\diamond$ and G does not vanish in a neighbourhood of $Z(F)$. By curve selection ([vdD-Lew], (1.8)), there is some $\varepsilon > 0$ and an M-definable map $\tau : (0, \varepsilon) \longrightarrow X \setminus Z(G)$ such that $\lim\limits_{x \searrow 0} \tau(x) = a \in Z(F)$, where a is not in the interior of $Z(G)$. Let $f : M \longrightarrow M$ be an M-definable, continuous map with $f(0) = 0$ and $\delta > 0$ such that $|f(G(\tau))| \geq \sqrt{F(\tau)}$ on $(0, \delta)$ (for example we can take $f = G(\tau)^{-1} \circ \sqrt{F(\tau)}$ inside some suitable $(0, \delta)$). By Proposition 5.5 we have $f(G) \in \mathfrak{a}$, thus there is some $H \in \mathcal{C}(X)$ such that $f(G) = H \cdot F$. Consequently $0 < \sqrt{F(\tau)} \leq |f(G(\tau))| = |H(\tau)| \cdot F(\tau)$ on $(0, \delta)$, a contradiction. $\square$

REMARK: The assumption that $\mathfrak{a}$ is T-radical and $Z(\mathfrak{a})$ is definable and nonempty does not imply that $\mathfrak{a}$ contains all maps $G \in \mathcal{C}(X)$ which vanish in a neighbourhood of $Z(\mathfrak{a})$. To see this, let $\mathfrak{a}$ be the set of all $F \in \mathcal{C}(M^2)$ which vanish in $M \times (-\infty, \varepsilon]$ for some $\varepsilon > 0$. Certainly $\mathfrak{a}$ is a T-radical ideal and $Z(\mathfrak{a}) = M \times (-\infty, 0]$. Let Y be the complement of $\{(a, b) \in M^2 \mid b > \frac{1}{a} > 0\}$. Then $Z(\mathfrak{a})$ is in the interior of Y, but $\mathfrak{a}$ does not contain the distance function of Y.

LEMMA 7.2. *If X is closed and $F \in \mathcal{C}(X)$, then there are some $\emptyset$-definable, continuous map $s : M \longrightarrow M$ and some $c \in M$ such that $|F(\bar{a})| \leq c \cdot s(||\bar{a}||)$ for all $\bar{a} \in X$.*

PROOF. (see also [B-C-R], Prop 2.6.2) Let $X_t := \{\bar{a} \in X \mid ||\bar{a}|| = t\}$ and let $f(t) := \sup |F|(X_t)$ if $X_t \neq \emptyset$, $f(t) = 0$ otherwise. Certainly f is M-definable. Take $s : M \longrightarrow M$ continuous, $\emptyset$-definable, $s \geq 1$ such that $|f| \leq s$ on $\{|x| \geq r\}$ and choose $c \in M$ with $|F(\bar{a})| \leq c$ on $X \cap \{||\bar{a}|| \leq r\}$. $\square$

THEOREM 7.3.*(Division Property in o-minimal structures)*
Let $\mathfrak{N} \subseteq \mathfrak{M}$ such that for each homeomorphism $s \in \mathfrak{M}$, there is some $t \in \mathfrak{N}$ with $|t(x)| \leq |s(x)|$ for small positive x. If X is locally closed and $F, G \in \mathcal{C}(X)$ with $\{F = 0\} \subseteq \{G = 0\}$, then there are some $H \in \mathcal{C}(X)$ and some $t \in \mathfrak{N}$ such that $t(|G|) = F \cdot H$.

PROOF. From Theorem 7.1(i) and Proposition 5.11(ii). $\square$

THEOREM 7.4.*(Łojasiewicz inequality in o-minimal structures)*
Let $X \subseteq M^n$ be closed. Let $(f_i \mid i \in I)$ be a set of $\emptyset$-definable maps $(0, \infty) \longrightarrow (0, \infty)$ such that for all $\emptyset$-definable $g : M \longrightarrow M$ there is some i with $|g(x)| \leq f_i(x)$ for sufficiently large x. If $F, G : X \longrightarrow M$ are continuous and M-definable, such that each zero of F is a zero of G, then there are $i, j \in I$ and some $c \in M$ such that

$$\frac{1}{f_i(\frac{1}{|G(\bar{x})|})} \leq |F(\bar{x})| \cdot c \cdot (1 + f_j(||x||)) \text{ for all } \bar{x} \in X$$

where we define the left hand side to be 0 if $G(\bar{x}) = 0$.

PROOF. We may assume that $\lim_{x\to\infty} f_i(x) = \infty$ for all i. Let $t_i \in \mathfrak{M}$ such that $t_i(x) = 1/f_i(1/x)$ for sufficiently small $x > 0$. The assumption on the family (f_i) says that $\{t_i \mid i \in I\}$ is a full set. By Theorem 7.3, there is some i and some $H \in \mathcal{C}(X)$ such that $t_i(|G|) = F\cdot H$. Since X is closed our assumption and Lemma 7.2 yields an $j \in I$ with $|H(\bar{x})| \leq c\cdot(1 + f_j(||\bar{x}||))$ for all $\bar{x} \in X$. □

8. Exponentially and polynomially bounded expansions. An o-minimal expansion T of $Th(\mathbb{R}_{\exp})$ (of the theory of real closed fields respectively) is called exponentially bounded (polynomially bounded) if for all $\emptyset$-definable maps $s : M \longrightarrow M$ there is some $k \in \mathbb{N}$ such that $|f(x)| \leq \exp_k(x)$ ($|f(x)| \leq x^k$) finally. By [vdD-M-M], 2.10. we have that $\mathbb{R}_{\exp}$ is exponentially bounded. This means we have certain full sets by definition. For these sets we collect the results of this paper. By the Growth dichotomy ([Mi1]) if T is not polynomially bounded and $\mathrm{cl}\,\emptyset$ is archimedean, then the exponential is definable in M. This means, that T has a model M, which expands $\mathbb{R}_{\exp}$. Therefore the theorems below are sharp in the sense that there is no intermediate Division Property between the classical and the exponential division property in o-minimal expansions of $\mathbb{R}$. Furthermore, at the present stage, it is not known if there is an o-minimal expansion of $\mathbb{R}$ which is not exponentially bounded.

THEOREM 8.1.*(Łojasiewicz inequality for polynomially bounded theories)*
Let T be polynomially bounded, let M be a model of T and let $X \subseteq M^n$ be M-definable and locally closed. Then each radical ideal of $\mathcal{C}(X)$ is T-radical and

$$\langle X\rangle \overset{1-1}{\longleftrightarrow} \operatorname{Sper}\mathcal{C}(X) \cong \operatorname{Spec}\mathcal{C}(X).$$

If $F, G \in \mathcal{C}(X)$ such that $\{F = 0\} \subseteq \{G = 0\}$, then there are some $k \in \mathbb{N}$ and some $H \in \mathcal{C}(X)$ such that $G^k = F\cdot H$. If in addition X is closed, then there are $m \in \mathbb{N}$, $c \in M$ such that $|G(\bar{a})| \leq |F(\bar{a})| \cdot c \cdot (1 + ||\bar{a}||)^k$ for all $\bar{a} \in X$.

PROOF. Let $\mathfrak{N} := \{x^k \mid k \in \mathbb{N}\}$ and $\mathfrak{M} := \{\sqrt[k]{|x|} \mid k \in \mathbb{N}\}$. Since T is polynomially bounded, $\mathfrak{M}$ is a full set. Furthermore $\mathfrak{N}$ has the properties required in Theorem 7.3 and Theorem 7.4. Now the Theorem follows from the latter theorems, from Corollary 5.7 and from Proposition 6.2. □

Let $s : M \longrightarrow M$ be definable such that $\lim_{x\to+\infty} = +\infty$. We write $s_k(x)$ for the k-th iterated $s(s_{k-1}(x))$ of s. Let $t(x) := 1/s(1/x)$, where we take $t(x) = 0$ if $x = 0$. By easy induction we have $t_k(x) = 1/s_k(1/x)$ for small $x > 0$. For large x the map s has an inverse and for small $x > 0$ the map t has an inverse. For sufficiently small $x > 0$ it follows $t^{-1}(x) = 1/s^{-1}(1/x)$.

Now suppose that T is an exponentially bounded expansion of $Th(\mathbb{R}_{\exp})$. Let $E(x) := 1/\exp(1/x^2)$. Let $L(x) := -1/\log(x^2)$ for $|x| < \frac{1}{e}$ and $L(x) = \frac{1}{2}$ for $|x| \geq \frac{1}{2}$. From the remark above, we get
(i) $\{L_k \mid k \in \mathbb{N}\}$ is a full set.
(ii) $\{(E_k)^{-1} \mid k \in \mathbb{N}\}$ is a full set, hence $\{E_k \mid k \in \mathbb{N}\}$ fulfills the requirements on the set $\mathfrak{N}$ in Theorem 7.3 and Theorem 7.4.

Now Corollary 5.7, Proposition 6.2, Theorem 7.3 and Theorem 7.4 say:

THEOREM 8.2.(*Exponential Division Property*)
Let T be an exponentially bounded expansion of $Th(\mathbb{R}_{\exp})$, let M be a model of T and let $X \subseteq M^n$ be M-definable and locally closed. Then an ideal $\mathfrak{a}$ of $\mathcal{C}(X)$ is T-radical if and only if $L(\mathfrak{a}) \subseteq \mathfrak{a}$. The T-radical prime ideals of $\mathcal{C}(X)$ are in 1-1 correspondence with the n-types of M containing X. Furthermore If $F, G \in \mathcal{C}(X)$ such that $\{F = 0\} \subseteq \{G = 0\}$, then there are some $H \in \mathcal{C}(X)$ and some $k \in \mathbb{N}$ such that $E_k(G) = H \cdot F$. If in addition X is closed, then there is some $m \in \mathbb{N}$ and a constant $c \in M$ such that

$$E_k(G(\bar{x})) \leq |F(\bar{x})| \cdot c \cdot \exp_m(||\bar{x}||) \text{ for all } \bar{x} \in X \qquad \square$$

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Marcus Tressl, NWF-I Mathematik, 93040 Universität Regensburg, Germany
e-mail: marcus.tressl@mathematik.uni-regensburg.de