ELEMENTARY PROPERTIES OF MINIMAL AND MAXIMAL POINTS IN ZARISKI SPECTRA

NIELS SCHWARTZ, MARCUS TRESSL

ABSTRACT. We investigate connections between arithmetic properties of rings and topological properties of their prime spectrum. Any property that the prime spectrum of a ring may or may not have, defines the class of rings whose prime spectrum has the given property. We ask whether a class of rings defined in this way is axiomatizable in the model theoretic sense. Answers are provided for a variety of different properties of prime spectra, e.g., normality or complete normality, Hausdorffness of the space of maximal points, compactness of the space of minimal points.

CONTENTS

1. INTRODUCTION

In commutative ring theory one studies the prime spectrum of a ring. This is a functorial construction that associates a topological space $\operatorname{Spec} A$ with a ring A. It serves at least two important purposes: Firstly, it is an invariant that encodes information about the ring. Secondly, it helps translate algebraic information into geometric language, and vice versa. This second aspect of prime spectra is the basis of their application in algebraic geometry via schemes, where $\operatorname{Spec} A$ is equipped with a structure sheaf (cf. Grothendieck's EGA, or some introductory text about algebraic geometry, such as [Ha]). Concerning the first aspect, the usefulness of spectra as invariants depends to a large extent on understanding how properties of a ring correspond to properties of its prime spectrum: Given a ring A with some arithmetical property, does $\operatorname{Spec} A$ have a corresponding topological property?

In the present paper the converse question is addressed, i.e.: If $\operatorname{Spec} A$ has some particular topological property, how is this property reflected in the arithmetic of

---

Date: July 28, 2008.
2000 *Mathematics Subject Classification.* Primary: 14A25, Secondary: 08C10.
Key words and phrases. commutative ring, prime ideal, spectral space, axiomatizability.
Both authors were partially supported by the DFG project no. SCHW 287/17-1.

A? Some of the most fundamental notions of commutative ring theory are instances of the correspondence between arithmetic and topology; e.g., the property "Spec A is irreducible" says that A modulo its nilradical is a domain (equivalently: if $a\cdot b = 0$ then there is some $k \in \mathbb{N}$ such that $a^k = 0$ or $b^k = 0$); the property "Spec A has a unique closed point" says that the non-units of A form an additive subgroup of A (i.e., the ring is local); the property "Spec A is connected" means that the ring has only trivial idempotents.

Let $\mathcal{P}$ be a topological property that prime spectra may or may not have. We ask whether the class $\mathcal{R}(\mathcal{P})$ of those rings whose prime spectrum has property $\mathcal{P}$ is first order-axiomatizable in the language $\mathscr{L} = \{+, -, \cdot, 0, 1\}$ of rings. We are interested in explicit arithmetical descriptions of the class $\mathcal{R}(\mathcal{P})$.

We focus on properties of spectra that are concerned with the space of maximal ideals or with the space of minimal prime ideals, or with how these spaces sit inside the full prime spectrum. Here is a selection:

- The spectrum is **normal**, i.e., every prime ideal is contained in a unique maximal ideal, or
- the spectrum is **completely normal**, i.e., the set of prime ideals that contain a given prime ideal form a chain with respect to inclusion, or
- The spectrum is **inversely normal**, i.e., every prime ideal contains a unique minimal ideal, or
- the set of maximal points is a Hausdorff space, or
- the set of minimal points is a compact space.

Experience shows that spectra with these properties abound in real algebra.

Whenever we prove axiomatizability of a class of rings we also provide an explicit set of axioms. But we do not develop a general method that decides upon input $\mathcal{P}$ whether the class $\mathcal{R}(\mathcal{P})$ is elementary.

For each question there are two different variants: One may ask the question for *all* rings or only for *reduced rings*. If the class of rings whose spectrum has property $\mathcal{P}$ is axiomatizable then the same is clearly true for the class of reduced rings. This is the case, for example, if $\mathcal{P}$ says that the spectrum is normal. On the other hand, if $\mathcal{P}$ means that the spectrum is completely normal then neither the class of rings, nor the class of reduced rings is axiomatizable. But if the prime spectrum has only one point then the answers are different for all rings and for reduced rings: Everybody knows that the class of *reduced* rings with only one prime ideal is the class of fields, which is clearly an axiomatizable class. But the class of *all* rings with only one prime ideal is not axiomatizable (cf. 6.8 and 6.7).

In section 11 a table summarizes our axiomatizability results, as well as some well-known classical answers to the type of question we study. Most of the answers that we present are new. Our answers are based upon a few key results and constructions. The first one is Theorem 4.3, which shows that the rings with normal prime spectrum form an axiomatizable class. (This, in fact, is not a new result, cf. [Co1], Theorem 4.1. We still include an extensive discussion of rings with normal spectrum. The proofs seem to be new, the results, as well as their presentation, are more comprehensive and play a key role later on the paper. More comments on the literature are given in sections 4 and 5.) Without much additional effort this leads to the fact that the classes of rings whose space of maximal ideals is Hausdorff, or is Boolean, or is a pro-constructible subspace of the full prime spectrum are all axiomatizable as well.

In section 3 we introduce the notion of *pseudo elementary classes of structures*, which is a more general notion than axiomatizability. If a class of structures is known to be pseudo elementary then it is possible to prove or disprove axiomatizability via judiciously chosen numerical invariants. We shall apply the technique several times. The basic procedure is always the same: First we associate subsets of $\mathbb{N}$ with elements of the ring. Then, for each element, we form the infimum of this set in $\mathbb{N}\cup\{\omega,\infty\}$, where $\mathbb{N} < \omega < \infty$. Thus, we have a numerical invariant for each element of the ring, which is either in $\mathbb{N}$ or is ∞. Finally we associate a number in $\mathbb{N}\cup\{\omega,\infty\}$ with the ring by forming the supremum of the set of invariants of the ring elements. Then the pseudo elementary class is axiomatizable if and only if the invariants of the rings of the class have a uniform upper bound in $\mathbb{N}$ (cf. 3.2).

We use this method to show that the class of rings with completely normal prime spectrum is not axiomatizable. In this case we denote the numerical invariant of the ring A by $\mathrm{CN}(A)$. In 6.7 we show that for every axiomatizable class $\mathcal{R}$ of rings with completely normal spectrum, there is an upper bound in $\mathbb{N}$ for all the $\mathrm{CN}(A)$, $A \in \mathcal{R}$. Then we construct a sequence of rings $(A_n)_{n\in\mathbb{N}}$ such that $\operatorname{Spec} A_n$ is a singleton and the $\mathrm{CN}(A_n)$ are an unbounded sequence of integers. Consequently, no axiomatizable class of rings with completely normal prime spectrum contains all the A_n. This also gives non-axiomatizability of the rings with only one prime ideal, or of the rings with boolean spectrum, or of the rings with linearly ordered spectrum.

The rings $(A_n)_{n\in\mathbb{N}}$ are not reduced. But we use them to construct, in 6.11, a sequence $(B_n)_{n\in\mathbb{N}}$ of domains with exactly two prime ideals such that $(\mathrm{CN}(B_n))_{n\in\mathbb{N}}$ is an unbounded sequence of integers. This then proves that the class of reduced rings with completely normal spectrum is non-axiomatizable as well.

These explanations account for many entries in the table of section 11. In section 7, we show that the class of all rings with *inversely normal spectrum* is not axiomatizable, whereas the class of all reduced rings with inversely normal spectrum is axiomatizable. (Recall that $\operatorname{Spec} A$ is *inversely normal* if every prime ideal of A contains a unique minimal prime ideal.)

The most difficult issue that remains is the question of compactness of the minimal prime spectrum. The model theoretic method for proving non-axiomatizability is the same as before: We associate a numerical invariant $\mathrm{AS}(A) \in \mathbb{N}\cup\{\omega,\infty\}$ with every ring A (cf. 10.1) as follows: For $a \in A$ we define the **annihilator size** $\mathrm{AS}(a)$ of a as the infimum (formed in $\mathbb{N}\cup\{\omega,\infty\}$) of the set

$$\{k \in \mathbb{N} \mid \exists b_1,\dots,b_k \in \mathrm{Ann}(a) : \mathrm{Ann}(a, b_1,\dots,b_k) = (0)\}$$

Then we define $\mathrm{AS}(A) := \sup\{\mathrm{AS}(a) \in \mathbb{N}\cup\{\omega\}\cup\{\infty\} \mid a \in A\}$, hence $\mathrm{AS}(A) = \omega$ if and only if $\{AS(a) \mid a \in A\}$ is an unbounded subset of $\mathbb{N}$. It turns out that (cf. 10.2)

- $\operatorname{Spec} A$ has compact minimal spectrum if and only if $\mathrm{AS}(A) \le \omega$.
- Every axiomatizable class $\mathcal{R}$ of rings with compact minimal spectrum must have a common upper bound in $\mathbb{N}$ for all the invariants $\mathrm{AS}(A)$, $A \in \mathcal{R}$.

In section 10 we modify and extend a construction due to Quentel to produce a ring A with $\mathrm{AS}(A) = \omega$ (cf. 10.16). It follows that A has compact minimal prime spectrum, but there is no axiomatizable class of rings with compact minimal prime spectrum that contains the ring A. There is an ultrapower of A whose minimal prime spectrum is not compact.

2. Preliminaries on spectral spaces

In this section we set up notation and terminology for spectra and present some results that will be used throughout. The theory of spectral spaces was started by Hochster with his paper [Hoc]. Section 2 of [Tr] is a convenient place to look up more basic notions and facts.

Notation 2.1. Let X be a topological space. If $x, y \in X$ we write $x \rightsquigarrow y$ if $y \in \overline{\{x\}}$ and we say y is a **specialization** of x or x is a **generalization** of y. Moreover we define

$$\begin{array}{rcl}
\overset{\circ}{\mathcal{K}}(X) & := & \{U \subseteq X \mid U \text{ is quasi-compact and open}\} \\
\overline{\mathcal{K}}(X) & := & \{X \setminus U \mid U \in \overset{\circ}{\mathcal{K}}(X)\} \\
\mathcal{K}(X) & := & \text{the Boolean algebra of subsets of } X \text{ generated by } \overset{\circ}{\mathcal{K}}(X); \\
 & & \text{the elements of } \mathcal{K}(X) \text{ are called } \textbf{constructible} \\
X^{\min} & := & \{x \in X \mid x \text{ does not have a proper generalization}\} \\
X^{\max} & := & \{x \in X \mid x \text{ does not have a proper specialization}\} \\
X_{\mathrm{con}} & := & \text{the set } X \text{ equipped with the constructible topology,} \\
 & & \text{which, by definition, has } \mathcal{K}(X) \text{ as a basis;} \\
 & & \text{the closed subsets of } X_{\mathrm{con}} \text{ are called } \textbf{proconstructible} \\
\overline{Y}^{\mathrm{con}} & := & \text{the closure of a subset } Y \subseteq X \text{ in the constructible topology} \\
X_{\mathrm{inv}} & := & \text{the set } X \text{ equipped with the } \textbf{inverse} \text{ topology,} \\
 & & \text{which, by definition, has } \overset{\circ}{\mathcal{K}}(X) \text{ as a basis of } \textit{closed} \text{ sets}
\end{array}$$

We emphasize that, for a subset $Y \subseteq X$, the set $Y^{\max}$ is the set of maximal points of the subspace Y of X. In general $Y^{\max}$ is different from $X^{\max} \cap Y$. The same clarification applies to $Y^{\min}$.

For any subset $Y \subseteq X$, let

$$\mathrm{Gen}(Y) := \{x \in X \mid x \rightsquigarrow y \text{ for some } y \in Y\}$$

be the set of generalizations of Y in X; we refer to this set as the **generic closure** of Y.

For any ring A let $\operatorname{Spec} A$ be the prime spectrum of A. We use the standard notations $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\}$ $(S \subseteq A)$ and $V(a_1, ..., a_n) = V(\{a_1, ..., a_n\})$ $(a_1, ..., a_n \in A)$. Moreover, for each element $a \in A$ we define $D(a) = \{\mathfrak{p} \in \operatorname{Spec} A \mid a \notin \mathfrak{p}\} = \operatorname{Spec} A \setminus V(a)$. The sets $V(a)$ are the **principal closed subsets**, the sets $D(a)$ are the **principal open subsets** of $\operatorname{Spec} A$.

Remark 2.2. Let Y be a subset of an arbitrary topological space X.

(*i*) $\mathrm{Gen}(Y) = \bigcap\{U \subseteq X \mid U \text{ open and } Y \subseteq U\}$.
(*ii*) $\mathrm{Gen}(Y)$ is **generically closed**, i.e., closed under generalization.
(*iii*) $\mathrm{Gen}(Y)^{\max} = Y^{\max}$.
(*iv*) $\mathrm{Gen}(Y) \supseteq \mathrm{Gen}(Y^{\max})$.
(*v*) $\mathrm{Gen}(Y) = \mathrm{Gen}(Y^{\max})$ if and only if $Y \subseteq \mathrm{Gen}(Y^{\max})$.
(*vi*) If Y is T_0 then the following are equivalent:
  (a) Y is quasi-compact.
  (b) $\mathrm{Gen}(Y)$ is quasi-compact.

(c) $Y^{\max}$ is quasi-compact and $Y \subseteq \mathrm{Gen}(Y^{\max})$.

In particular, every point in a quasi-compact T_0-space specializes to a maximal point in that space.

Proof. (i)-(v) are obvious. We give the proof of (vi). (a) and (b) are equivalent by (i). (c)⇒(a). First note that $Y \subseteq \mathrm{Gen}(Y^{\max})$ means $\mathrm{Gen}(Y) = \mathrm{Gen}(Y^{\max})$ (by (v)). Then we apply the implication "(a)⇒(b)" to the quasi-compact set $Y^{\max}$.

(a)⇒(c). Let $y \in Y$. We show $y \in \mathrm{Gen}(Y^{\max})$. By Zorn there is a maximal chain $Z \subseteq Y$ in the set of specializations of y. The intersection of finitely many sets of the form $\overline{\{z\}} \cap Y$, with $z \in Z$, is non empty. Since Y is quasi-compact the intersection of *all* these sets is nonempty, hence contains a point z_0. This is a maximal point of Z (as Y is T_0). However, Z is a maximal specialization chain in Y, thus $z_0 \in Y^{\max}$. This shows $Y \subseteq \mathrm{Gen}(Y^{\max})$, in other words: $\mathrm{Gen}(Y) = \mathrm{Gen}(Y^{\max})$, and, using the equivalence "(a)⇔(b)", we conclude that $Y^{\max}$ is quasi-compact. □

Recall from [Hoc] that a topological space X is called **spectral** if X is quasi-compact, T_0, $\mathring{\mathcal{K}}(X)$ is a basis of the topology and is closed under finite intersections, and each closed irreducible subset A of X has a (unique) generic point $x \in A$, i.e., $\overline{\{x\}} = A$. A map between spectral spaces is called a **spectral map** if preimages of quasi-compact open sets are quasi-compact open.

We mention that a subset Y of X is proconstructible if and only if Y is a **spectral subspace** of X, i.e., Y together with the topology inherited from X is spectral and the inclusion is a spectral map.

Proposition 2.3. *Let X be a spectral space and let $Y \subseteq X$. The following are equivalent.*

(i) *Y is quasi-compact.*
(ii) *$\mathrm{Gen}(Y)$ is quasi-compact.*
(iii) *$Y^{\max}$ is quasi-compact and $Y \subseteq \mathrm{Gen}(Y^{\max})$.*
(iv) *$\mathrm{Gen}(Y)$ is proconstructible.*
(v) *$\mathrm{Gen}(Y) = \bigcap\{U \in \mathring{\mathcal{K}}(X) \mid Y \subseteq U\}$.*

Proof. The implications (v)⇒(iv)⇒(iii) are trivial; items (i), (ii), (iii) are equivalent in every T_0-space by 2.2(vi). Hence it remains to show (i)⇒(v). Let Y be quasi-compact. Clearly $\mathrm{Gen}(Y) \subseteq \bigcap\{U \in \mathring{\mathcal{K}}(X) \mid Y \subseteq U\}$. Conversely, pick $x \in X \setminus \mathrm{Gen}(Y)$. For each $y \in Y$ we have $x \not\rightsquigarrow y$. Since $\mathring{\mathcal{K}}(X)$ is a basis of the topology of X, there is some $U_y \in \mathring{\mathcal{K}}(X)$ with $x \notin U_y \ni y$. Therefore Y is covered by all the U_y and since Y is quasi-compact there is some $U \in \mathring{\mathcal{K}}(X)$ with $x \notin U \supseteq Y$. □

Applying 2.3 to X and X_{inv} gives the following consequences, which can also be found in [Tr], Cor. (2.7) as a consequence of the so-called separation lemma ([Tr], Thm. (2.6)).

Corollary 2.4. *Let X be a spectral space and let $Y, Z \subseteq X$. Then*

(i) *Y is quasi-compact in the inverse topology if and only if $\overline{Y} = \bigcup_{y \in Y} \overline{\{y\}}$.*
(ii) *If Y is closed, Z is quasi-compact and disjoint from Y, then there is a closed, constructible subset A of X with $Y \subseteq A$ and $A \cap Z = \emptyset$.*

(iii) *If Y and Z are quasi-compact in the inverse topology and if there are no points $y \in Y$, $z \in Z$ that have a common specialization in X, then there are closed and constructible subsets A, B of X with $Y \subseteq A$, $Z \subseteq B$ and $A \cap B = \emptyset$.*

(iv) *If Y and Z are quasi-compact and if there are no points $y \in Y$, $z \in Z$ which have a common generalization in X, then there are open quasi-compact subsets U, V of X with $Y \subseteq U$, $Z \subseteq V$ and $U \cap V = \emptyset$.*

If X is any topological space and $Y \subseteq X$, then $\operatorname{int}(Y)$ denotes the interior of Y.

Lemma 2.5. *Let X be a spectral space and let $x \in X$. The following are equivalent.*

(i) $x \in X^{\min}$.

(ii) *If $V \in \overline{\mathcal{K}}(X)$ with $x \in V$, then $x \in \operatorname{int}(V)$.*

(iii) *If $Y \subseteq X$ is open in the constructible topology with $x \in Y$, then $x \in \operatorname{int}(Y)$ (w.r.t. the spectral topology).*

Proof. (iii)⇒(ii)⇒(i) are obvious. We prove (i)⇒(iii). Let the subset $Y \subseteq X$ be open in the constructible topology and let $x \in Y$. If $x \notin \operatorname{int}(Y)$, then for all $U \in \mathring{\mathcal{K}}(X)$ with $x \in U$ we have $U \cap (X \setminus Y) \neq \emptyset$. Since X_{con} is compact we get an element $y \in \bigcap_{x \in U \in \mathring{\mathcal{K}}(X)} U \cap (X \setminus Y)$. This shows that y is a proper generalization of x, a contradiction to (i). □

Corollary 2.6. *Let X be a spectral space.*

(i) *If $Y \subseteq X$ is open in the constructible topology, then $Y \cap X^{\min} = \operatorname{int}(Y) \cap X^{\min}$. In particular, the topologies induced by X and X_{con} on $X^{\min}$ are the same.*

(ii) *If $Y \subseteq X$ is proconstructible, then $X^{\min} \subseteq Y$ iff Y is dense in X.*

Proof. (i) holds by 2.5(iii).

(ii). Obviously we have ⇒. Conversely if Y is dense in X, then $X \setminus Y$ has empty interior, so for each $x \in X^{\min}$ we have $x \notin X \setminus Y$, by 2.5(i)⇒(iii). □

It is a consequence of 2.6 that, in a spectral space X, the subspace of minimal points is always a Tychonoff space (i.e., a completely regular space, or , equivalently, a subspace of a compact Hausdorff space). This is so, since by 2.6, $X^{\min}$ is a subspace of X_{con}. In particular, if a spectral space does not have any proper specializations, then $X^{\min} = X_{\text{con}}$, and the space is boolean.

Corollary 2.7. *Let X be a spectral space. Then $X^{\min}$ is quasi-compact (hence compact) if and only if $X^{\min}$ is proconstructible, if and only if*

$$X^{\min} = \bigcap \{U \in \mathring{\mathcal{K}}(X) \mid U \text{ is dense in } X\}.$$

Proof. By 2.6, we know that for every $U \in \mathring{\mathcal{K}}(X)$, $X^{\min} \subseteq U$ iff U is dense in X. Therefore the corollary follows from 2.3, (i)⇔(iv)⇔(v). □

Lemma 2.8. *Let Y be a subset of a spectral space X. Then the minimal points of the closure of Y are the same as the minimal points of the constructible closure of Y. In particular, if A is a ring, $X = \operatorname{Spec} A$ and Y is a set of prime ideals, then the minimal points of $V(I)$, $I := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$, are contained in the constructible closure of Y (note that $\overline{Y}$ is $V(I)$).*

Proof. The constructible topology is finer than the spectral topology. Therefore $\overline{Y}^{\text{con}} \subseteq \overline{Y}$. Using 2.4(i) we conclude that

$$\bigcup_{y\in\overline{Y}^{\min}} \overline{\{y\}} = \overline{Y} = \overline{\overline{Y}^{\text{con}}} = \bigcup_{y\in\overline{Y}^{\text{con}}} \overline{\{y\}} = \bigcup_{y\in\left(\overline{Y}^{\text{con}}\right)^{\min}} \overline{\{y\}}.$$

It follows that $\overline{Y}^{\min} = \left(\overline{Y}^{\text{con}}\right)^{\min}$. □

3. Pseudo elementary classes

We shall use basic notions from model theory (cf. [Ho]).

Definition 3.1. Let $\mathscr{L}$ be a first order language and let $\mathcal{C}$ be a class of $\mathscr{L}$-structures. Let $Th(\mathcal{C})$ be the theory of $\mathcal{C}$, i.e., $Th(\mathcal{C})$ is the set of all $\mathscr{L}$-sentences, that are valid in all structures from $\mathcal{C}$. We call $\mathcal{C}$ **pseudo elementary** if there is an index set I and $\mathscr{L}$-formulas $\varphi_{i,k}(x_1, ..., x_{n(i)}, y_1, ..., y_{l(i,k)})$ $(i \in I,\ k \in \mathbb{N})$ with, at most, the free variables $x_1, ..., x_{n(i)}, y_1, ..., y_{l(i,k)}$, such that for every model M of $Th(\mathcal{C})$ we have $M \in \mathcal{C}$ if and only if for each $i \in I$ and all $\bar{a} \in M^{n(i)}$ there are $k \in \mathbb{N}$ and some $\bar{b} \in M^{l(i,k)}$ such that $M \models \varphi_{i,k}(\bar{a}, \bar{b})$.

The formulas $\varphi_{i,k}$ $(i \in I, k \in \mathbb{N})$ are called **witnesses** of $\mathcal{C}$.

We shall write $\bar{x}_i, \bar{y}_{i,k}$ for the tuples $(x_1, ..., x_{n(i)})$, $(y_1, ..., y_{l(i,k)})$, respectively. If I is a singleton, we suppress the subscript i.

For example, the class of finite $\mathscr{L}$-structures is pseudo elementary, where I is a singleton and the witnesses

$$\varphi_k = \exists v_1, ..., v_k\ \forall u\ u = v_1 \vee ... \vee u = v_k,$$

have no free variables.

Proposition 3.2. *Let $\mathcal{C}$ be a pseudo elementary class of $\mathscr{L}$-structures with witnesses $\varphi_{i,k}(\bar{x}_i, \bar{y}_{i,k})$ $(i \in I, k \in \mathbb{N})$. The following are equivalent:*

- (i) *$\mathcal{C}$ is axiomatizable, in other words every model of $Th(\mathcal{C})$ is in $\mathcal{C}$.*
- (ii) *$\mathcal{C}$ is closed under (countable) ultraproducts.*
- (iii) *For every $i \in I$, there is a natural number K such that for every $M \in \mathcal{C}$ and every $\bar{a} \in M^{n(i)}$ there are some $k \leq K$ and some $\bar{b} \in M^{l(i,k)}$ with $M \models \varphi_{i,k}(\bar{a}, \bar{b})$.*

Proof. This holds by basic model theory; for the convenience of the reader we include a proof:
(iii)⇒(i). Condition (iii) says that the sentences

$$\forall \bar{x}_i \exists \bar{y}_{i,1}, ..., \bar{y}_{i,K}\ \varphi_{i,1}(\bar{x}_i, \bar{y}_{i,1}) \vee ... \vee \varphi_{i,K}(\bar{x}_i, \bar{y}_{i,K})\ (i \in I)$$

are in $Th(\mathcal{C})$. As $\mathcal{C}$ is pseudo elementary, it follows that every model of $Th(\mathcal{C})$ is in $\mathcal{C}$.

(i)⇒(ii) holds by Corollary 9.5.10 of [Ho]. It remains to show (ii)⇒(iii). Fix $i \in I$ and suppose there is no bound K as in (iii). For each $K \in \mathbb{N}$ pick some $M_K \in \mathcal{C}$ and $\bar{a}(K) \in M_K^{n(i)}$ such that

$$M_K \models \forall \bar{y}_{i,k} \neg\varphi_{i,k}(\bar{a}(K), \bar{y}_{i,k})\ (1 \leq k \leq K).$$

Let $\mathscr{M} = \prod_K M_K/\mathscr{U}$, where $\mathscr{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Let $\bar{a} := (\bar{a}(K))/\mathscr{U} \in \mathscr{M}^{n(i)}$. By (ii), $\mathscr{M}$ is in $\mathcal{C}$, hence there are $k \in \mathbb{N}$ and some $\bar{b} =$

$(\bar{b}(K))/\mathscr{U} \in \mathscr{M}^{l(i,k)}$ such that $\mathscr{M} \models \varphi_{i,k}(\bar{a}, \bar{b})$ (where each $\bar{b}(K)$ is a tuple from $M_K^{l(i,k)}$). Since $\mathscr{U}$ is nonprincipal there is some $K \geq k$ such that

$$M_K \models \varphi_{i,k}(\bar{a}(K), \bar{b}(K)),$$

which contradicts the choice of M_K and $\bar{a}(K)$. □

Observe that (ii) does not imply (i) in Proposition 3.2, without the assumption that $\mathcal{C}$ is pseudo elementary; e.g. if $\mathcal{C}$ is the class of all uncountable structures in a countable language, then (ii) holds, but not (i).

Note that every elementary class $\mathcal{C}$ is also pseudo elementary. Every sequence $\varphi_k = \varphi_k(\bar{x}, \bar{y}_k)$ of formulas ($\bar{x}$ of length n, $\bar{y}_k$ of length $l(k)$, as above) trivially serves as a sequence of witnesses if it satisfies the following condition:

(∗) For each $M \in \mathcal{C}$ and for all $\bar{a} \in M^n$ there are $k \in \mathbb{N}$ and a tuple $\bar{b} \in M^{l(k)}$ such that $M \models \varphi_k(\bar{a}, \bar{b})$.

We shall use the following consequence of 3.2:

Corollary 3.3. *Let $\mathcal{C}$ be an elementary class of $\mathscr{L}$-structures and let $(\varphi_k(\bar{x}, \bar{y}_k))_{k\in\mathbb{N}}$ be a sequence of $\mathscr{L}$-formulas that satisfies condition (∗). Then there is some $K \in \mathbb{N}$ such that for every $M \in \mathcal{C}$ and every $\bar{a} \in M^n$ there are some $k \leq K$ and $\bar{b} \in M^{l(k)}$ with $M \models \varphi_k(\bar{a}, \bar{b})$.* □

The example of finite structures above shows that the existence of bounds K as in 3.2(iii) that depend on the selected structure, but are independent from the choice of tuples $\bar{a} \in M^{\bar{x}}$, does not imply that $\mathcal{C}$ is axiomatizable. The following proposition characterizes those situations where a bound exists for a particular structure from $\mathcal{C}$.

Proposition 3.4. *Let $\mathcal{C}$ be a pseudo elementary class of $\mathscr{L}$-structures with witnesses $\varphi_{i,k}(\bar{x}_i, \bar{y}_{i,k})$ ($i \in I$, $k \in \mathbb{N}$). The following are equivalent for every $\mathscr{L}$-structure M.*

(i) *$M \in \mathcal{C}$, and for each $i \in I$ there is some $K \in \mathbb{N}$ such that for all $\bar{a} \in M^{n(i)}$ there are $k \leq K$ and some $\bar{b} \in M^{l(i,k)}$ with $M \models \varphi_{i,k}(\bar{a}, \bar{b})$.*

(ii) *Every (countable) ultrapower of M is in $\mathcal{C}$.*

Proof. (i)⇒(ii). If (i) holds, then in M the sentences

$$\forall \bar{x}_i \exists \bar{y}_{i,1}, ..., \bar{y}_{i,K} \bigvee_{k=1}^{K} \varphi_{i,k}(\bar{x}_i, \bar{y}_{i,k})$$

hold true. By the theorem of Łos (cf. [Ho], Theorem 9.5.1), this sentence also holds in every ultrapower $M^{\mathscr{U}}$ of M, which implies that $M^{\mathscr{U}} \in \mathcal{C}$, as the $\varphi_{i,k}(\bar{x}_i, \bar{y}_{i,k})$ are witnesses of $\mathcal{C}$.

(ii)⇒(i) holds by the same proof as 3.2(ii)⇒(iii), where each M_K is equal to M. □

4. Axiomatizing rings with normal spectrum

Recall that a topological space X is called **normal** if for all disjoint closed subsets Y, Z of X, there are disjoint open subsets U, V of X with $Y \subseteq U$ and $Z \subseteq V$. If X is a spectral space, then X is normal if and only if every point in X has a unique specialization in $X^{\max}$. Equivalently, for every $y \in X^{\max}$, $\operatorname{Gen} y$ is closed. All this is well known (cf. [Ca-Co], Proposition 2) and follows quickly from 2.4. Also recall that closed subspaces of normal spaces are normal again and that the set of maximal points of a normal spectral space is Hausdorff.

Rings with normal Zariski spectrum are called **Gel'fand rings** (cf. [Joh] p.199) and have been studied by several authors, e.g., [Ca], [Ca-Co], [Co1], [Co2] and [dM-Or].

Lemma 4.1. *Let X be a spectral space and let $Y \subseteq X$ such that for all $x \in X$, $y_1, y_2 \in Y$ with $x \rightsquigarrow y_1, y_2$ we have $y_1 = y_2$. Then*

(i) *For all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ there are $U_1, U_2 \in \mathring{\mathcal{K}}(X)$ with $y_i \in U_i$ and $U_1 \cap U_2 = \emptyset$ - in particular Y is Hausdorff.*

(ii) *The map $r : \mathrm{Gen}(Y) \longrightarrow Y$ that sends z to the unique $y \in Y$ with $z \rightsquigarrow y$ is a closed map.*

(iii) *If Y is quasi-compact, then $\mathrm{Gen}(Y)$ is a spectral subspace, $Y = \mathrm{Gen}(Y)^{\max}$ and r is continuous (cf.* [Ca-Co]*, Proposition 3)*

Proof. Item (i) holds by 2.4(iv).

(ii) r is closed, since for a closed subset A of X, $r(\mathrm{Gen}(Y) \cap A) = A \cap Y$, which is closed in Y.

(iii) If Y is quasi-compact, then by 2.3, $\mathrm{Gen}(Y)$ is a spectral subspace and $Y = \mathrm{Gen}(Y)^{\max}$. Thus, in order to prove that r is continuous we may assume that $Y = X^{\max}$ and $\mathrm{Gen}(Y) = X$. We show that r is continuous: If $A \subseteq X^{\max}$ is closed, then A is quasi-compact, hence $r^{-1}(A) = \mathrm{Gen}(A)$ is proconstructible by 2.3. The assumption implies that $\mathrm{Gen}(A)$ is closed under specialization, hence $r^{-1}(A)$ is closed by 2.4(i). □

By Hochster's Theorem ([Hoc]), every spectral space is homeomorphic to $\mathrm{Spec}\, A$ for some ring A. The ring of course imposes a lot of additional structure on X. A simple, but crucial, separating property in terms of the principal open sets $D(f)$, $f \in A$ is the following.

Lemma 4.2. *Let A be a ring. If $V \subseteq \mathrm{Spec}\, A$ is closed and $U \subseteq \mathrm{Spec}\, A$ is open with $V \subseteq U$, then there are $f, g \in A$ with $V \subseteq V(f) \subseteq D(g) \subseteq U$.*

Proof. Let I, J be ideals of A with $V = V(I)$ and $\mathrm{Spec}\, A \setminus U = V(J)$. Since $V \subseteq U$ we have $V(I + J) = V(I) \cap V(J) = \emptyset$, in other words $1 \in I + J$. Take $f \in I$, $g \in J$ with $1 = f + g$. Then $V \subseteq V(f)$, $V(J) \subseteq V(g)$ and $V(f) \cap V(g) = \emptyset$, which gives the assertion. □

In the next theorem we extend the list of characterizations of Gel'fand rings given in [Ca-Co], Proposition 3 and [Joh], p. 199. The equivalence of conditions (i) and (iv) is Contessa's Theorem 4.1, [Co1]. The implication (i)⇒(iii) is essentially Lemma 3.1 of [Co1].

Theorem 4.3. *Let A be a ring. The following are equivalent:*

(i) *A is a Gel'fand ring.*

(ii) *If $V, ..., V_n \subseteq \mathrm{Spec}\, A$ are closed with $V_1 \cap ... \cap V_n = \emptyset$, then there are $c_1, ..., c_n \in A$ with $V_i \subseteq D(c_i)$ and $\overline{D(c_1)} \cap ... \cap \overline{D(c_n)} = \emptyset$.*

(iii) *for all $a, b \in A$ with $V(a) \cap V(b) = \emptyset$ there are $c, d \in A$ with $V(a) \subseteq D(c)$, $V(b) \subseteq D(d)$ such that $D(c) \cap D(d) = \emptyset$.*

(iv) *for all* $a,b \in A$ *with* $1 \in (a,b)$ *there are* $c,d \in A$ *with* $1 \in (a,c)$, $1 \in (b,d)$ *such that* $c\cdot d = 0$.

(v) $A \models \forall a\ \exists x,x'\ (1-xa)\cdot(1-x'(1-a)) = 0$.

Hence the class of rings with normal spectrum is axiomatizable. Normality of Spec A *can be characterized by a strict Horn formula (cf.* [Ho], *section 9.1) in the language of rings.*

Proof. (i)⇒(ii). We first show that there is some $c_1 \in A$ with $V_1 \subseteq D(c_1)$ and $\overline{D(c_1)} \cap W = \emptyset$, where $W = V_2 \cap ... \cap V_n$. Since Spec A is normal and V_1, W are disjoint and closed we can apply 2.4(iv) to find open and disjoint sets $O \supseteq V$, $U \supseteq W$. By 4.2 there is some $c_1 \in A$ with $V_1 \subseteq D(c_1) \subseteq O$. Then $\overline{D(c_1)} \subseteq \overline{O} \subseteq \operatorname{Spec} A \setminus U$, and $\overline{D(c_1)} \cap W = \emptyset$.

Applying this argument again to V_2 and $\overline{D(c_1)} \cap V_3 \cap ... \cap V_n$ gives $c_2 \in A$ with $V_2 \subseteq D(c_2)$ and $\overline{D(c_1)} \cap \overline{D(c_2)} \cap V_3 \cap ... \cap V_n = \emptyset$. Continuing in this way we get the elements $c_1, ..., c_n$ as desired.

(ii)⇒(iii) is a weakening.

(iii)⇒(iv). Let $a,b \in A$ with $1 \in (a,b)$. Then $V(a) \cap V(b) = \emptyset$. Hence by (iii) there are $c,d \in A$ with $V(a) \subseteq D(c)$, $V(b) \subseteq D(d)$ such that $D(c) \cap D(d) = \emptyset$. Now $D(c) \cap D(d) = \emptyset$ says $c^k \cdot d^k = 0$ for some $k \in \mathbb{N}$. Note that $V(a) \subseteq D(c) = D(c^k)$ implies $1 \in (a, c^k)$. Similarly one proves $1 \in (b, d^k)$. The elements c^k, d^k have the properties required in (iv).

(iv)⇒(v). By (iv), there are $c,d \in A$ with $1 \in (a,c)$, $1 \in (1-a,d)$ and $c\cdot d = 0$. Pick $x,x',y,y' \in A$ with $1 = xa + yc$, $1 = x'(1-a) + y'd$. Then

$$(1-xa)\cdot(1-x'(1-a)) = ycy'd = 0.$$

(v)⇒(i). Let $\mathfrak{m}, \mathfrak{n}$ be distinct maximal ideals of A. Take $a \in \mathfrak{m}$, $b \in \mathfrak{n}$ with $1 = a + b$. By (v) there are $x, x' \in A$ with $(1-xa)\cdot(1-x'(1-a)) = 0$. Hence a common generalization $\mathfrak{p}$ of $\mathfrak{m}$ and $\mathfrak{n}$ will contain $1-xa$ or $1-x'(1-a)$, say $1-xa \in \mathfrak{p}$. Then $a, 1-xa \in \mathfrak{m}$, so $1 \in \mathfrak{m}$, a contradiction. This shows that distinct maximal ideals of A do not have a common generalization, which proves (i). □

The Theorem says that the class of all rings with normal spectrum is axiomatizable. We shall apply this result to the factor rings $A/\operatorname{Jac} A$, where Jac A is the Jacobson radical. The class of rings with normal Spec($A/\operatorname{Jac} A$) is axiomatizable as well. The argument we use is a special instance of the “interpretation method”, which is explained in [Ho], section 5. We sketch the method since it will appear several times later on.

To start with, recall that Jac A is the intersection of the maximal ideals of A and

$$\operatorname{Jac} A = \{a \in A \mid \forall x\ \exists u\ 1 = u\cdot(1+xa)\}.$$

Hence Jac A is the subset of A defined by the formula

$$\iota(v) := \forall x\ \exists u\ 1 = u\cdot(1+xv).$$

In this sense $A/\operatorname{Jac} A$ is a “definable residue ring” of A.

Proposition 4.4. *If* $\mathcal{C}$ *is an axiomatizable class of rings, then the class* $\mathcal{D}$ *of all rings* A *with* $A/\operatorname{Jac} A \in \mathcal{C}$ *is axiomatizable, too. Moreover, any explicitly given set of axioms of* $\mathcal{C}$ *can be explicitly translated into a set of axioms for* $\mathcal{D}$.

Proof. We give an outline the proof: Let T be the theory of $\mathcal{C}$. For each quantifier free ring-formula $\varphi(v_1, ..., v_n)$, let φ_{Jac} be the ring formula obtained from φ by replacing a term equality $t(\bar{v}) = 0$ with $\iota(t(\bar{v}))$. Then for each n-tuple $\bar{a} \in A^n$ we certainly have

$$(*) \qquad A/\operatorname{Jac} A \models \varphi(\bar{a} \mod \operatorname{Jac} A) \iff A \models \varphi_{\mathrm{Jac}}(\bar{a}).$$

By induction on the number of quantifiers, we extend the assignment $\varphi \mapsto \varphi_{\mathrm{Jac}}$ to all ring-formulas. It is straightforward to check that $(*)$ remains true for all formulas. This proves the proposition, since now we know that $\{\varphi_{\mathrm{Jac}} \mid \varphi \in T\}$ axiomatizes the class of rings A with $A/\operatorname{Jac} A \in \mathcal{C}$. □

We give an application of 4.4 using 4.3. First recall from [Ca-Co], p.230 for every spectral space X: If $X^{\max}$ is Hausdorff and dense in X, then X is normal (If $x, y \in X^{\max}$ are distinct points, then take $U, V \in \mathring{\mathcal{K}}(X)$, $x \in U, y \in V$ and $U \cap V \cap X^{\max} = \emptyset$. The density of $X^{\max}$ implies $U \cap V = \emptyset$, in particular x, y do not have a common generalization in X.)

Corollary 4.5. *Let A be a ring. We set $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec}(A/\operatorname{Jac} A)$. Then $X^{\max}$ is a Hausdorff space if and only if Y is normal. Hence, by* 4.4 *and* 4.3*, the class of rings A with* $\operatorname{Spec} A$ *Hausdorff is first order axiomatizable.*

Proof. We identify Y canonically with a closed subspace of X. Note that $X^{\max} = Y^{\max}$. By the above remark, $Y^{\max}$ is Hausdorff if and only if Y is normal. □

The class of rings such that the maximal points form a proconstructible subset of the spectrum is axiomatizable as well:

Corollary 4.6. *Let A be a ring. Then* $(\operatorname{Spec} A)^{\max}$ *is proconstructible if and only if $A/\operatorname{Jac} A$ has boolean spectrum. Since $A/\operatorname{Jac} A$ has boolean spectrum if and only if it is von Neumann regular, the property "*$(\operatorname{Spec} A)^{\max}$ *is proconstructible" defines an axiomatizable class of rings.*

Proof. Since $\operatorname{Spec} A/\operatorname{Jac} A$ is a proconstructible subset of $\operatorname{Spec} A$, $(\operatorname{Spec} A)^{\max}$ is proconstructible if $A/\operatorname{Jac} A$ has boolean spectrum.

Conversely, if $(\operatorname{Spec} A)^{\max}$ is proconstructible, then by 2.8, $V(\operatorname{Jac} A)^{\min}$ is contained in $(\operatorname{Spec} A)^{\max}$, which shows that $A/\operatorname{Jac} A$ has boolean spectrum. □

Remark 4.7. In [Sch-Tr] we give an elementary description of the property

"$(\operatorname{Spec} A)^{\max}$ is boolean",

namely $(\operatorname{Spec} A)^{\max}$ is boolean if and only if in the ring $A/\operatorname{Jac} A$ is an **exchange ring**, i.e., every element is a sum of a unit and an idempotent (cf. [Joh], p. 187; another name appearing in the literature is **clean ring**).

5. Partition of unity and local characterization of normality

For any subset S of a ring A let

$$D(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\}.$$

Hence $D(S)$ is a generically closed subset of $\operatorname{Spec} A$, in fact $D(S) = \bigcap_{s \in S} D(s)$ is inversely closed (and thus proconstructible). Note also that not every inversely closed subset is of this form. For example $D(a) \cup D(b) = D(S)$ for some set S if and only if $D(a) \cup D(b) = D(c)$ for some $c \in A$.

Recall that for any multiplicatively closed subset S of A, the localization map $\iota_S : A \longrightarrow A_S$ induces an homeomorphism $\operatorname{Spec} A_S \longrightarrow D(S)$.

Theorem 5.1. *(Partition of unity in Gel'fand rings) Given a ring A, $\operatorname{Spec} A$ is normal if and only if A has partitions of unity, i.e., for every open cover $\operatorname{Spec} A = U_1 \cup ... \cup U_n$ there are $f_1, \dots, f_n \in A$ with $1 = f_1 + ... + f_n$ such that $\overline{D(f_i)} \subseteq U_i$ $(1 \leq i \leq n)$.*

Proof. First suppose A has partitions of unity. Take $a, b \in A$ with $V(a) \cap V(b) = \emptyset$. We show that there are open disjoint neighborhoods of $V(a)$ and $V(b)$ in $\operatorname{Spec} A$. From the characterization 4.3(iii) and 4.2 this proves normality of $\operatorname{Spec} A$. Since $D(a) \cup D(b) = \operatorname{Spec} A$ and A has partitions of unity, there are $f, g \in A$ with $f + g = 1$ such that $\overline{D(f)} \subseteq D(a)$ and $\overline{D(g)} \subseteq D(b)$. Then $V(a) \subseteq \operatorname{Spec} A \setminus \overline{D(f)}$, $V(b) \subseteq \operatorname{Spec} A \setminus \overline{D(g)}$ and $(\operatorname{Spec} A \setminus \overline{D(f)}) \cap (\operatorname{Spec} A \setminus \overline{D(g)}) = \emptyset$, since $\operatorname{Spec} A = D(f) \cup D(g) \subseteq \overline{D(f)} \cup \overline{D(g)}$. Thus $\operatorname{Spec} A$ is normal.

Conversely assume $\operatorname{Spec} A$ is normal. Take an open cover $\operatorname{Spec} A = U_1 \cup ... \cup U_n$. Let $V_i := \operatorname{Spec} A \setminus U_i$. Then $V_1 \cap ... \cap V_n = \emptyset$ and by 4.3(ii), there are $c_i \in A$ with $V_i \subseteq D(c_i)$ and $\overline{D(c_1)} \cap ... \cap \overline{D(c_n)} = \emptyset$. Let $I_i \subseteq A$ be an ideal with $V(I_i) = \overline{D(c_i)}$. Then $V(I_1) \cap ... \cap V(I_n) = \emptyset$, which means $1 \in I_1 + ... + I_n$. Pick $f_i \in I_i$ with $1 = f_1 + ... + f_n$. Then $V_i \subseteq D(c_i) \subseteq V(I_i) \subseteq V(f_i)$, thus V_i is in the interior of $V(f_i)$, in other words $\overline{D(f_i)} \subseteq U_i$ $(1 \leq i \leq n)$. □

Lemma 5.2. *If $S \subseteq A$ is multiplicatively closed, then ι_S is surjective if and only if $D(S)$ is closed.*

Proof. If ι_S is surjective, then the image of $\operatorname{Spec} \iota_S$ is $V(\operatorname{Ker} \iota_S)$, which is closed.

Conversely assume $D(S)$ is closed. Take $s \in S$. As $D(S) \subseteq D(s)$ and $D(S)$ is closed, 4.2 gives us some $a \in A$ with $D(S) \subseteq V(a) \subseteq D(s)$. Since $V(a) \subseteq D(s)$, there are $\alpha, \beta \in A$ with $\alpha a + \beta s = 1$. Since $D(S) \subseteq V(a)$ we have $\frac{a^k}{1} = 0$ in A_S for some k. Then $1 = (\alpha a + \beta s)^k = \alpha^k \cdot a^k + s \cdot c$ for some $c \in A$. Hence $\frac{sc}{1} = 1$ in A_S, which shows that $\frac{1}{s} \in A_S$ is in the image of ι_S, as desired. □

For any subset X of $\operatorname{Spec} A$ we write $S(X)$ for the multiplicatively closed set $\{s \in A \mid X \subseteq D(s)\}$. Observe that $S(X) = S(\operatorname{Gen} X)$.

Corollary 5.3. *If X is closed and generically closed, then $X = D(S(X))$. Hence by 5.2, the localization map $\iota_{S(X)} : A \to A_{S(X)}$ is surjective.*

Proof. Obviously $X \subseteq D(S(X))$. Conversely if $\mathfrak{p} \notin X$, then $X \cap \overline{\{\mathfrak{p}\}} = \emptyset$, since X is generically closed. Since X is closed we may apply 4.2 to $X \subseteq \operatorname{Spec} A \setminus \overline{\{\mathfrak{p}\}}$ and there is some $s \in A$ with $X \subseteq D(s)$ and $\mathfrak{p} \in V(s)$. This means $\mathfrak{p} \notin D(S(X))$. □

As a remark we give the following characterization of Gel'fand rings. Contessa proved the result in [Co2], Theorem 1.2. It is also related to [Joh], section 3.8, p. 199, Lemma. The proof is an easy application of our previous considerations.

Remark 5.4. The following are equivalent for every ring A.

(i) A is a Gel'fand ring.
(ii) For every maximal ideal $\mathfrak{m}$ of A, the localization map $A \longrightarrow A_{\mathfrak{m}}$ is surjective

(iii) For all mutually disjoint, quasi-compact subsets $K_1, ..., K_n$ of $(\operatorname{Spec} A)^{\max}$, the product of the localization maps

$$(\iota_1, ..., \iota_n) : A \longrightarrow \prod_{i=1}^{n} A_{S(K_i)}$$

is surjective.

Proof. Clearly (iii) implies (ii): take $n = 1$ and $K_1 = \{\mathfrak{m}\}$. If (ii) holds, then for every maximal ideal $\mathfrak{m}$ of $\operatorname{Spec} A$, $D(A \setminus \mathfrak{m})$ is closed by 5.2. But $D(A \setminus \mathfrak{m})$ is the set of generalizations of $\mathfrak{m}$ in $\operatorname{Spec} A$. This shows that $\operatorname{Spec} A$ is normal.

It remains to show that (i) implies (iii). Let $V_i := \operatorname{Gen}(K_i)$ and let $K := K_1 \cup ... \cup K_n$. Since $\operatorname{Spec} A$ is normal, all these sets are closed and generically closed, and $\operatorname{Gen} K$ is the disjoint union of the V_i. By 5.3 we know that $\iota_{S(K)}$ is surjective and it remains to show that the natural map

$$A_{S(K)} \longrightarrow \prod_{i=1}^{n} A_{S(K_i)}$$

is an isomorphism. Since ι_i is surjective, we know that $A_{S(K_i)} \cong A/I_i$, where I_i is the kernel of ι_i. Since the $V(I_i) = \operatorname{Gen}(K_i)$ are mutually disjoint, the assertion follows from the Chinese Remainder Theorem. □

Remark 5.4 can be used to show that for every Gel'fand ring the natural map between the boolean algebras of idempotents of A and $A/\operatorname{Jac} A$ is onto. This will be discussed (and proved) in greater generality in [Sch-Tr].

By 4.3(v), a ring A is a Gel'fand ring if and only if for every $a \in A$, the equation $(1 - Xa)\cdot(1 - Y(1 - a)) = 0$ has a solution (x, y) in $A \times A$. One may ask if there is an overring C of A which is Gel'fand (in other words: which has solutions for these equations) and which is in some sense minimal with this property.

In fact by successively adjoining solutions to A for the equations above in a universal way one can easily show the following: For every ring A, there is an overring N of A, N Gel'fand, such that whenever $\varphi : A \longrightarrow B$ is a ring homomorphism and B is Gel'fand, then there is a ring homomorphism $\psi : N \longrightarrow B$ extending φ. In general ψ will not be uniquely determined by φ and N. Moreover there are many overrings N with these properties and it is unlikely that there is a "Gel'fand hull" for every ring A.

Nevertheless there are canonical constructions that produce a Gel'fand extension for any ring. Below we exhibit such a construction. The question whether Gel'fand hull exists has also been studied in [Co2]. Contessa arrived at the same construction that we give below (cf. [Co2], Theorem 5.11 and Theorem 6.3), but again we present a different proof.

First recall that for every ring A, the natural map $A \longrightarrow B := \prod_{\mathfrak{m} \in (\operatorname{Spec} A)^{\max}} A_{\mathfrak{m}}$ is an embedding and B is a Gel'fand ring, since products of Gel'fand rings are again Gel'fand (cf. 4.3(v)). We construct a small Gel'fand ring C between A and B:

Example 5.5. Firstly, if A is a local ring, then for every function $\varphi : A \longrightarrow A$ which extends $A^\times \longrightarrow A^\times$, $a \mapsto a^{-1}$, we have $(1 - \varphi(a)\cdot a)\cdot(1 - \varphi(1 - a)\cdot(1 - a)) = 0$, since $a \in A^\times$ or $1 - a \in A^\times$ for every $a \in A$. In particular the function $\varphi_A : A \longrightarrow A$ defined by

$$\varphi_A(a) := \begin{cases} a^{-1} & \text{if } a \in A^\times \\ 0 & \text{if } a \notin A^\times \end{cases}$$

provides the solution $(\varphi_A(a), \varphi_A(1-a))$ of $(1-Xa)\cdot(1-Y(1-a)) = 0$. Observe that φ_A is a multiplicative map $A \longrightarrow A$ which extends $a \mapsto a^{-1}$.

Now let $(A_x \mid x \in X)$ be a family of local rings and let $B := \prod_{x\in X} A_x$. We define $\varphi : B \longrightarrow B$ by $\varphi((a_x)) := (\varphi_{A_x}(a_x))$ and we see that for every $b \in B$, $(\varphi(b), \varphi(1-b))$ solves $(1-Xb)\cdot(1-Y(1-b)) = 0$. Again φ is a multiplicative map $B \longrightarrow B$.

Let A be a subring of B and let C be the subring of B, generated by A and all the $\varphi(a)$ $(a \in A)$. We claim that C is closed under φ, in particular C is Gel'fand (since the restriction of φ to C provides solutions for all the equations $(1-Xc)\cdot(1-Y(1-c)) = 0$ where $c \in C$).

Proof. Since φ is multiplicative and $\varphi(1) = 1$, every element c of C is of the form

$$c = a_1\varphi(s_1) + ... + a_n\varphi(s_n) \text{ for some } a_i, s_i \in A.$$

For $\varepsilon \in \{0,1\}^n$ and $\varepsilon(i) = 1$ let $s_{\varepsilon,i} := \prod_{j=1,\varepsilon(j)=1,j\neq i}^n s_j$. We claim that

$$\varphi(c) = \sum_{\varepsilon\in\{0,1\}^n} d_\varepsilon,$$

where

$$d_\varepsilon := (\prod_{i=1,\varepsilon(i)=1}^n s_i^2\varphi(s_i))\cdot\varphi(\sum_{i=1,\varepsilon(i)=1}^n s_{\varepsilon,i}a_i)\cdot\prod_{i=1,\varepsilon(i)=0}^n (1 - s_i\varphi(s_i)) \in C.$$

To see this, fix $x \in X$. Let $\varepsilon \in \{0,1\}^n$ be defined by $\varepsilon(i) = 1 \iff s_{i,x} \in A_x^\times$. Clearly, if $\varepsilon' \in \{0,1\}^n$ with $\varepsilon' \neq \varepsilon$, then $d_{\varepsilon',x} = 0$. Moreover we have

$$d_{\varepsilon,x} = (\prod_{i=1,\varepsilon(i)=1}^n s_{i,x}^2\varphi_{A_x}(s_{i,x}))\cdot\varphi_{A_x}(\sum_{i=1,\varepsilon(i)=1}^n s_{\varepsilon,i,x}a_{i,x}).$$

Since $\prod_{i=1,\varepsilon(i)=1}^n s_{i,x}^2\varphi_{A_x}(s_{i,x}) = \varphi_{A_x}(\prod_{i=1,\varepsilon(i)=1}^n s_{i,x}^{-1})$ and φ_{A_x} is multiplicative we get

$$d_{\varepsilon,x} = \varphi_{A_x}(\sum_{i=1,\varepsilon(i)=1}^n s_{i,x}^{-1}a_{i,x}).$$

But $\sum_{i=1,\varepsilon(i)=1}^n s_{i,x}^{-1}a_{i,x} = c_x$ by definition of ε, which shows that $\varphi(c)_x = \varphi_{A_x}(c_x) = d_{\varepsilon,x}$ as desired. □

6. Non-axiomatizability of rings with completely normal spectrum

Recall that a spectral space is **completely normal** if the closure of every point is a specialization chain.

Lemma 6.1. *Suppose that A is a ring and the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable. Then there are elements $s, t \in A$ such that $\mathfrak{p}, \mathfrak{q} \in D(s)$, $\mathfrak{p}, \mathfrak{q} \in V(t)$, there is no common specialization in $D(s)$ and there is no common generalization in $V(t)$.*

Proof. We pick elements $a \in \mathfrak{p} \setminus \mathfrak{q}$ and $b \in \mathfrak{q} \setminus \mathfrak{p}$. Then $s = a + b$ and $t = a \cdot b$ meet the requirements. □

Corollary 6.2. *A ring A has completely normal spectrum if and only if $D(s)$ is normal for every $s \in A$.*

Proof. It follows directly from the definition that every spectral subspace of a completely normal spectral space is completely normal, as well. This applies to the $D(s)$.

Conversely suppose $\operatorname{Spec} A$ is not completely normal. Pick $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ that are incomparable w.r.t. inclusion and have a common generalization $\mathfrak{r}$ in $\operatorname{Spec} A$. Pick $s \in A$ as in 6.1. Then Dp and Dq have distinct maximal specializations in $D(s)$, and Dr specializes to both of them. Thus, $D(s)$ is not normal. □

The Corollary suggests that, in order to characterize rings with completely normal spectrum algebraically, we should first describe the property "$D(s)$ is completely normal" in algebraic terms.

Proposition 6.3. *Let A be a ring and let $s \in A$. The following are equivalent:*

(i) *$D(s)$ is a normal spectral space.*
(ii) *for all $a, b \in A$ with $D(s) \cap V(a) \cap V(b) = \emptyset$ there are $c, d \in A$ with $D(s) \cap V(a) \subseteq D(c)$, $D(s) \cap V(b) \subseteq D(d)$ such that $D(s) \cap D(c) \cap D(d) = \emptyset$.*
(iii) *for all $a, b \in A$ with $s \in \sqrt{(a,b)}$ there are $c, d \in A$ with $s \in \sqrt{(a,c)}$, $s \in \sqrt{(b,d)}$ such that $s \cdot c \cdot d$ is nilpotent.*
(iv) *for all $p \in \mathbb{N}$ and all $a \in A$ there are $c, d \in A$ and $k \in \mathbb{N}$ with $s^k \in (a,c)$, $s^k \in (s^p - a, d)$ and $c \cdot d = 0$.*

Proof. (i)⇔(ii) follows easily from 4.3, since $D(s)$ is canonically homeomorphic to $\operatorname{Spec} A_s$ and $V_{\operatorname{Spec} A_s}(\frac{a}{s^k})$ is mapped onto $D(s) \cap V(a)$.

(ii)⇔(iii) follows from the following translation table:

(a) $D(s) \cap V(a) \cap V(b) = \emptyset \iff V(a) \cap V(b) \subseteq V(s) \iff s \in \sqrt{(a,b)}$.
(b) $D(s) \cap V(a) \subseteq D(c) \iff V(c) \subseteq D(a) \cup V(s) \iff V(c) \cap V(a) \subseteq V(s) \iff s \in \sqrt{(a,c)}$.
(c) $D(s) \cap D(c) \cap D(d) = \emptyset \iff s \cdot c \cdot d$ is nilpotent.

(iii)⇒(iv). Let $a \in A$ and take $b = s^p - a$. Then $s \in \sqrt{(a,b)}$ and by (iii) there are $c_0, d_0 \in A$ and $l \in \mathbb{N}$ such that $(s \cdot c_0 \cdot d_0)^l = 0$, $s^l \in (a, c_0)$ and $s^l \in (b, d_0)$. Take $k := 2 \cdot l^2$, $c = (sc_0)^l$ and $d = (sd_0)^l$. Then $c \cdot d = 0$, $s^k = (s^{2l})^l \in (a, s^l \cdot c_0)^l \subseteq (a,c)$ and similarly $s^k \in (b,d) = (s^p - a, d)$.

(iv)⇒(iii). Take $a, b \in A$ with $s \in \sqrt{(a,b)}$, hence $s^p = xa + yb$ for some $p \in \mathbb{N}$ and some $x, y \in A$. Define $a_0 := xa$. By (iv) there are $c, d \in A$ and $k \in \mathbb{N}$ with $s^k \in (a_0, c)$, $s^k \in (s^p - a_0, d)$ and $c \cdot d = 0$. Since $(a_0, c) \subseteq (a,c)$ and $(s^p - a_0, d) \subseteq (b,d)$ we get (iii). □

Corollary 6.4. $\operatorname{Spec} A$ *is completely normal if and only if for all $s, a \in A$ there are $x, x' \in A$ and $k \in \mathbb{N}$ such that*

$$(*) \qquad (s^k - xsa) \cdot (s^k - x'(s^2 - sa)) = 0.$$

Hence the class of rings with completely normal spectrum is pseudo elementary with witnesses $\varphi_k(x_1, x_2, y_1, y_2) := (x_1^k - y_1 x_1 x_2) \cdot (x_1^k - y_2(x_1^2 - x_1 x_2)) = 0$.

Proof. If $\operatorname{Spec} A$ is completely normal, then $(*)$ holds by 6.3(i)⇒(iv) applied to $p = 2$ and $s \cdot a$.

Conversely if $(*)$ holds, then item (iv) of 6.3 holds for every $s \in A$: Pick $p \in \mathbb{N}, a \in A$ and apply $(*)$ to s^p and a. Then straightforward checking shows that

$c = s^{kp} - xs^p a$ and $d = s^{kp} - x'(s^{2p} - s^p a)$ satisfy $s^{kp} \in (a, c)$, $s^{kp} \in (s^p - a, d)$ and $c \cdot d = 0$.

Therefore, condition $(*)$ implies that all $D(s)$ are normal and by 6.2, $\operatorname{Spec} A$ is completely normal. □

Remark 6.5. To compare the classes of rings with normal spectrum and with completely normal spectrum, note that condition $(*)$ applied to $a = 1$ and any element $a \in A$ yields the condition of 4.3(v). Conversely, 4.3(v) implies that, given $s, a \in A$, there are $x, x' \in A$ with $(s - xsa)\cdot(s - x'(s - sa)) = 0$.

Definition 6.6. Let A be a ring. We define

- $k(s, a) = \inf\{k \in \mathbb{N} \mid \exists x, x' \; (s^k - xsa)\cdot(s^k - x'(s^2 - sa)) = 0\} \in \mathbb{N} \cup \{\omega, \infty\}$ for all pairs of elements $a, s \in A$;
- $\operatorname{CN}(A) = \sup\{k(s, a) \mid s, a \in A\} \in \mathbb{N} \cup \{\omega, \infty\}$.

By Corollary 6.4, the spectrum of A is completely normal if and only if $\operatorname{CN}(A) \leq \omega$.

Observe that $k(s, a) = k(s, s - a)$. Since the product $(s^k - xsa)\cdot(s^k - x'(s^2 - sa))$ is a multiple of s it is trivially true that $k(0, a) = 1$. Moreover we have $k(s, 0) = k(s, s) \leq 2$ for all s. If A is local then $k(s, a) = 1$ for each $s \in A^\times$. For, one defines $x = a^{-1}$, $x' = 1$ if a is a unit and $x = 1, x' = (s - a)^{-1}$ if a is not a unit. If A is a domain, then $(s^k - xsa)\cdot(s^k - x'(s^2 - sa)) = 0$ if and only if $s = 0$ or $a \neq 0$, $\frac{s^{k-1}}{a} \in A$ or $s \neq a$, $\frac{s^{k-1}}{s-a} \in A$.

Corollary 6.7. *Let T be a theory extending commutative rings in a language extending the language of rings, such that every model of T has completely normal spectrum. Then there is a natural number k such that for every model A of T and all $s, a \in A$ there are $x, x' \in A$ with $(s^k - xsa)\cdot(s^k - x'(s^2 - sa)) = 0$. In other words,* $\operatorname{CN}(A)$ *is bounded by a natural number when A runs through the models of T.*

Proof. By 3.2 and 6.4. □

In the rest of this section we use the following notation: We pick a field F and consider the polynomial ring $F[X] = F[X_1, \ldots, X_n]$. The quotient field is denoted by $F(X)$. The variables generate the maximal ideal $\mathfrak{m} \subset F[X]$. If $P = \sum_{i \in \mathbb{N}_0^n} \alpha_i \cdot X^i$ is a polynomial then $\operatorname{ord}(P) = \inf\{|i| \mid \alpha_i \neq 0\}$ is the *order* of P. This is an integer or ∞.The map $v : F(X) \to \mathbb{Z} \cup \{\infty\}$, $v(\frac{P}{Q}) = \operatorname{ord}(P) - \operatorname{ord}(Q)$ is a valuation; let V be the valuation ring, $\mathfrak{n}$ its maximal ideal. Then $\mathfrak{n} \cap F[X] = \mathfrak{m}$, hence the localization $F[X]_\mathfrak{m}$ is contained in V. The maximal ideal of $F[X]_\mathfrak{m}$ is denoted by $\mathfrak{m}_\mathfrak{m}$; its powers are $\mathfrak{m}_\mathfrak{m}^k$. If $\frac{P}{Q} \in F[X]_\mathfrak{m}$ then $v(\frac{P}{Q}) = k \in \mathbb{N}$ if and only if $\frac{P}{Q} \in \mathfrak{m}_\mathfrak{m}^k \setminus \mathfrak{m}_\mathfrak{m}^{k+1}$.

Example 6.8. We pick some $k \in \mathbb{N}$ and define $A := F[X]/\mathfrak{m}^k$. The residue class of $P \in F[X]$ is denoted by $P + \mathfrak{m}^k$. We claim that $\operatorname{CN}(A) = \lceil \frac{k}{2} \rceil$.

Proof. Note that A is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{m}^k$. Pick any two elements $s, a \in A$. We show that $k(s, a) \leq \lceil \frac{k}{2} \rceil$: If $s \in A^\times$ then $k(s, a) = 1$. If $s \notin A^\times$ then $s^{2 \cdot l} = 0$ for all $l \geq \lceil \frac{k}{2} \rceil$. Setting $x = x' = s^{l-1}$ we obtain $(s^l - xsa)\cdot(s^l - x'(s^2 - sa)) = 0$, and this implies $k(s, a) \leq \lceil \frac{k}{2} \rceil$. It has been proved that $\operatorname{CN}(A) \leq \lceil \frac{k}{2} \rceil$

It remains to exhibit elements $s, a \in A$ with $k(s, a) = \lceil \frac{k}{2} \rceil$. We set $s = X_1 + \mathfrak{m}^k$ and $a = X_2 + \mathfrak{m}^k$. We need to show that the existence of $c, d \in A$ with

$(s^k - csa)\cdot(s^l - d(s^2 - sa)) = 0$ implies $2l \geq k$. We show this by looking at representatives in the polynomial ring. Suppose that there are polynomials $C, D \in F[X]$ such that $v(X_1^l - C\cdot X_1\cdot X_2)\cdot(X_1^l - D\cdot(X_1^2 - X_1\cdot X - 2)) \geq k$. The order of $X_1^l - C\cdot X_1\cdot X_2$ is at most l. The automorphism of $F[X]$ that preserves all variables except X_2 and maps $X_2 \to X_1 - X_2$ preserves the order of polynomials. Therefore $\operatorname{ord}(X_1^l - D\cdot(X_1^2 - X_1\cdot X - 2)) \leq l$ as well. We conclude that, for any choice of polynomials C and D, $(X_1^l - C\cdot X_1\cdot X_2)\cdot(X_1^l - D\cdot(X_1^2 - X_1\cdot X - 2)) \notin \mathfrak{m}^k$ if $2\cdot l < k$. □

Several non-axiomatizability results follow immediately from the example:

Corollary 6.9. *The following classes of rings are not axiomatizable:*

- *Rings with singleton spectrum.*
- *Rings with Boolean spectrum.*
- *Rings with totally ordered spectrum.*
- *Rings with completely normal spectrum.*

Proof. From 6.8 and 6.7. □

Remark 6.10. For every ring A and each ideal I of A we have $\mathrm{CN}(A) \geq \mathrm{CN}(A/I)$.

Proof. Every equation of type $(*)$ in 6.4, remains valid when applying the residue map $A \to A/I$. Thus, by definition of $\mathrm{CN}(A)$ and $\mathrm{CN}(A/I)$ we get $\mathrm{CN}(A) \geq \mathrm{CN}(A/I)$. □

The ring in Example 6.8 is not reduced. Therefore it cannot be applied directly to decide axiomatizability of the class of *reduced* rings with completely normal spectrum. However, we shall now construct a domain B_k which has exactly one prime ideal besides (0) and a factor ring that is isomorphic to $F[X]/\mathfrak{m}^k$. By 6.10 and 6.8 this implies $\mathrm{CN}(B_k) \geq \frac{k}{2}$. In fact, we shall prove the stronger result that $\mathrm{CN}(B_k) = k+2$. One concludes that the set of invariants $CN(A)$ is not bounded by a natural number as A varies in the class of reduced rings with completely normal spectrum.

In the following proposition we use a construction, that is closely related to the so-called "D+M-construction" (cf. [Gi], Appendix 2).

Proposition 6.11. *We fix a natural number k and set $I_k = \{z \in V \mid v(z) \geq k\}$. We claim that the ring $B_k := F[X]_{\mathfrak{m}} + I_k$ has the following properties:*

(i) *B_k is a local domain with maximal ideal $\mathfrak{n}_k = \mathfrak{m}_{\mathfrak{m}} + I_k$; the maximal ideal and (0) are the only prime ideals of B.*
(ii) *The residue ring B_k/I_k is isomorphic to $F[X]/\mathfrak{m}^k$.*
(iii) *If $n \geq 2$, then $\mathrm{CN}(B_k) = k + 2$.*

Proof. First note that $I_k \subset V$ is an ideal. Thus B_k is a subring of V. Clearly, B_k is a domain. Both I_k and $\mathfrak{n}_k$ are ideals of B_k; note that $\mathfrak{n}_k = B_k \cap \mathfrak{n}$. It follows from $I_k \cap F[X]_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}^k$ that

$$B_k/I_k \simeq F[X]_{\mathfrak{m}}/(I_k \cap F[X]_{\mathfrak{m}}) = F[X]_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^k \simeq F[X]/\mathfrak{m}^k.$$

This proves claim (ii). The ideal $\mathfrak{n}_k/I_k \subset B_k/I_k$ corresponds to the ideal $\mathfrak{m}/\mathfrak{m}^k \subset F[X]/\mathfrak{m}^k$ under this isomorphism. It follows that

$$B_k/\mathfrak{n}_k \simeq (B_k/I_k)/(\mathfrak{n}_k/I_k) \simeq (F[X]/\mathfrak{m}^k)/(\mathfrak{m}/\mathfrak{m}^k) \simeq F[X]/\mathfrak{m} \simeq F,$$

which shows that $\mathfrak{n}_k$ is a maximal ideal. In order to prove that it is the only maximal ideal it is enough show that $a \in \mathfrak{n}_k$ implies $1 - a \in B_k^\times$:

It follows from $v(a) \geq 1$ that $v(a^k) \geq k$, hence $a^k \in I_k$. Since I_k is a proper ideal in the valuation ring V, the set $1 + I_k$ is a multiplicative subgroup of $V^\times$. The set $1 + I_k$ is also contained in B_k, hence it is also a multiplicative subgroup of $B_k^\times$. The identity $(1-a)\cdot(1 + a + a^2 + ... + ak - 1) = 1 - a^k$ shows that there is a multiple of $1 - a$ that is a unit, hence $1 - a$ is a unit as well.

Next we show that the B_k does not have prime ideals other than (0) and $\mathfrak{n}_k$. We pick an element $s \in \mathfrak{n}_k$, $s \neq 0$ and prove that $\mathfrak{n}_k = \sqrt{s\cdot B_k}$: Let $t \in \mathfrak{n}_k$ and pick some $d \in \mathbb{N}$ with $d\cdot v(t) \geq v(s) + k$. Then $v(\frac{t^d}{s}) \geq k$, hence $\frac{t^d}{s} \in I_k$, we see that $t^d \in s\cdot B_k$. This proves our claim and finishes the proof of (i).

It remains to prove (iii): The first step is to show $k(s, a) \leq k+2$ for all $s, a \in B_k$. (This statement is also true for $n = 1$). If $s = 0$ or if $s \in B_k^\times$, then we know $k(s, a) = 1$ since B_k is local. If $a = 0$ or $a = s$, then $k(s, a) \leq 2$. So suppose that $s \notin B_k^\times$, $s \neq 0$ and $a \neq s, 0$. Since we are in a domain it is enough to show $\frac{s^{k+1}}{a} \in B_k$ or $\frac{s^{k+1}}{s-a} \in B_k$. Notice that $v(s) > 0$.

Since $v(s) \geq \min\{v(a), v(s - a)\}$ we have $v(a) \leq v(s)$ or $v(a - s) \leq v(s)$, say $v(a) \leq v(s)$. Then

$$v\left(\frac{s^{k+1}}{a}\right) = (k+1)\cdot v(s) - v(a) \geq k \cdot v(s) \geq k,$$

and $\frac{s^{k+1}}{a} \in I_k \subset B_k$, as desired.

It remains to verify that $k(s, a) = k + 2$ for suitable $s, a \in B_k$. We show $k(X_1, X_2) \geq k + 2$. Since B_k is a domain, we have to show for $l \in \mathbb{N}$ that both $\frac{X_1^l}{X_2} \in B_k$ and $\frac{X_1^l}{X_1 - X_2} \in B_k$ imply $l \geq k + 1$.

If $\frac{X_1^l}{X_2} \in B_k$ then we write $\frac{X_1^l}{X_2} - \frac{P}{Q} \in I_k$, where $Q(0) \neq 0$. The polynomial $X_1^l \cdot Q$ contains the monomial X_1^l with non-zero coefficient. This monomial is not canceled in the polynomial $X_1^l\cdot Q - X_2\cdot P$, hence $v(X_1^l\cdot Q - X_2\cdot P) \leq l$. We conclude that $k \leq v\left(\frac{X_1^l}{X_2} - \frac{P}{Q}\right) = v(X_1^l\cdot Q - X_2\cdot P) - v(X_2) - v(Q) \leq l - 1$, which proves the desired inequality.

Finally we define σ to be the F-automorphism of $F(X)$ defined by $X_2 \to X_1 - X_2$, $X_i \to X_i$ otherwise. Then σ is an involution, preserves the valuation and restricts to an automorphism of B_k. Therefore, supposing that $\frac{X_1^l}{X_1 - X_2} \in B_k$, we apply σ to show that $\frac{X_1^l}{X_2} \in B_k$, and this is a case that has already been dealt with. □

Corollary 6.12. *The classes of reduced rings with totally ordered spectrum, or with totally ordered spectrum of length bounded by some natural number $l \geq 2$, or with completely normal spectrum are all not first order axiomatizable.*

Proof. The rings B_k constructed in 6.11 belong to all classes. As $\mathrm{CN}(B_k) = k + 2$ it follows that there is no axiomatizable class of rings all of whose members have completely normal spectrum and that contains all the rings B_k. □

Finally in this section, we exhibit a reduced ring A with $\mathrm{CN}(A) = \omega$. The ring will be constructed from the sequence of rings B_k defined in Proposition 6.11.

Each of the rings B_k is a local F-algebra with residue field F. Thus, $B_k = F \oplus \mathfrak{n}_k$. We form the direct product $B = \prod_{k\in\mathbb{N}} B_k$. This is an F-algebra, and we consider F

as a subring. The direct sum $\mathfrak{n} = \bigoplus_{k\in\mathbb{N}} \mathfrak{n}_k$ of the maximal ideals of the components is an ideal of B. Then $A := F \oplus \mathfrak{n}$ is a subring of B and $\mathfrak{n}$ is the largest proper ideal of A. Hence A is a reduced local ring with maximal ideal $\mathfrak{n}$. We shall identify $\mathfrak{n}_l$ with the ideal $\prod_{k\in\mathbb{N},k\neq l}\{0\} \times \mathfrak{n}_l$. The projections $\mathrm{pr}_k : A \to B \to B_k$ are surjective, their kernels are denoted by $\mathfrak{p}_k$.

The prime ideals $\mathfrak{p}_k$ are incomparable. We show that $\operatorname{Spec} A = \{\mathfrak{p}_k \mid k \in \mathbb{N}\} \cup \{\mathfrak{n}\}$: Suppose that $\mathfrak{p} \in \operatorname{Spec} A \setminus \mathfrak{n}$, i.e., $\mathfrak{p} \subsetneq \mathfrak{n}$. There are some $k \in \mathbb{N}$ and $a \in \mathfrak{n}_k$ with $a \notin \mathfrak{p}$. For each $b \in \mathfrak{p}_k$ we have $a\cdot b \in \mathfrak{p}_k \cap \mathfrak{n}_k = (0)$, hence $a\cdot b = 0 \in \mathfrak{p}$. This implies $b \in \mathfrak{p}$, hence $\mathfrak{p}_k \subseteq \mathfrak{p} \subsetneq \mathfrak{n}$. But then $(0) = \mathrm{pr}_k(\mathfrak{p}_k) \subseteq \mathrm{pr}_k(\mathfrak{p}) \subsetneq \mathrm{pr}_k(\mathfrak{n}) = \mathfrak{n}_k$ is a sequence of prime ideals in B_k. As we know all the prime ideals of B_k (6.11) we conclude that $\mathrm{pr}_k(\mathfrak{p}) = (0)$, hence $\mathfrak{p}_k = \mathfrak{p}$.

The space $\operatorname{Spec} A$ is completely normal (since its structure has been determined completely). It follows that $\mathrm{CN}(A) \leq \omega$. On the other hand, $\mathrm{CN}(A) \geq \mathrm{CN}(A/\mathfrak{p}_k) = \mathrm{CN}(B_k) \geq k + 2$ for each k (by 6.11).This shows:

Proposition 6.13. *The ring A is reduced and local and satisfies $\mathrm{CN}(A) = \omega$. Its spectrum is completely normal, but some ultra power of A does not have completely normal spectrum (by 3.4).*

7. Rings with inversely normal spectrum

Definition 7.1. A spectral space X is called **inversely normal** if X_{inv} is normal, i.e., every point has a unique minimal generalization in X. Recall that X_{inv} is the set X, equipped with the inverse topology (cf. 2.1).

Remark 7.2. For every ring A, $\operatorname{Spec} A$ is inversely normal if and only if for all distinct minimal prime ideals $\mathfrak{p}, \mathfrak{q}$ of A there is some $a \in A$ with $a \in \mathfrak{p}$ and $1 - a \in \mathfrak{q}$.

Proof. $\operatorname{Spec} A$ is inversely normal if and only if for all distinct $\mathfrak{p}, \mathfrak{q} \in (\operatorname{Spec} A)^{\min}$ we have $\overline{\{\mathfrak{p}\}}$ and $\overline{\{\mathfrak{q}\}}$ are disjoint, if and only if for all distinct $\mathfrak{p}, \mathfrak{q} \in (\operatorname{Spec} A)^{\min}$ we have $\mathfrak{p} + \mathfrak{q} = A$. □

Lemma 7.3. *Let X be a spectral space.*

(i) *If there is a specialization preserving retraction $r : X \longrightarrow X^{\min}$ of $X^{\min} \longrightarrow X$, then X is inversely normal and $r(x) \rightsquigarrow x$ for all $x \in X$. (cf.* [Ca-Co], *Proposition 3).*

(ii) *If X is inversely normal, then $X^{\min}$ is quasi-compact if and only if the map $r : X \longrightarrow X^{\min}$ that maps x to the unique minimal point $z \in X$ with $z \rightsquigarrow x$, is continuous. If this is the case then r is a spectral map.*

Proof. (i). Let $x \in X$. Take $y \in X^{\min}$ with $y \rightsquigarrow x$. By assumption $y = r(y) \rightsquigarrow r(x) \in X^{\min}$, hence $y = r(x)$.

(ii). If r is continuous, then $X^{\min}$ is the image of a quasi-compact space under a continuous map, hence is quasi-compact. Conversely suppose $X^{\min}$ is quasi-compact. Then, by 2.7, $X^{\min}$ is a proconstructible subset of X. We prove that r is continuous: Let $A \in \overline{\mathcal{K}}(X)$. Then

$$r^{-1}(A \cap X^{\min}) = \bigcup_{x\in A\cap X^{\min}} \overline{\{x\}}.$$

Since $X^{\min}$ and A are proconstructible, the latter set is $\overline{A \cap X^{\min}}$, which is closed. Hence r is continuous.

As $X^{\min}$ is proconstructible, 4.1(iii) applied to X_{inv} says that r is continuous w.r.t. the inverse topology. As it is also continuous, r is spectral. □

Proposition 7.4. *The following are equivalent for every ring A:*

(i) $\operatorname{Spec} A$ *is inversely normal.*
(ii) For all $a, b \in A$ with $D(a) \cap D(b) = \emptyset$ there is some $c \in A$ such that $D(a) \subseteq V(c)$ and $D(b) \subseteq V(1-c)$.
(iii) $\forall a, b \in A \exists c \in A \exists n \in \mathbb{N} : a \cdot b = 0 \rightarrow a^n \cdot c = 0 = b^n \cdot (1-c)$.

If A is reduced, then (i)-(iii) are also equivalent to

(iv) $\forall a, b \in A \exists c \in A : a \cdot b = 0 \rightarrow a \cdot c = 0 = b \cdot (1-c)$.

Proof. (ii)⇒(i). Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ be minimal prime ideals with $\mathfrak{p} \neq \mathfrak{q}$. We must show that $\mathfrak{p} + \mathfrak{q} = A$. Since both ideals are minimal, there are $a, b \in A$ such that $D(a) \cap D(b) = \emptyset$, $\mathfrak{p} \in D(a)$ and $\mathfrak{q} \in D(b)$. Pick c as in (ii). Then $c \in \mathfrak{p}$, $1 - c \in \mathfrak{q}$, and $1 \in \mathfrak{p} + \mathfrak{q}$ as desired.

(i)⇒(ii). Let $a, b \in A$ with $D(a) \cap D(b) = \emptyset$. Since $\operatorname{Spec} A$ is inversely normal. there are disjoint inversely open subsets V, W of $\operatorname{Spec} A$ such that $D(a) \subseteq V$, $D(b) \subseteq W$. Since $D(a), D(b)$ are quasi-compact in the inverse topology we may assume that $V = V(c_1, ..., c_k)$ and $W = V(d_1, ..., d_l)$. Since $V \cap W = \emptyset$, there are $c \in (c_1, ..., c_k)$, $d \in (d_1, ..., d_l)$ with $c + d = 1$. Then $d = 1 - c$, and $D(a) \subseteq V \subseteq V(c)$, $D(b) \subseteq V(d_1, ..., d_l) \subseteq V(1-c)$.

(ii)⇒(iii). If $a \cdot b = 0$, then by (ii) there are $d, e \in A$ with $D(a) \subseteq V(d)$ and $D(b) \subseteq V(e)$, $V(d) \cap V(e) = \emptyset$. Hence $a \cdot d$ and $b \cdot e$ are nilpotent and there is some $n \in \mathbb{N}$ with $a^n d^n = b^n e^n = 0$. We now replace d by d^n and e by e^n and still have $V(d) \cap V(e) = \emptyset$. Thus $1 = xd + ye$ for some $x, y \in A$. Now choose $c := xd$. Then $a^n \cdot c = a^n \cdot xd = 0 = b^n \cdot ey = b^n \cdot (1-c)$.

The implication (iii)⇒(ii) is straightforward. Moreover the proof of (ii)⇒(iii) also shows that we can choose $n = 1$ if A is reduced, hence (i)-(iii) are equivalent to (iv) if A is reduced. □

The spectral space X is called **inversely completely normal** if the inverse topology is completely normal. We use 6.1 to obtain an inverse version of the characterization of complete normality in 6.2:

Corollary 7.5. *The spectrum of A is inversely completely normal if and only if each principle closed subspace $V(s)$ is inversely normal.*

Proof. Spectral subspaces of inversely completely normal spaces are clearly inversely completely normal, hence are inversely normal. Conversely, assume that *SpecA* is not inversely completely normal, i.e., there are incomparable prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ that are contained in a prime ideal $\mathfrak{r}$. By 6.1 there is a set $V(s)$ that contains both prime ideals, but no common generalization. Then $\mathfrak{r} \in V(s)$ has two distinct minimal generalizations in $V(s)$. Thus, $V(s)$ is not normal. □

Corollary 7.6. (i) *The property "A is reduced and* $\operatorname{Spec} A$ *is inversely normal" is elementary.*
(ii) *The property "*$\operatorname{Spec} A$ *is inversely normal" is not elementary.*
(iii) *The property "A is reduced and* $\operatorname{Spec} A$ *is completely inversely normal" is not elementary.*

Proof. (i) follows immediately from (i) $\iff$ (iv) in 7.4.

(iii). If the property "A is reduced and $\operatorname{Spec} A$ is completely inversely normal" is elementary, then also the property "A is reduced and $\operatorname{Spec} A$ is totally ordered" is elementary, since $\operatorname{Spec} A$ is totally ordered if and only if A is local and $\operatorname{Spec} A$ is completely inversely normal. On the other hand 6.12 shows that "A is reduced and $\operatorname{Spec} A$ is totally ordered" is not elementary: a contradiction.

(ii). Assume that the class of rings with inversely normal spectrum is elementary. Then by 7.4(i) $\iff$ (iii) and 3.2, there is a bound N for the numbers $n \in \mathbb{N}$ that occur in 7.4(iii). Thus, a ring A has inversely normal spectrum if and only if $A \models \varphi$, where φ is the sentence

$$\forall a, b \, \exists c : \ a \cdot b = 0 \to a^N \cdot c = 0 = b^N \cdot (1 - c).$$

Let $\psi(x, s)$ be the formula $\exists y : \ x = y \cdot s$ and let γ be the sentence

$$\forall s \, \forall a, b \, \exists c : \ \psi(a \cdot b, s) \to \psi(a^N \cdot c, s) \wedge \psi(b^N \cdot (1 - c), s).$$

Then A satisfies γ if and only if $A/s \cdot A \models \varphi$ for all $s \in A$. Consequently, A satisfies γ if and only if for all $s \in A$, the ring $A/s \cdot A$ has inversely normal spectrum, if and only if $V(s)$ is inversely normal for all $s \in A$, if and only if $\operatorname{Spec} A$ is inversely completely normal (7.5). Therefore γ axiomatizes rings with completely inversely normal spectrum. This contradicts (iii). □

The condition (cf. 7.4(iv)) that expresses the property "A is reduced and $\operatorname{Spec} A$ is inversely normal" is a Horn sentence (cf. [Ho], section 9.1). This implies, in particular, that products of reduced rings with inversely normal spectrum again have inversely normal spectrum. Products of domains have this property, for example.

8. Minimal points of spectral spaces

The remainder of the paper is devoted to the study of compactness of the space of minimal prime ideals of a ring. In the present section we characterize this property by topological conditions concerning the spectrum itself and by properties of distributive lattices. In the next section we take a ring theoretic point of view.

Lemma 8.1. *Let X be a topological space and let $O, Y \subseteq X$, O open. Then*

$$\overline{O \cap Y} = \overline{O \cap \overline{Y}}.$$

In particular, if O is open and closed, then $\overline{O \cap Y} = O \cap \overline{Y}$.

Proof. Suppose $x \in X \setminus \overline{O \cap Y}$. Then there is an open set $U \subseteq X$ with $x \in U$ such that $U \cap O \cap Y = \emptyset$. Since $U \cap O$ is open, $U \cap O \cap \overline{Y} = \emptyset$, so $x \notin \overline{O \cap \overline{Y}}$. □

Corollary 8.2. *Let X be a spectral space, let $K, P, O, Y \subseteq X$ such that K is constructible, P is proconstructible and O is open in the constructible topology of X. Let Z be the closure of Y in the constructible topology of X. Then*

$$O \cap Y \subseteq K \cap Y \subseteq P \iff O \cap Z \subseteq K \cap Z \subseteq P. \qquad \square$$

Definition 8.3. Let X be a spectral space and let $K \in \mathcal{K}(X)$. If there is some $U \in \mathring{\mathcal{K}}(X)$ with $K \cap X^{\min} = U \cap X^{\min}$, then we say that K has **generically constructible interior**. In this case, every $U \in \mathring{\mathcal{K}}(X)$ with $K \cap X^{\min} = U \cap X^{\min}$ is called a **generic interior** of K.

Proposition 8.4. *Let X be a spectral space, let $K \in \mathcal{K}(X)$ and let $U \in \mathring{\mathcal{K}}(X)$. The following are equivalent.*

(i) *There is a dense subset $Z \subseteq X$ with $U \cap Z \subseteq \operatorname{int}(K) \cap Z \subseteq \overline{U}$.*
(ii) *There is a dense subset $Z \subseteq X$ with $\operatorname{int}(K) \cap Z = U \cap Z$.*
(iii) *There is a dense subset $Z \subseteq X$ with $\operatorname{int}(K) \cap Z \subseteq U \cap Z \subseteq K$.*
(iv) *U is a generic interior of K, i.e. $K \cap X^{\min} = U \cap X^{\min}$.*

If this is the case and $K \cap U$ is closed in U, then $U \subseteq K$.

Proof. (i)⇒(ii). Let $Z \subseteq X$ be dense with $U \cap Z \subseteq \operatorname{int}(K) \cap Z \subseteq \overline{U}$. Let $Y := Z \setminus (\operatorname{int}(K) \cap Z \setminus U)$. Then Y is dense in X: take $O \subseteq X$ open and some $z \in Z \cap O$. If $z \notin \operatorname{int}(K) \cap Z \setminus U$, then $z \in Y$. If $z \in \operatorname{int}(K) \cap Z \setminus U$, then $z \in \operatorname{int}(K) \cap Z \subseteq \overline{U}$, so $O \cap U \neq \emptyset$. Hence there is some $y \in Z \cap O \cap U$, and $y \in Y$. This shows that Y is dense in X. Since $U \cap Z \subseteq \operatorname{int}(K) \cap Z \subseteq \overline{U}$ it follows that $U \cap Y = \operatorname{int}(K) \cap Y$.

(ii)⇒(iii) is trivial.

(iii)⇒(iv). Let $Z \subseteq X$ be dense with $\operatorname{int}(K) \cap Z \subseteq U \cap Z \subseteq K$. By 8.2 we may assume that Z is proconstructible. Since Z is dense in X, 2.6 implies that $X^{\min} \subseteq Z$. Hence $\operatorname{int}(K) \cap X^{\min} \subseteq U \cap X^{\min} \subseteq K \cap X^{\min}$. But $K \cap X^{\min} = \operatorname{int}(K) \cap X^{\min}$ by 2.6, so $K \cap X^{\min} = U \cap X^{\min}$ as desired.

(iv)⇒(i). By 2.6 we have $\operatorname{int}(K) \cap X^{\min} = U \cap X^{\min}$, so we can take $Z = X^{\min}$.

This shows the equivalence of (i)-(iv). Now let U be a generic interior of K and assume that $K \cap U$ is closed in U. If $x \in U$ and $y \in U \cap X^{\min}$ with $y \rightsquigarrow x$, then $y \in K \cap U$, hence $x \in K$, since $K \cap U$ is closed in U. Thus $U \subseteq K$. □

Proposition 8.4 is inspired by the fact that, in practice, many rings occur as rings of functions, i.e., $A \subseteq K^T$, where T is a set and K is a field. Let $\hat{t} : A \to K$ be the evaluation map at $t \in T$. Then the set $\hat{T} = \{\ker(\hat{t}) \mid t \in T\}$ is dense in $\operatorname{Spec} A$, and each of the equivalent conditions (i)-(iii) of 8.4 may be used to decide whether a constructible subset of $\operatorname{Spec} A$ has generic interior or not. To illustrate this method, consider the following example. Here, and also later on, shall use the following notation: If $f \in K^T$ then $\mathrm{Z}_T(f) = \{t \in T \mid f(t) = 0\}$ is the **zero set** of f, and $\mathrm{Coz}_T(f) = T \setminus \mathrm{Z}_T(f)$ is the **cozero set** of f.

Example 8.5. Let T be a Tychonov space and let $A = C(T, \mathbb{R})$ be the ring of continuous functions with values in $\mathbb{R}$. We identify T with the subspace $\hat{T} \subseteq \operatorname{Spec} C(Z)$. Pick $f \in A$ and suppose the interior of $\mathrm{Z}_T(f)$ is of the form $\mathrm{Coz}_T(g)$ for some $g \in A$. (Such a g exists always if X is a metric space.) Then 8.4(ii)⇒(iv) says that $V(f) \subseteq \operatorname{Spec} A$ has generic interior $D(g)$.

Lemma 8.6. *Let X be a spectral space and let O, P be subsets of X, O generically closed, P is quasi-compact in the inverse topology. Then*

$$\overline{O} \subseteq \overline{P} \text{ if and only if } O \cap X^{\min} \subseteq P \cap X^{\min}.$$

Proof. We know from 2.4(i) that $\overline{P} = \bigcup_{x \in P} \overline{\{x\}}$. So, if $\overline{O} \subseteq \overline{P}$ then $O \cap X^{\min} \subseteq \overline{O} \cap X^{\min} \subseteq \overline{P} \cap X^{\min} = P \cap X^{\min}$.

Conversely, if $O \cap X^{\min} \subseteq P$ then $O \subseteq \bigcup_{x \in O \cap X^{\min}} \overline{\{x\}} \subseteq \bigcup_{x \in P} \overline{\{x\}} = \overline{P}$, and we conclude that $\overline{O} \subseteq \overline{P}$. □

Proposition 8.7. *Let X be a spectral space. The following are equivalent.*

(i) $X^{\min}$ *is quasi-compact, hence compact.*

(ii) $X^{\min} = \bigcap\{U \in \overset{\circ}{\mathcal{K}}(X) \mid U$ *is dense in* $X\}$, *equivalently* $X^{\min}$ *is proconstructible.*

(iii) *For all* $K \in \mathcal{K}(X)$ *there is some* $U \in \overset{\circ}{\mathcal{K}}(X)$ *with* $K \cap X^{\min} \subseteq U \subseteq K$. *For each such* U *we have* $U \subseteq \operatorname{int}(K) \subseteq \overline{U}$.

(iv) *Every constructible subset of* X *has a generically constructible interior.*

(v) *Every closed constructible subset of* X *has a generically constructible interior.*

Suppose that the subset $\mathfrak{B} \subseteq \overline{\mathcal{K}}(X)$ *has the property that every element* $\overline{\mathcal{K}}(X)$ *is a finite intersection of elements in* $\mathfrak{B}$ (*e.g. if* $X = \operatorname{Spec} A$ *and* $\mathfrak{B} = \{V(f) \mid f \in A\}$), *then* (i)-(v) *are equivalent to*

(vi) *Every* $B \in \mathfrak{B}$ *has a generically constructible interior.*

Proof. By 2.7 we know already (i)$\Leftrightarrow$(ii).

(ii)$\Rightarrow$(iii). Let $K \subseteq X$ be constructible. Let $x \in K \cap X^{\min}$. By 2.6, $\operatorname{int}(K) \cap X^{\min} = K \cap X^{\min}$, hence there is some $U_x \in \overset{\circ}{\mathcal{K}}(X)$ with $x \in U_x \subseteq K$. By (ii), $K \cap X^{\min}$ is proconstructible, thus there are finitely many $x_i \in K \cap X^{\min}$ with $K \cap X^{\min} \subseteq \bigcup_i U_{x_i}$ and $U := \bigcup_i U_{x_i} \in \overset{\circ}{\mathcal{K}}(X)$ fulfills $K \cap X^{\min} \subseteq U \subseteq K$. Hence $U \subseteq K$ is a generic interior of K, and by, 8.6, we conclude that $\operatorname{int}(K) \subseteq \overline{\operatorname{int}(K)} \subseteq \overline{U}$.

(iii)$\Rightarrow$(iv) follows from 8.4(i)$\Leftrightarrow$(iv). The implication (iv)$\Rightarrow$(v) is trivial.

(v)$\Rightarrow$(ii). Let $y \in X \setminus X^{\min}$ and take $x \in X^{\min}$ with $x \rightsquigarrow y$. Let $V \in \overline{\mathcal{K}}(X)$ with $y \in V$, $x \notin V$. Take $U \in \overset{\circ}{\mathcal{K}}(X)$ with $V \cap X^{\min} = U \cap X^{\min}$. Then $V \setminus U$ is closed, constructible, with empty interior. Hence $X \setminus (V \setminus U)$ is open, constructible and dense. So $X^{\min} \subseteq X \setminus (V \setminus U)$. Since $x \rightsquigarrow y$, $y \in V \setminus U$ (otherwise $y \in U$, so $x \in U \cap X^{\min} \subseteq V$, a contradiction), thus $y \notin X^{\min} \subseteq X \setminus (V \setminus U)$.

Hence (i)-(v) are equivalent and since (v)$\Rightarrow$(vi) is trivial, it remains to show (vi)$\Rightarrow$(v). Take $A \in \overline{\mathcal{K}}(X)$ and $B_1, ..., B_n \in \mathfrak{B}$ with $A = B_1 \cap ... \cap B_n$. Take a generic constructible interior U_i of B_i. Then clearly $U_1 \cap .... \cap U_n$ is a generic constructible interior of A. □

Let X be a spectral space. If $x \in X \setminus X^{\min}$, then the complement of $\overline{\{x\}}$ in X is an open and dense subset of X containing $X^{\min}$. Hence $X^{\min}$ is the intersection of all open subsets of X containing $X^{\min}$. We know (cf. 2.6(ii)) that the closure of $X^{\min}$ with respect to the constructible topology is contained in $\bigcap\{U \in \overset{\circ}{\mathcal{K}}(X) \mid \overline{U} = X\}$. In general the inclusion $\overline{X^{\min}}^{\text{con}} \subseteq \bigcap\{U \in \overset{\circ}{\mathcal{K}}(X) \mid \overline{U} = X\}$ is proper:

Example 8.8. Let $A = \operatorname{Spec} C([0,1], \mathbb{R})$, and let X be the inverse spectral space of $\operatorname{Spec} A$. We denote the constructible closure of $X^{\min}$ by Z; this is the same as the constructible closure of $(\operatorname{Spec} A)^{\max}$, thus Z corresponds to the prime z-filters of closed subsets of $[0,1]$ (equivalently: to the prime z-ideals) and $Z \neq \operatorname{Spec} A$ (cf. [Schw], section 3 or [Tr],p. 145). On the other hand, if $U \subseteq X$ is open, quasi-compact and dense, then $(\operatorname{Spec} A)^{\max} = X^{\min} \subseteq U$, and U is closed in $\operatorname{Spec} A$. We conclude that $U = \operatorname{Spec} A = X$ (since $(\operatorname{Spec} A)^{\max}$ is dense in $\operatorname{Spec} A$). □

In general, it is not true that an open and dense subset of a spectral space Y contains $Y^{\min}$, even if $Y^{\min}$ is compact: Suppose that $y \in Y^{\min}$ is not an isolated point. Then $Y^{\min} \setminus \{y\}$ is dense in $Y^{\min}$, and it follows that $Y \setminus \overline{\{y\}}$ is dense and open in Y. Such a point y always exists if $Y^{\min}$ is compact and infinite. An example is provided by the ring $C([0,1],\mathbb{R})$. Observe that $\operatorname{Spec} C([0,1],\mathbb{R})$ has compact minimal spectrum by 8.7(i)⇔(v) and 8.5).

We conclude this section with the description of the generic interior in terms of lattices. Recall that every spectral space X is canonically homeomorphic to the spectral space of prime filters of the distributive lattice $E = \overline{\mathcal{K}}(X)$. In what follows X denotes the spectral space of prime filters of a lattice E with $\top$ and $\bot$. Given $a \in E$ we denote by $V(a)$ the set of all $x \in X$ containing a and $D(a) = X \setminus V(a)$. A general reference for distributive lattices and spectral spaces is Johnstone's book [Joh].

Lemma 8.9. *Let $a_1, b_1, ..., a_n, b_n \in E$. Then for every $c \in E$ we have*

$$D(c) \cap X^{\min} \subseteq \bigcup_{i=1}^{n} V(a_i) \cap D(b_i) \text{ if and only if}$$

$$E \models \forall x\ [(a_1 \le b_1 \vee x)\ \&...\&\ (a_n \le b_n \vee x)] \ \longrightarrow\ c \vee x = \top.$$

Observe that this formula is strict universal Horn in the language $\{\wedge, \vee, \bot, \top\}$ of lattices with top and bottom.

Proof. In E we have for all α, β, γ: $V(\alpha) \cap D(\beta) \subseteq V(\gamma) \iff \alpha \le \beta \vee \gamma$. By 8.6 we know that $D(c) \cap X^{\min} \subseteq \bigcup_{i=1}^{n} V(a_i) \cap D(b_i)$ if and only if for all closed constructible subsets A of X with $\bigcup_{i=1}^{n} V(a_i) \cap D(b_i) \subseteq A$ we have $D(c) \subseteq A$. Since the closed constructible subsets of X are exactly the sets of the form $V(x)$ with $x \in E$ we get $D(c) \cap X^{\min} \subseteq \bigcup_{i=1}^{n} V(a_i) \cap D(b_i)$ if and only if

$$E \models \forall x\ [(a_1 \le b_1 \vee x)\ \&...\&\ (a_n \le b_n \vee x)] \ \longrightarrow\ c \vee x = \top.$$

□

Corollary 8.10. *Let E be a distributive lattice and let X be the spectral space attached to X. Let $a, b \in E$. Then*

(i) *$D(a) \cap X^{\min} \subseteq V(b) \cap X^{\min}$ if and only if $E \models a \vee b = \top$.*
(ii) *$D(a) \cap X^{\min} \subseteq D(b) \cap X^{\min}$ if and only if $E \models \forall x\ x \vee b = \top \longrightarrow x \vee a = \top$.*
(iii) *$D(a) \cap X^{\min} = D(b) \cap X^{\min}$ if and only if $E \models \forall x\ x \vee b = \top \leftrightarrow x \vee a = \top$.*

All these formulas are strict universal Horn. □

Corollary 8.11. *Let E be a distributive lattice and let X be the spectral space attached to X. Let $a, b \in E$. Then $D(a)$ is a generic interior of $V(b)$ if and only if*

$$E \models a \vee b = \top\ \&\ \forall z\ [z \vee b = \top \rightarrow \forall x\ (x \vee a = \top \rightarrow x \vee z = \top)].$$

Proof. By 8.10 the formula holds for a, b if and only if $D(a) \cap X^{\min} \subseteq V(b) \cap X^{\min}$ and for every $z \in E$ with $D(z) \cap X^{\min} \subseteq V(b) \cap X^{\min}$ we have $D(z) \cap X^{\min} \subseteq D(a) \cap X^{\min}$. By 2.5 this is equivalent to $D(a) \cap X^{\min} = V(b) \cap X^{\min}$. □

Corollary 8.12. *Let E be a distributive lattice and let X be the spectral space attached to X. Then $X^{\min}$ is compact if and only if*

$$E \models \forall b\ \exists a\ \{a \vee b = \top\ \&\ \forall z\ [z \vee b = \top \rightarrow \forall x\ (x \vee a = \top \rightarrow x \vee z = \top)]\}.$$

Proof. By 8.7 and 8.11. □

9. The generic interior in Zariski Spectra

Recall that an ideal $I \subseteq A$ is called **dense** if the annihilator $\mathrm{Ann}(I)$ of I is 0. If A is a reduced ring, then one checks without difficulty that

$$V(\mathrm{Ann}(I)) = \overline{\mathrm{Spec}\, A \setminus V(I)},$$

hence I is dense if and only if $\mathrm{Spec}\, A \setminus V(I)$ is dense in $\mathrm{Spec}\, A$.

Proposition 9.1. *Let A be a reduced ring and let $f_1, ..., f_k, g_1, ..., g_n \in A$. Then $D(g_1) \cup ... \cup D(g_n)$ is a generic interior of $V(f_1, ..., f_k)$ if and only if*

$$\mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n)) = \mathrm{Ann}(f_1, ..., f_k).$$

Proof. First assume that $D(g_1) \cup ... \cup D(g_n)$ is a generic interior of $V(f_1, ..., f_k)$. By 8.4, we have $D(g_1) \cup ... \cup D(g_n) \subseteq V(f_1, ..., f_k)$. Thus $D(g_j) \subseteq V(f_i)$ for all i, j which means $f_i \cdot g_j = 0$ (since A is reduced). Thus $f_1, ..., f_k \in \mathrm{Ann}(g_1, ..., g_n)$, which shows that $\mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n)) \subseteq \mathrm{Ann}(f_1, ..., f_k)$. Conversely let $a \in \mathrm{Ann}(f_1, ..., f_k)$ and suppose that $a \cdot b \neq 0$ for some $b \in \mathrm{Ann}(g_1, ..., g_n)$. Since A is reduced, there is a minimal prime ideal $\mathfrak{p}$ of A not containing ab. Since $a \notin \mathfrak{p}$ and $a \cdot (f_1, ..., f_k) = 0$ it follows that $(f_1, ..., f_k) \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in V(f_1, ..., f_k) \cap (\mathrm{Spec}\, A)^{\min}$ and by assumption there is some j such that $\mathfrak{p} \in D(g_j)$. Since $b \notin \mathfrak{p}$, we then get $b \cdot g_j \notin \mathfrak{p}$, which contradicts $b \in \mathrm{Ann}(g_1, ..., g_n)$.

Conversely suppose $\mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n)) = \mathrm{Ann}(f_1, ..., f_k)$. Since each g_j is in $\mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n))$ we get $g_j \cdot (f_1, ..., f_k) = 0$, hence $D(g_j) \subseteq V(f_1, ..., f_k)$. Conversely let $\mathfrak{p} \in V(f_1, ..., f_k) \cap (\mathrm{Spec}\, A)^{\min}$. Since

$$V(\mathrm{Ann}(f_1, ..., f_k)) = \overline{\mathrm{Spec}\, A \setminus V(f_1, ..., f_k)}$$

we have $\mathfrak{p} \notin V(\mathrm{Ann}(f_1, ..., f_k))$. Hence $\mathfrak{p} \notin V(\mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n)))$, which means that there is some $b \in \mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n))$ with $b \notin \mathfrak{p}$. Suppose $g_1, ..., g_n \in \mathfrak{p}$. Then again $\mathfrak{p} \notin V(\mathrm{Ann}(g_1, ..., g_n))$. This means that $h \notin \mathfrak{p}$ for some $h \in \mathrm{Ann}(g_1, ..., g_n)$. But then $b \cdot h \notin \mathfrak{p}$ either, which contradicts $b \in \mathrm{Ann}(\mathrm{Ann}(g_1, ..., g_n))$. □

Combining 8.7(i)⇔(vi) with 9.1 we obtain a result due to Mewborn (cf. [Me2], see also [Gl], Theorem 4.2.15).

Corollary 9.2. *If A is a reduced ring, then $(\mathrm{Spec}\, A)^{\min}$ is compact if and only if for all $a \in A$ there are $k \in \mathbb{N}$ and $b_1, ..., b_k \in \mathrm{Ann}(a)$ such that the ideal $(a, b_1, ..., b_k)$ is dense.* □

Observe that, for any two ideals $I, J \subseteq$ in a reduced ring, $\mathrm{Ann}(\mathrm{Ann}(I)) = \mathrm{Ann}(J)$ if and only if $I \cdot J = 0$ and $I + J$ is dense. We define $\varphi_k(x, y_1, ..., y_k)$ to be the following formula in the language of rings:

$$x \cdot y_1 = 0 \wedge ... \wedge x \cdot y_k = 0 \wedge \forall z\ (z \cdot x = z \cdot y_1 = ... = z \cdot y_k = 0 \rightarrow z = 0).$$

It expresses that $y_1, ..., y_k \in \mathrm{Ann}(a)$ and $(a, y_1, ..., y_k)$ is dense. Now 9.2 says that the class of all reduced rings with compact minimal spectrum is pseudo elementary with witnesses φ_k.

The first equivalence of the next theorem can be found as Theorem 3.4. in [He-Je].

Theorem 9.3. *The following are equivalent for every reduced ring A:*

(i) *The set $\{D(f) \cap (\mathrm{Spec}\, A)^{\min} \mid f \in A\}$ is closed under finite unions, and $(\mathrm{Spec}\, A)^{\min}$ is compact*

(ii) *For all* $f \in A$ *there is some* $g \in A$ *such that* $f \cdot g = 0$ *and* $f + g$ *is a non zero-divisor of* A.
(iii) *The total ring of quotients* $\operatorname{Tot} A$ *of* A *is von Neumann regular.*

Proof. (i)⇔(ii) is Theorem 3.4. in [He-Je].

(ii)⇒(iii). Take $f \in A$. It is enough to show that there is $y \in \operatorname{Tot}(A)$ with $\frac{f^2}{1} \cdot y = \frac{f}{1}$ in $\operatorname{Tot}(A)$. By (ii), there is $g \in A$ such that $f \cdot g = 0$ and $\frac{f+g}{1}$ is a unit in $\operatorname{Tot}(A)$. Hence $f(f+g) = f^2$ and $y = \frac{1}{f+g} \in \operatorname{Tot}(A)$ has the required property.

(iii)⇒(ii). Take $f \in A$. By assumption, there are $a, s \in A$, s a non zero-divisor of A with $\frac{f}{1}^2 \cdot \frac{a}{s} = \frac{f}{1}$ in $\operatorname{Tot}(A)$. Hence $fs - f^2a = 0$ and $g := s - fa$ satisfies $fg = 0$. It remains to show that $f + g = f + s - fa$ is not contained in any minimal prime ideal $\mathfrak{p}$ of A. As $f \cdot g = 0$ we have $f \in \mathfrak{p}$ or $s - fa = g \in \mathfrak{p}$. Since $s \notin \mathfrak{p}$ this is only possible if $f + s - fa \notin \mathfrak{p}$. □

10. Non-axiomatizability of the compactness of minimal primes

We have seen that the class of reduced rings with compact minimal spectrum is pseudo elementary, cf. the remark following 9.2. In this section we shall show that the class is not axiomatizable.

Notation 10.1. Let A be a ring. For $a \in A$ we define the **annihilator size** of a to be

$$\operatorname{AS}(a) = \inf\{k \in \mathbb{N} \mid \exists\, b_1, ..., b_k \in \operatorname{Ann}(a) : \operatorname{Ann}(a, b_1, ..., b_k) = (0),$$

which is an element of $\mathbb{N} \cup \{\infty\}$. Moreover, we define

$$\operatorname{AS}(A) := \sup\{\operatorname{AS}(a) \in \mathbb{N} \cup \{\omega, \infty\} \mid a \in A\}.$$

It is clear form the definition that $\operatorname{AS}(A) = \omega$ if and only if $\{AS(a) \mid a \in A\}$ is an unbounded subset of $\mathbb{N}$. Moreover, $\operatorname{AS}(A) = \infty$ if and only if there is some $a \in A$ such that $AS(a) = \infty$, which means that the

$$\{k \in \mathbb{N} \mid \exists\, b_1, ..., b_k \in \operatorname{Ann}(a) : \operatorname{Ann}(a, b_1, ..., b_k) = (0) = \emptyset.$$

Corollary 10.2. (i) *A reduced ring* A *has compact minimal spectrum if and only if* $\operatorname{AS}(A) \leq \omega$.
(ii) *A pseudo-elementary class* $\mathcal{C}$ *of reduced rings with witnesses* φ_k *from 9.2 is elementary if and only if there is some* $K \in \mathbb{N}$ *with* $\operatorname{AS}(A) \leq K$ *for all* $A \in \mathcal{C}$.

Proof. (i) holds by 9.2, and (ii) holds by 3.2. □

Remark 10.3. Suppose $(\operatorname{Spec} A)^{\min}$ is compact, $K \in \mathbb{N}$ and for all $f_1, ..., f_{K+1} \in A$ there are $g_1, ..., g_K \in A$ such that

$$(D(f_1) \cup ... \cup D(f_{K+1})) \cap (\operatorname{Spec} A)^{\min} = (D(g_1) \cup ... \cup D(g_K)) \cap (\operatorname{Spec} A)^{\min}.$$

Then $\operatorname{AS}(A) \leq K$, as follows from 9.2 and 9.1.

If $K = 1$, then the converse of the implication in the Remark also holds true, cf. Theorem 9.3. Note that $\operatorname{AS}(A) = 1$ is equivalent to item (ii) of 9.3.

10.4. Open problem. *Let A be a ring with $\mathrm{AS}(A) \in \mathbb{N}$. Does there exist some $K \in \mathbb{N}$ such that for all $f_1, ..., f_{K+1} \in A$ there are $g_1, ..., g_K \in A$ with*

$$(D(f_1) \cup ... \cup D(f_{K+1})) \cap (\operatorname{Spec} A)^{\min} = (D(g_1) \cup ... \cup D(g_K)) \cap (\operatorname{Spec} A)^{\min}?$$

We are asking for a weak converse of Remark 10.3.

For a while it was an open question whether $\mathrm{AS}(A) = 1$ is implied by the compactness of $(\operatorname{Spec} A)^{\min}$. However, Quentel constructed a ring A with compact minimal prime spectrum such that $\mathrm{AS}(A) \geq 2$ (cf. [Qu]; see also [Gl], p.117 ff). We present a construction that is a considerably more general than Quentel's, but was inspired by his method. We construct a reduced ring A with compact minimal spectrum such that $\mathrm{AS}(A) = \omega$ (cf. Theorem 10.16 below). Recall from 3.4, that $\mathrm{AS}(A) = \omega$ is equivalent to saying that some (countable) ultra power of A does not have compact minimal spectrum. In particular the existence of our ring shows that the class of reduced rings with compact minimal spectrum is not elementary.

We start by setting up a framework for our construction. This includes the notion of so-called *T-algebras*, as well as some of their basic properties.

Throughout, C denotes an algebraically closed field of arbitrary characteristic. Given any set I, we consider the C-algebra C^I of functions from I to C. Any set map $p : J \to I$ defines the homomorphism $p^* : C^I \to C^J$, $a \to a \circ p$. We consider C-algebras A together with injective homomorphisms $\varphi_A : A \to C^I$. Such homomorphisms are called *representations as function rings.* A map from one representation $\varphi_A : A \to C^I$ to another one, $\varphi_B : B \to C^J$, consists of a homomorphism $f : A \to B$ and a set map $p : J \to I$, such that $p^* \circ \varphi_A = \varphi_B \circ f$.

The evaluation at an element $i \in I$ is a homomorphism $\hat{i} : A \to C$. We define $\hat{I} := \{\ker \hat{i} \mid i \in I\}$.

Lemma 10.5. (i) $\hat{I} \subseteq (\operatorname{Spec} A)^{\max}$.

(ii) $\bigcap \hat{I} = \{0\}$, *in particular* $\operatorname{Jac} A = \{0\}$.

(iii) $(\operatorname{Spec} A)^{\min} \subseteq \overline{\hat{I}}^{\mathrm{con}}$.

Proof. (i) holds since $C \subseteq A$, hence every C-algebra homomorphism $A \longrightarrow C$ is surjective. (ii) holds since $f \neq 0$ means $f(i) \neq 0$ for some $i \in I$, thus $f \notin \ker(\hat{i})$. (iii) follows from (ii) and 2.8. □

The notation for zero sets and co-zero sets of elements of C^I has been introduced before: Given $a \in C^I$ we write $\mathrm{Z}_I(a) = \{i \in I \mid a(i) = 0\}$ and $\mathrm{Coz}_I(a) = I \setminus \mathrm{Z}_I(a)$. If $a \in A$ then we set $\mathrm{Z}_I(a) = \mathrm{Z}_I(\varphi(a))$ and $\mathrm{Coz}_I(a) = \mathrm{Coz}_I(\varphi(a))$.

We consider the following condition on A:

(+) for every $a \in A$ there are $n \in \mathbb{N}$ and $b_1, ..., b_n \in A$ with $\mathrm{Z}_I(a) = \mathrm{Coz}_I(b_1) \cup ... \cup \mathrm{Coz}_I(b_n)$.

Lemma 10.6. *If A satisfies condition* (+), *then* $(\operatorname{Spec} A)^{\min} = \overline{\hat{I}}^{\mathrm{con}}$, *and it follows that* $(\operatorname{Spec} A)^{\min}$ *is a boolean space.*

Proof. We show that, by assumption (+), there are no proper specializations in the spectral subspace $\overline{\hat{I}}^{\mathrm{con}}$ of $\operatorname{Spec} A$: Assume by way of contradiction that $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ is a proper specialization in $\overline{\hat{I}}^{\mathrm{con}}$. Then there is some $a \in A$ with $a \in \mathfrak{q} \setminus \mathfrak{p}$. Since the prime ideals belong to $\overline{\hat{I}}^{\mathrm{con}}$ we conclude that $\mathrm{Z}_I(a) \neq \emptyset$ and $\mathrm{Coz}_I(a) \neq \emptyset$.

According to condition (+) we write $\mathrm{Z}_I(a) = \bigcup_{\nu=1}^n D(b_\nu)$. It follows that $a \cdot b_\nu = 0$ for all ν, hence $D(a) \cap D(b_\nu) = \emptyset$, and $V(a) \supseteq \bigcup_{\nu=1}^n D(b_\nu)$.

We claim that the sets $V(a) \cap \overline{\hat{I}}^{\mathrm{con}}$ and $(\bigcup_{\nu=1}^n D(b_\nu)) \cap \overline{\hat{I}}^{\mathrm{con}}$ are equal. Assume that there is some element

$$\mathfrak{r} \in V(a) \cap \overline{\hat{I}}^{\mathrm{con}} \setminus \left(\bigcup_{\nu=1}^n D(b_\nu) \right) = V(a) \cap \bigcap_{\nu=1}^n V(b_\nu) \cap \overline{\hat{I}}^{\mathrm{con}}.$$

Since we are inside $\overline{\hat{I}}^{\mathrm{con}}$ there must be some element $i \in \mathrm{Z}_I(a) \cap \bigcap_{\nu=1}^n \mathrm{Z}_I(b_\nu)$. But this is impossible by the choice of $b_1, \ldots, b_n$. The contradiction yields $V(a) \cap \overline{\hat{I}}^{\mathrm{con}} = (\bigcup_{\nu=1}^n D(b_\nu)) \cap \overline{\hat{I}}^{\mathrm{con}}$.

Because $\mathfrak{q} \in V(a)$ we now conclude that $y \in \bigcup_{\nu=1}^n D(b_\nu)$. The set $\bigcup_{\nu=1}^n D(b_\nu)$ is open, hence closed under generalization. Therefore it contains the generalization $\mathfrak{p}$ of $\mathfrak{q}$. But then there is some ν with $\mathfrak{p} \in D(a) \cap D(b_\nu)$, which is impossible. This contradiction proves our claim. There are no proper specializations in $\overline{\hat{I}}^{\mathrm{con}}$.

From 10.5(iii) we know that $(\mathrm{Spec}\, A)^{\min} \subseteq \overline{\hat{I}}^{\mathrm{con}}$. If the containment is proper then any element $\mathfrak{q} \in \overline{\hat{I}}^{\mathrm{con}} \setminus (\mathrm{Spec}\, A)^{\min}$ has a proper generalization $\mathfrak{p}$ in $(\mathrm{Spec}\, A)^{\min}$, and there is a proper specialization in $\overline{\hat{I}}^{\mathrm{con}}$. But we have seen that this is impossible. □

From 10.6 and 10.5(i) we see that, assuming (+), $\hat{I}$ consists of points that are both minimal and maximal in $\mathrm{Spec}\, A$.

Definition 10.7. A representation $\varphi_A : A \to C^I$ is called a **T-algebra**, if every non-constant function from A has a zero in I.

Suppose that $\varphi_A : A \to C^I$ is a representation of a C-algebra and that $p : J \to I$ is a surjective map of sets. Then the homomorphism $p^* : C^I \to C^J$ is injective and the composition $p^* \circ \varphi_A : A \to C^J$ is a representation as well. Moreover for every $a \in A$ we have $\mathrm{Z}_J(a) = p^{-1}(\mathrm{Z}_I(a))$, i.e., if φ_A is a T-algebra, then $p^* \circ \varphi_A$ is a T-algebra as well.

Observe that T-algebras exist: Let φ be the canonical monomorphism from the polynomial ring $C[X]$ into the ring of functions C^C. Since C is algebraically closed, every non-constant polynomial has a zero in C. Thus, $\varphi : C[X] \to C^C$ is a T-algebra. If $p : I \to C$ is a surjective map, then the composition $p^* \circ \varphi : C[X] \to C^I$ is also a T-algebra.

Here are some simple properties of T-algebras:

Lemma 10.8. *If $\varphi_A : A \to C^I$ is a T-algebra then*

(i) *an element $a \in A$ is constant, i.e., belongs to C, if and only if there is some $b \in A$ such that $Z_I(a) = \mathrm{Coz}_I(b)$;*
(ii) *the only idempotent elements of A are* 0 *and* 1*.*

In particular, $\mathrm{Spec}\, A$ *is connected and A is von Neumann regular if and only if $A = C$.*

Proof. (i). If $a \in C$, then trivially $Z_I(fa) = \mathrm{Coz}_I(b)$ where $b = 0$ if $a \neq 0$ and $g = 1$ if $a = 0$. Conversely, let $Z_I(a) = \mathrm{Coz}_I(b)$. Pick some $c \in C \setminus \{0, 1\}$. Then $a - b$ and $b - c \cdot a$ both do not have any zeroes in I. Since A is a T-algebra, this implies that $a - bg, b - c \cdot a \in C$. It follows $a = (1-c)^{-1} \cdot (1-c)a = (1-c)^{-1} \cdot ((a-b) + (b - c \cdot a)) \in C$.

(ii). If $e \in A$ with $e^2 = e$, then $Z_I(e) = \mathrm{Coz}_I(1-e)$ and (i) says $e \in C$. □

In [Qu], Quentel constructs a T-algebra A satisfying (+) such that $A \neq C$. Then $\mathrm{Spec}\, A^{\min}$ is compact (by 10.6) and we note that $\mathrm{AS}(A) \geq 2$: Given $a \in A \setminus C$, assume that $V(a) \cap (\mathrm{Spec}\, A)^{\min} = D(b) \cap (\mathrm{Spec}\, A)^{\min}$ for some $b \in A$. Then $Z_I(a) = \mathrm{Coz}_I(b)$ (10.6), and a is constant by 10.8(i), a contradiction.

We present another construction that was inspired by Quentel's method. It leads to T-algebras whose behavior with regard to zero sets and co-zero sets can be prescribed rather freely.

We fix a representation $\varphi_A : A \to C^I$ of a C-algebra, a non-empty subset M of $A \setminus C$ and an integer $k \geq 2$. Starting from these data we construct an extension of φ_A:

- We consider the affine space C^k and its subset $\mathcal{T} := C^k \setminus \{0\}$. Then we form the possibly infinite dimensional affine space $(C^k)^M$ and its subset $\mathcal{T}^M$.
- The affine space $(C^k)^M$ has projections onto the coordinates, which are denoted by $t_{\kappa,a}$, $\kappa \in \{1, \dots k\}$, $a \in M$. The restriction of a coordinate function to $\mathcal{T}^M$ is also denoted by $t_{\kappa,a}$. Given an element $x \in (C^k)^M$ we write $x_{\kappa,a} = t_{\kappa,a}(x)$. Thus, x will be represented by the family $(x_{\kappa,a})_{(\kappa,a)\in\{1,\dots,k\}\times M}$. The coordinate functions $t_{\kappa,a}(x)$ belong to the ring $C^{\mathcal{T}^M}$.
- We form the product $I \times \mathcal{T}^M$ and consider the projection $\pi_I : I \times \mathcal{T}^M \to I$. The rings $C^{I\times\mathcal{T}^M}$ and $(C^{\mathcal{T}^M})^I$ will be identified canonically. The projection π_I defines the ring homomorphism $\pi_I^* : C^I \to C^{I\times\mathcal{T}^M} = (C^{\mathcal{T}^M})^I$. The maps $\pi_i : \mathcal{T}^M \to \{i\} \times \mathcal{T}^M \hookrightarrow I \times \mathcal{T}^M$ define homomorphisms $\pi_i^* : C^{I\times\mathcal{T}^M} = (C^{\mathcal{T}^M})^I \to C^{\mathcal{T}^M}$. If we view $(C^{\mathcal{T}^M})^I$ as a direct product of rings then this is the projection onto the i-th component. We observe that the composition $\pi_i^* \circ \pi_I^* \circ \varphi_A : A \to C^{\mathcal{T}^M}$ maps $a \in A$ to the constant function $a(i) \in C^{\mathcal{T}^M}$.
- The projection $\pi_M : I \times \mathcal{T}^M \to \mathcal{T}^M$ yields a ring homomorphism $\pi_M^* : C^{\mathcal{T}^M} \to C^{I\times\mathcal{T}^M}$. From the point of view of the direct product $(C^{\mathcal{T}^M})^I$ this is the diagonal map. The images of the coordinate functions $t_{\kappa,a}$ are denoted by $T_{\kappa,a}$.
- If $a \in M$ then we define $\chi_a \in C^{I\times\mathcal{T}^M}$ to be the characteristic function of $\mathrm{Z}_{I\times\mathcal{T}^M}(a) = \mathrm{Z}_{I\times\mathcal{T}^M}(\pi_I^* \circ \varphi_A(a))$.
- For $\kappa \in \{1, ..., k\}$ and $a \in M$ we define
$$S_{\kappa,a} = \chi_a \cdot T_{\kappa,a} \in C^{I\times\mathcal{T}^M}.$$
Hence for $x \in \mathcal{T}^M$ and $i \in I$ we have
$$S_{\kappa,a}(i, x) = \begin{cases} x_{\kappa,a} & \text{if } a(i) = 0 \\ 0 & \text{if } a(i) \neq 0. \end{cases}$$
- We define A_M to be the C-subalgebra
$$\pi_M^* \circ \varphi_A(A)[S_{\kappa,a} \mid (\kappa, a) \in \{1, ..., k\} \times M]$$
of $C^{I\times\mathcal{T}^M}$. The inclusion homomorphism is denoted by φ_{A_M}. Each element $b \in A_M$ has a representation $b = P(S_{\kappa,a} \mid (\kappa, a) \in \{1, ..., k\} \times M)$, where $P \in A[X_{\kappa,a} \mid (\kappa, a) \in \{1, ..., k\} \times M]$ is a polynomial in the variables $X_{\kappa,a}$. The image of A_M under the homomorphism π_i^* is the polynomial ring
$$C[t_{\kappa,a} \mid (\kappa, a) \in \{1, ..., k\} \times M,\ a(i) = 0].$$

We shall now analyze the properties of the C-algebra A_M and its representation φ_{A_M}.

The entire construction we have exhibited depends on the chosen integer k and the chosen set M. The integer will be fixed when we study A_M, but we shall vary the set M. If $N \subseteq M$ then $\mathcal{T}^M = \mathcal{T}^N \times \mathcal{T}^{M \setminus N}$. The projection $\pi_{N,M} : I \times \mathcal{T}^M \longrightarrow I \times \mathcal{T}^N$ is surjective and yields the injective homomorphism $\pi^*_{N,M} : C^{I\times\mathcal{T}^N} \longrightarrow C^{I\times\mathcal{T}^M}$. We identify A_N with its image under $\pi^*_{N,M}$. Then A_M is the union of the directed set of sub-algebras A_N, where N varies in the set of finite subsets of M. Note that every finite subset of A_M is contained in some A_N, $N \subseteq M$ finite. Therefore in many arguments that involve only finitely many elements of A_M we may assume that M itself is finite.

The following results are concerned mostly with zero sets and co-zero sets of elements of A_M .

Remark 10.9. If M is finite, then $\mathcal{T}^M$ is a Zariski open subset of $(C^k)^M$, and the Zariski dimension of $(C^k)^M \setminus \mathcal{T}^M$ is $k \cdot (|M| - 1)$. Moreover, if $r \leq k - 1$ and $f_1, ..., f_r$ are polynomial functions on $(C^k)^M$ that have a common zero, then the zero set $Z_{(C^k)^M}(f_1, ..., f_r)$ has dimension at least $k\cdot|M|-r \geq k\cdot|M|-(k-1)$. Hence $Z_{(C^k)^M}(f_1, ..., f_r)$ can not be contained in $(C^k)^M \setminus \mathcal{T}^M$, and $Z_{\mathcal{T}^M}(f_1, ..., f_r) \neq \emptyset$.

Lemma 10.10. *An element $b \in A_M$ belongs to A if and only if every polynomial $\pi^*_i(b)$ is constant.*

Proof. If $b \in A$ then $\pi^*_i(b)$ is the constant polynomial $b(i)$. Conversely, suppose that each $\pi^*_i(b)$ is constant. We write $b = P(S_{\kappa,a} \mid \kappa, a$, where

$$P = \sum a_\omega X^\omega \in A\left[X_{\kappa,a} \mid \kappa, a\right]$$

and ω is a multi-index. If $a_\omega = 0$ for all $\omega \neq 0$, then $b = a_0 \in A$. Otherwise, pick $\omega \neq 0$ with $a_\omega \neq 0$. There is some $i \in I$ with $a_\omega(i) \neq 0$. Then the polynomial $\pi^*_i(b) = \sum a_\omega(i)X^\omega$ is non constant, which contradicts our assumption. □

Lemma 10.11. *If $a \in M$, then*

$$\mathrm{Z}_{I\times\mathcal{T}^M}(a) = \bigcup_{\kappa=1}^{k} \mathrm{Coz}_{I\times\mathcal{T}^M}(S_{\kappa,a}).$$

Proof. Observe that $\mathrm{Z}_{I\times\mathcal{T}^M}(a) = \pi_I^{-1}(\mathrm{Z}_I(a))$. If $(i,x) \in \mathrm{Z}_{I\times\mathcal{T}^M}(a)$, then $a(i) = 0$ and $S_{\kappa,a}(i,x) = x_{\kappa,a}$. Since $x \in \mathcal{T}^M$, there is some $\kappa \in \{1, ..., k\}$ with $S_{\kappa,a}(i,x) = x_{\kappa,a} \neq 0$. We see that $(i,x) \in \mathrm{Coz}_{I\times\mathcal{T}^M}(S_{\kappa,a})$.

For the reverse inclusion, suppose that $(i,x) \notin \mathrm{Z}_{I\times\mathcal{T}^M}(a)$, i.e., $a(i) \neq 0$. Then $S_{\kappa,a}(i,x) = 0$ for all κ. □

Lemma 10.12. *If $b \in A_M \setminus A$, then $\mathrm{Z}_{I\times\mathcal{T}^M}(b)$ is not a finite union of co-zero sets of elements of A_M.*

Proof. Assume by way of contradiction that

$$\mathrm{Z}_{I\times\mathcal{T}^M}(b) = \bigcup_{\rho=1}^{r} \mathrm{Coz}_{I\times\mathcal{T}^M}(a_\rho).$$

There is a finite subset $N \subseteq M$ such that $b, a_1, \ldots, a_r \in A_N$. We may replace M by N, i.e., we may assume that M is finite.

Since $b \notin A$, 10.10 gives some $i \in I$ such that

$$\pi_i^*(b) \in C[t_{\kappa,a} \mid (\kappa, a) \in \{1, ..., k\} \times M, a(i) = 0]$$

is not a constant polynomial. Hence $\mathrm{Z}_{(C^k)^M}(\pi_i^*(b)) \subseteq (C^k)^M$ is a proper Zariski-closed set, and, by 10.9, $\mathrm{Z}_{\mathcal{T}^M}(\pi_i^*(b)) \neq \emptyset$, which means that there is some $x \in \mathcal{T}^M$ with $(i, x) \in \mathrm{Z}_{I\times\mathcal{T}^M}(b)$. The assumption yields an index $\rho \in \{1, ..., r\}$ with $(i, x) \in \mathrm{Coz}_{I\times\mathcal{T}^M}(a_\rho) \subseteq \mathrm{Z}_{I\times\mathcal{T}^M}(b)$. This shows that the proper Zariski-closed subset $\mathrm{Z}_{(C^k)^M}(\pi_i^*(b)) \subseteq (C^k)^M$ contains the nonempty Zariski open set $\mathrm{Coz}_{\mathcal{T}^M}(\pi_i^*(a_\rho))$, which is impossible. This contradiction finishes the proof. □

Lemma 10.13. *If $b \in A$ and $\mathrm{Z}_{I\times\mathcal{T}^M}(b)$ is the union of $k-1$ co-zero sets of elements of A_M, then $\mathrm{Z}_I(b)$ is the union of $k-1$ co-zero sets of elements of A.*

Proof. Again we may assume that M is finite. Let

$$\mathrm{Z}_{I\times\mathcal{T}^M}(b) = \bigcup_{\rho=1}^{k-1} \mathrm{Coz}_{I\times\mathcal{T}^M}(b_\rho)$$

with $b_\rho \in A_M$ and let

$$P_\rho = \sum_\omega a_{\rho,\omega} X^\omega \in A\,[X_{\kappa,a} | \kappa, a]$$

with $b_\rho = P_\rho(S_{\kappa,a} \mid \kappa, a)$. We show that $\mathrm{Z}_I(b) = \bigcup_{\rho=1}^{k-1} \mathrm{Coz}_I(a_{\rho,0})$.

If $a_{\rho,0}(i) \neq 0$, then the constant term of the polynomial $\pi_i^*(b_\rho)$ is not 0, and there is some $x \in \mathcal{T}^M$ with $\pi_i^*(b_\rho)(x) \neq 0$. Then $(i, x) \in \mathrm{Coz}_{I\times\mathcal{T}^M}(b_\rho) \subseteq \mathrm{Z}_{I\times\mathcal{T}^M}(b)$, which means $b(i) = 0$.

Conversely suppose $b(i) = 0$. Then $\mathcal{T}^M = \mathrm{Z}_{\mathcal{T}^M}(\pi_i^*(b)) \subseteq \bigcup_{\rho=1}^{k-1} \mathrm{Coz}_{\mathcal{T}^M}(\pi_i^*(b_\rho))$. Therefore

$$\bigcap_{\rho=1}^{k-1} \mathrm{Z}_{(C^k)^M}(\pi_i^*(b_\rho)) \subseteq (C^k)^M \setminus \mathcal{T}^M.$$

By 10.9 we conclude that $\bigcap_{\rho=1}^{k-1} Z_{(C^k)^M}(\pi_i^*(b_\rho)) = \emptyset$. Then the constant coefficient of one of the polynomials $\pi_i^*(b_\rho)$ is not 0. The constant coefficient of $\pi_i^*(b_\rho)$ is $a_{\rho,0}(i) \neq 0$, and it follows that $i \in \bigcup_{\rho=1}^{k-1} \mathrm{Coz}_I(a_{\rho,0})$. □

Now the construction of the algebra A_M, together with its representation in $C^{I\times\mathcal{T}^M}$, will be applied recursively to produce an increasing sequence of C-algebras with representation: Let $2 \leq k_0 \leq k_1 \leq ...$ be an increasing sequence of integers. We define

- $A_{-1} = C$.
- $I_0 = C$, $A_0 = C[X]$, the univariate polynomials considered as a subalgebra of C^C.
- Suppose that $A_n \subseteq C^{I_n}$ has been defined. Then we apply the construction above using the following data:
  - The C-algebra $A_n \subseteq C^{I_n}$, and
  - the integer k_n, and
  - the subset $M_n = A_n \setminus A_{n-1}$.

  With $\mathcal{T}_n = C^{k_n} \setminus \{0\}$ and $I_{n+1} = I_n \times \mathcal{T}_n^{M_n}$ the construction yields the C-algebra $A_{n+1} \subseteq C^{I_{n+1}}$.

Finally we form the union of the increasing sequence $(A_n \subseteq C^{I_n})_{-1\leq n}$ of C-algebras with representation: First we define the representation set, which is the projective limit $I_\infty = \varprojlim I_n$, where the transition maps are the canonical projection $\pi_{n,n+1} : I_{n+1} \to I_n$. Let $\pi_{n,\infty} : I_\infty \to I_n$ be the canonical maps from the projective limit to the components. We use the injective homomorphisms $\pi^*_{n,\infty}$ to consider each C^{I_n} as a sub-algebras of C^{I_∞}. Thus, the C^{I_n} form an increasing sequence of sub-algebras of C^{I_∞}. We identify A_n with its image and define $A_\infty = \bigcup_{n=-1}^{\infty} A_n$. It is obvious that A_∞ is the direct limit of the sequence $(A_n)_{-1\leq n}$.

Lemma 10.14. *Consider the following statements about an element $a \in A_\infty$.*

(a) $a \in A_n$.

(b) *There are elements $a_1, ..., a_{k_n} \in A_\infty$ such that $\mathrm{Z}_{I_\infty} = \bigcup_{\rho=1}^{k_n} \mathrm{Coz}_{I_\infty}(a_\rho)$.*

It is always true that (a) *implies* (b). *If $k_n < k_{n+1}$, then also* (b) *implies* (a).

Proof. (a)⇒(b). Suppose that $a \in A_m \setminus A_{m-1}$, where $m \leq n$. By 10.11, $\mathrm{Z}_{I_{m+1}}(a)$ is the union of k_m cozero sets of elements from A_{m+1}. Pulling this back to I_∞ via $\pi_{m+1,\infty}$ we obtain (b) (note that $k_m\ leqk_n$).
(b)⇒(a). Assume $a \notin A_n$, say $a \in A_{r+1} \setminus A_r$ with $r \geq n$. Suppose that $s \geq r+1$ is the least number such that $\mathrm{Z}_{I_s}(a) = \bigcup_{\rho=1}^{k_n} \mathrm{Coz}_{I_s}(a_\rho)$ with $a_1, \ldots, a_{k_n} \in A_s$. By 10.12 we know that $r+1 < s$, in particular $k_n < k_{n+1} \leq k_{r+1} \leq k_s$. Since $a \in A_{r+1} \subseteq A_{s-1}$, 10.13 implies that $\mathrm{Z}_{I_{s-1}}(a)$ is a union of k_n cozero sets of elements of A_{s-1}, which contradicts the minimality of s. □

We shall now show that the construction we have presented produces T-algebras if it is applied to T-algebras:

Proposition 10.15. (i) *Suppose $\varphi_A : A \to C^I$ is a* T*-algebra, that $2 \leq k \in \mathbb{N}$ and $M \subseteq A$ is nonempty. Then $\varphi_{A_M} : A_M \to C^{I\times\mathcal{T}^M}$ is a* T*-algebra.*

(ii) *If $(\varphi_n : A_n \to C^I)_n$ is a monotonically increasing sequence of* T*-algebras with representations in the same ring of functions then $\varphi : \bigcup_n \varphi_n A_n \to C^I$ is a T-algebra.*

Proof. (i). We have to show $Z_{I\times\mathcal{T}^M}(b) \neq \emptyset$ for every $b \in A_M \setminus C$. We may assume that M is finite. If $b \in A$, then the claim follows from the hypothesis that A is a T-algebra. Assume now that $b \notin A$. By 10.10, there is some $i \in I$ such that the polynomial $\pi_i^*(b) \in C[t_{\kappa,a} \mid \kappa, a]$ is not constant. Hence $\mathrm{Z}_{(C^k)^M}(\pi_i^*(b)) \neq \emptyset$, and, by 10.9, also $\mathrm{Z}_{\mathcal{T}^M}(\pi_i^*(b)) \neq \emptyset$. - Part (ii) is obvious. □

Finally, the previous results are combined to show that the class of reduced rings with compact minimal prime spectrum is not axiomatizable. The main step is the following

Theorem 10.16. (i) *The representation $\varphi_\infty : A_\infty \longrightarrow C^{I_\infty}$ constructed above is a T-algebra satisfying condition (+).*

(ii) *If the sequence $(k_n)_n$ is unbounded then the number of cozero sets that are needed to write a zero set as a union of co-zero sets is unbounded. In particular,* $\mathrm{AS}(A_\infty) = \omega$.

(iii) *If the sequence $(k_n)_n$ is bounded with maximum k, then* $\mathrm{AS}(A_\infty) = k$.

Proof. (i). By 10.15, $\varphi_\infty : A_\infty \longrightarrow C^{I_\infty}$ is a T-algebra. Condition (+) is satisfied by the implication (a)⇒(b) of 10.14.

(ii) follows from 10.14, (b)⇒(a).
(iii). It follows from 10.11 that $\mathrm{AS}(A_\infty) \le k$, and 10.12 and 10.13 imply $\mathrm{AS}(A_\infty) \ge k$. □

Corollary 10.17. *A_∞ has compact minimal spectrum, but in case the sequence $(k_n)_n$ is unbounded, some ultrapower of A_∞ does not have compact minimal spectrum. In particular, the class of all reduced rings with compact minimal spectrum is not elementary.*

Proof. We know from 10.16 (i) and 10.6 that A_∞ has compact minimal spectrum. The remaining part of the assertion follows from 10.16 (ii) and 3.4. □

11. Summary of axiomatizability

We give a summary of our results about the axiomatizability of classes of rings defined by properties of their Zariski spectra. The table below is to be read as follows: The entries in the first column contain properties of Spec A. The second column contains the letters "Y" or "N", according as the class of *reduced* rings whose Zariski spectrum satisfies the property in the first column, is or is not first order in the language of rings. The third column has to be read in the same manner for the class of *all* rings. Note that, given an axiomatizable class $\mathcal{C}$ of rings, the class of all reduced rings in $\mathcal{C}$ is elementary, too.

After each entry in the second and third columns, we give a reference to the text, or, in the case of well-known facts, we just name the elementary class.

Spec A	A reduced	A not reduced
normal	Y	Y, 4.3
completeley normal	N, 6.12	N
boolean	Y, v. N. regular	N, 6.8 and 6.7
singleton	Y, fields	N, 6.8 and 6.7
finite	N, $(*)$	N
linear	N, 6.12	N
inversely normal	Y, 7.6(i)	N, 7.6(ii)
inversely completely normal	N, 7.6(iii)	N
minimal points compact	N, 10.17	N
minimal points singleton	Y, domains	N, 6.8 and 3.2
maximal points hausdorff	Y	Y, 4.5
maximal points boolean	Y	Y, 4.7
$(\mathrm{Spec}\,A)^{\max}$ proconstructible	Y	Y, 4.6
maximal points singleton	Y	Y, local rings

$(*)$. The class of reduced rings with finite spectrum is not elementary, since no free ultra product of a family $(A_n)_{\in\mathbb{N}}$, A_n a product of n fields, has finite spectrum.

We point out that all classes of rings in the table are pseudo elementary.

References

[Ar-Ma] G. Artico, U. Marconi, **On the compactness of minimal spectrum.** *Rend. Sem. Mat. Univ. Padova 56 (1976), 79–84 (1978)*

[Ca] M. Carral, **K-theory of Gel'fand rings.** *J. Pure Appl. Algebra 17 (1980), no. 3, 249–265*

[Ca-Co] M. Carral, M. Coste, **Normal spectral spaces and their dimensions.** *J. Pure Appl. Algebra 30 (1983), no. 3, 227–235*

[Co1] M. Contessa, **On pm-rings.** *Comm. Algebra 10 (1982), no. 1, 93–108*

[Co2] M. Contessa, **On certain classes of pm-rings.** *Comm. Algebra 12 (1984), no. 12, 1447–1469*

[Co] W. H. Cornish, **Normal lattices.** *J. Austral. Math. Soc. 14 (1972), 200–215*

[dM-Or] G. De Marco, A. Orsatti, **Commutative rings in which every prime ideal is contained in a unique maximal ideal.** *Proc. Amer. Math. Soc. 30 1971 459–466*

[Gi] R. Gilmer, **Multiplicative ideal theory.** *Queen's Papers in Pure and Applied Mathematics, no. 12; Queen's University, Kingston, ON, 1968*

[Gl] S. Glaz, **Commutative coherent rings.** *Lecture Notes in Mathematics, 1371. Springer-Verlag, Berlin, 1989. xii+347 pp*

[Ha] R. Hartshorne, **Algebraic Geometry.** *Graduate Texts in Mathematics, vol. 52, Springer, New York 1977*

[He-Je] M. Henriksen, M. Jerison, **The space of minimal prime ideals of a commutative ring.** *Trans. Amer. Math. Soc. 115, pp. 110–130 (1965)*

[He-Wo] M. Henriksen, R. G. Woods, **Cozero complemented spaces; when the space of minimal prime ideals of a $C(X)$ is compact.** *Topology Appl. 141 (2004), no. 1-3, 147–170*

[Hoc] M. Hochster, **Prime ideal structure in commutative rings.** *Trans. Amer. Math. Soc. 142 (1969) 43-60*

[Hoc2] M. Hochster, **The minimal prime spectrum of a commutative ring.** *Canad. J. Math. 23 (1971), 749–758*

[Ho] W. Hodges, **Model Theory.** *Encyclopedia of mathematics and its applications, vol. 42, 1993*

[Hu] J. A. Huckaba, **Commutative rings with zero divisors.** *Monographs and Textbooks in Pure and Applied Mathematics, 117. Marcel Dekker, Inc., New York, 1988. x+216 pp.*

[Joh] P.T. Johnstone, **Stone spaces.** *Reprint of the 1982 edition. Cambridge Studies in Advanced Mathematics, 3. Cambridge University Press, Cambridge, 1986. xxii+370 pp.*

[Ka] Kaplansky, Irving, **Rings of operators.** *W. A. Benjamin, Inc., New York-Amsterdam 1968 viii+151 pp. 46.65*

[Ki] J. Kist, **Two characterizations of commutative Baer rings.** *Pacific J. Math. 50 (1974), 125–134*

[La] D. Laksov, **Radicals and Hilbert Nullstellensatz for not necessarily algebraically closed fields.** *Enseign. Math. (2) 33 (1987), no. 3-4, 323–338*

[Me] Mewborn, Ancel C, **Regular rings and Baer rings.** *Math. Z. 121 (1971), 211–219*

[Me2] Mewborn, Ancel C, **Some conditions on commutative semiprime rings.** *J. Algebra 13 (1969) 422–431*

[Pi] Picavet, Gabriel, **Ultrafiltres sur un espace spectral - anneaux de Baer - anneaux spectre minimal compact.** *Math. Scand. 46 (1980), no. 1, 23–53*

[Qu] Y. Quentel, **Sur la compacité du spectre minimal d'un anneau.** *Bull. Soc. Math. France 99, 265 - 272 (1971)*

[Schw] N. Schwartz, **Rings of continuous functions as real closed rings.** *Ordered algebraic structures (Curaao, 1995), 277–313, Kluwer Acad. Publ., Dordrecht, 1997*

[Sch-Tr] N. Schwartz, M. Tressl, **Boolean factor spaces of Zariski spectra.** *In preparation*

[Tr] M. Tressl, **Computation of the z-radical in $C(X)$.** *Advances in Geometry 6 (2006), no. 1, 139-175*

Universität Passau, FIM, D-94030 Passau, Germany
E-mail address: schwartz@fmi.uni-passau.de

University of Manchester, School of Mathematics, Manchester M13 9PL, UK
E-mail address: marcus.tressl@manchester.ac.uk