PROJECTS FOR M.SC. IN PURE MATHEMATICS AND MATHEMATICAL LOGIC

MATH61000 PROJECT

Credits	20
Staff/student contact hours	7
Private study hours	193
Total study hours	200
Assessment	Written report and short oral examination
Level	MSc
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The project is written in the period
week 9 of **semester 1** – first day of the exam period of **semester 2**.

Contact details of project supervisors may be found on the Staff Directory

Project topics

- **Fusion systems of p-groups** (Cesare G. Ardito)
The project will be about fusion systems, which are new objects being studied in group theory and in particular representation theory. While defining a fusion system in a group is straightforward, the more general definition requires notions from category theory, whose basic concepts will form part of the project. The main goal is to understand how abstract fusion systems are defined, and how this definition is related to fusion systems that come from finite groups.
The project can then take many different directions in exploring classical or more recent applications of fusion systems in group theory and representation theory, or the existence of exotic fusion systems.
Fusion systems appeared in many different results, and they form one of the most promising and interesting new research direction in group theory. For instance, Aschbacher's new proof of the classification of finite simple groups (arguably the most important result of the 20th century in group theory) heavily relies on this theory.
References:
  - M. Linckelmann, An introduction to fusion systems.
  - D. Craven, Fusion Systems: Group theory, representation theory, and topology.
  - M. Aschbacher, R. Kessar, B. Oliver, Fusion Systems in Algebra and Topology.
  - S. MacLane, Categories for the Working Mathematician.
- **Hopf algebras** (Yuri Bazlov)
Please contact the project supervisor for more details of this project

- **Sieve methods and distribution of prime numbers** (Hung Bui)
Sieve methods dated back more than 2000 years ago from Eratosthenes of Cyrene to determine all the primes up to a certain point. The theory has had a remarkable development over the last 100 years, and is now a powerful tool to study the distribution of prime numbers. Most notably is the recent breakthrough of Zhang and Maynard on small gaps between prime numbers, which gives a partial answer to the famous Twin Prime Conjecture.

  Particular directions of the project can vary depending on the student's background and interests.

- **Polynomial methods in combinatorics** (Jan Dobrowolski)
The polynomial method is a technique allowing application of basic algebraic properties of polynomials to problems in combinatorics that a priori have no obvious connection to polynomials. In recent years, the method has proved very successful in providing surprisingly simple solutions to a number of problems in incidence geometry, i.e. a combinatorial study of the relation "a point lies on a line", especially in vector spaces over finite fields. The aim of the project is to understand this technique and to survey its main applications in combinatorics, with a focus on incidence geometry.

  Reference: A. Sheffer, Polynomial Methods and Incidence Theory, Cambridge University Press, 2022.

- **Fusion systems** (Charles Eaton)
Fusion in finite groups is about conjugacy (or fusion) of subgroups of a Sylow p-subgroup by elements of the group containing it. The classical result, by Burnside, is that if the Sylow p-subgroup is abelian, then two subgroups that are fused in the larger group must be fused within the normalizer of the Sylow p-subgroup. Study of patterns of fusion is fundamental in the classification of finite simple groups. A project would centre around the theory of fusion of finite groups. A subsequent dissertation would involve the modern theory of fusion systems, which are a categorical construction generalizing fusion in finite groups, with the possibility of contributing some new calculations.

  Fusion in finite groups is covered in Daniel Gorenstein's book Finite Groups. The theory of fusion systems is treated in David Craven's book (you've guessed it) "The theory of fusion systems".

- **Modular representation theory** (Charles Eaton)
The project would follow on from the Noncommutative Algebra course, specializing in the structure of modules of group algebras for finite groups touched on in that course. A dissertation would continue with basic algebras, which are minimal objects capturing the structure of the algebras being studied.

- **Group actions and invariant theory** (Florian Eisele)
If a group G acts on $\mathbb{C}^n$ we can construct an action of the same group on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The invariants of G are the polynomials fixed under this action, and they form a ring $\mathbb{C}[x_1, \ldots, x_n]^G$ called an invariant ring. These invariant rings have a particularly nice structure theory when the group G is finite, which you will explore in this project. You will also describe some examples of invariant rings explicitly.
References:
  - B. Sturmfels. *Algorithms in Invariant Theory.*

- **Path algebras and their representations** (Florian Eisele)
Similar to the notion of a group acting on a set, we can study actions of directed graphs Q, known as quivers in this context, on vector spaces like $\mathbb{C}^n$. Such an action is called a representation of Q. This leads to the definition of a path algebra $\mathbb{C}Q$, and modules over path algebras.
The first aim of this project would be to understand these algebraic structures. The next goal would be to understand the statement and at least part of the proof of Gabriel's theorem, which states that a quiver Q has only finitely many indecomposable representations (the "building blocks" of arbitrary representations) if and only if the undirected version of Q is a Dynkin diagram (these diagrams pop up in various areas of mathematics). You could then work out the indecomposable representations explicitly in some examples.
In a subsequent dissertation you would look at one or more other families of finite-dimensional algebras, e.g. string algebras, gentle algebras, Brauer tree algebras, ... (see third reference for more possibilities).
References:
  - D.J. Benson. *Representations and cohomology. I. Basic representation theory of finite groups and associative algebras.*
  - M. Auslander, I. Reiten, S.O. Smalø. *Representation Theory of Artin Algebras*
  - FD Atlas

- **The λ-calculus** (Nicola Gambino)
What does it mean for a function to be computable? One answer this question, given by A. Church, involves a logical system known as the λ-calculus, which is the theoretical basis of modern functional programming languages. In this project, you will explore the syntax of the λ-calculus (which involves studying reduction strategies, which correspond to the process of computing the value of an expression) and its models (which provide useful ways of understanding how functions in the λ-calculus correspond to usual functions). Depending on your interest, the project can be developed either towards pure mathematics (via categorical logic) or computer science (via functional programming).
References:
[1] H. Barendregt, The Lambda Calculus, its Syntax and Semantics, College Publications, 2012.
[2] J. L. Krivine, Lambda-calculus, Types and Models, Ellis Horwood, 1993.
[3] J. R. Hindley and J. P. Seldin, Lambda-Calculus and Combinators - An Introduction, Cambridge University Press, 2008.

- **Category Theory** (Nicola Gambino)
In mathematics, we work not only with mathematical objects (e.g. sets, groups, topological spaces), but also with the corresponding mappings between them (functions, homomorphisms, continuous functions, respectively). Category theory begins with the observation that considering mathematical structures together with the associated notion of mapping is very useful. Indeed, this point of view allows us to make precise some informal analogies and to discover some unexpected connections between different parts of mathematics. In this project, you will explore the basic concepts of category theory, namely categories, functors, natural transformations, and adjunctions. Examples and applications will be taken from

different areas of mathematics (e.g. logic, algebra, topology), according to your background and personal interests.

References:

[1] S. Awodey, Category theory, Oxford University Press, 2010.
[2] T. Leinster, Basic category theory, Cambridge University Press, 2014.
[3] S. Mac Lane, Categories for the working mathematician, Springer, 1998.
[4] E. Riehl, Category theory in context, Dover Publications, 2016.

- **A topic in semigroup theory** (Marianne Johnson)
The aim of the project would be to learn some structural semigroup theory [Green's relations, regularity, Green's lemma, Rees Theorem...] and to illustrate this general theory in the context of some concrete examples.
Prerequisites: Experience of working with a range of algebraic structures such as: vector spaces, groups, rings and modules. Firm understanding of basic notions from algebra (substructures, homomorphisms, direct products, quotients, isomorphism theorems, generators and relations). Some experience in working with combinatorial structures (e.g. graphs) may be advantageous.
References: Fundamentals of Semigroup theory, J. M. Howie.

- **Group Cohomology** (Radha Kessar)
To a group G and a G-module M can be associated a sequence of abelian groups, $H^n(G, M)$, $n = 0, 1, 2, \dots$ called the cohomology groups of G with coefficients in M. Roughly speaking, cohomology groups measure the impact that transforming a sequence of G-modules through taking fixed points has on the exactness of the sequence. Cohomology theory plays an important role in the study of extensions of groups and of modules, of homotopy theoretic properties of topological spaces, and in the study of Galois field extensions. The project, which will take an algebraic viewpoint, will focus on the basic notions of the cohomology of groups, along with rudiments of homological algebra, the interpretation of low dimensional cohomology groups, and some group theoretic applications. A future dissertation would develop these ideas in a direction of interest to the student. A flavour of what to expect may be found in Chapter 17 of "Abstract algebra" by Dummit and Foote.

- **Differential Algebra** (Omar León-Sánchez)
In this project we will look at rings equipped with an additive map satisfying the classical Leibniz rule from Calculus (from an algebraic point of view). We will pay close attention to the ring of differential polynomials which is the analogue of the classical polynomial ring and study its ideal structure. The elements of this ring are the algebraic formalism of differential equations.
Reference: Differential Algebra and Algebraic Groups. E. Kolchin. Academic Press, 1973.

- **Differential Galois theory and primitive element theory.** (Omar León-Sánchez)
In this topic we will at the differential counterpart of differential Galois theory, where one studies differential polynomial equations rather than algebraic polynomial equations. Mostly we will be looking at the so-called Picard-Vessiot theory but Kolchin's strongly normal theory might also be considered. Along the way, we may look at the differential analogue of the classical primitive element theorem (which is a powerful result with applications in algebraic geometry).

Reference: Galois theory of linear differential equations. M. van der Put and M. Singer. Springer, 2003.

- **Model theory of algebraically closed fields** (Omar León-Sánchez)
  We will explore the model-theoretic properties of algebraically closed fields, such as strong minimality, quantifier elimination, elimination of imaginaries, and could look at structural theorems of definable groups in this theory.
  Reference: Model theory of fields. D. Marker, M. Messmer, A. Pillay. Lecture notes in logic, 1996.
- **Model-theoretic stability and independence** (Omar León-Sánchez)
  This project is about a branch of model theory called geometric stability theory. Roughly speaking, it is a very general abstraction of algebraic geometry where the notion of independence between sets plays a crucial role.
  Reference: Model theory: an introduction. D. Marker. Springer, 2002.
- **Groups of finite Morley rank in model theory** (Omar León-Sánchez)
  In this project groups equipped with a certain rank-function (model-theoretic Morley rank) are studied. The existence of such rank has striking consequence (that resemble properties of the so-called algebraic groups). We will look at fundamental results such as the "DCC condition" and Zilber's "Indecomposability theorem".
  Reference: Stable groups. B. Poizat. Amer. Math. Soc., 2001.
- **p-adic numbers** (Gabor Megyesi)
  Let p be a prime. One can define a metric on the integers by setting $d(m,n) = p^{-k}$, where p^k is the highest power of p dividing $m - n$. It can be also extended to the rationals. The completion of the rationals with respect to the usual metric gives the real numbers, the completion with respect to this new metric gives a different field, the field of p-adic numbers. The project would involve construction of p-adic numbers, their basic properties and applications to solving Diophantine equations.
  Reference: JWS Cassels, Local fields, CUP 1986.
- **Algebraic curves** (Gabor Megyesi)
  This project would involve the application of the Riemann-Roch theorem to study projective algebraic curves.
  Reference: K. Hulek; Elementary algebraic geometry, AMS 2003
- **Oppenheim's Conjecture** (Donald Robertson)
  Representing numbers by quadratic forms has a long history. Oppenheim conjectured in the 1950s that a quadratic form in at least three variables represents a dense set of real numbers if it is indefinite, degenerate, and not a multiple of a rational form. In the 1980s Margulis affirmed Oppenheim's conjecture using dynamics on the homogeneous space $\mathsf{SL}(3,R)/\mathsf{SL}(3,Z)$. Over the course of the project you could cover questions such as why the hypotheses are necessary, what the problem looks like in terms of dynamics on homogeneous spaces, and how ergodic theory can help to answer the dynamical question.
  References
  - Ratner's Theorems on Unipotent Flows. D. Witte Morris University of Chicago Press, 2005.
  - Ergodic Theory with a view towards Number Theory. M. Einsiedler, T. Ward. Springer, 2011.

  - Discrete subgroups and ergodic theory. In Number theory, trace formulas and discrete groups (Oslo, 1987), Pages 377–398. G. A. Marguils Academic Press, 1989.

- **Finite simple groups** (Peter Rowley)
  Please contact the project supervisor for more details of this project

- **Fractals and their dimension** (Nikita Sidorov)
  In the theory of chaotic dynamical systems there are natural invariant sets (typically, attractors or repellers) which may have very beautiful and complicated 'self-similar' features. For example, this occurs with Iterated Function Systems and Julia Sets.

  A key quantity for fractal sets is their dimension for example, either the Hausdorff dimension or box-dimension. For linear dynamical systems, the dimension is often explicitly known (for example, for the middle third Cantor set it is $\log 2/\log 3$). For nonlinear dynamical systems one has to calculate the dimension numerically.

  This project deals with different types of fractals, their construction and methods for estimating their dimension.

- **Existential Closed Structures** (Nikesh Solanki)
  Algebraically closed fields are hugely important to mathematics. They have the wonderful property that given any system of equations with coefficients from that field, if there exist solutions of that system some (larger) field then the solutions already exist in the field. As such, algebraically closed fields are what model theorists call an "existential closed" structures. In turns out that, other kinds of mathematical structures are also existential closed. This raises some interesting questions:
  - We know from this existential closedness condition of algebraically closed fields, many other wonderful properties of follow. Do equivalent properties follow for other kinds of existentially closed structures?
  - If we have a first order theory, all whose models are existentially closed, what can we say about the theory?

  The purpose of this project is to address questions like these.
  References
  - Hodges, Wilfrid, "First-order Model Theory", The Stanford Encyclopedia of Philosophy (Summer 2005 Edition), Edward N. Zalta (ed.).
  - Marker, David (2002), Model theory: An introduction, Graduate Texts in Mathematics, 217, New York, NY: Springer-Verlag, ISBN 0-387-98760-6, Zbl 1003.03034

- **Self-similar groups** (Nora Szakacs)
  Self-similar groups appeared in the seventies, initially as examples of groups that are easy to define but that enjoy exotic properties. Early famous examples include the celebrated Grigorchuk 2-group, the first known group of intermediate growth (solving a well-known problem of Milnor), and the Gupta-Sidki p-groups; these are among the easiest to understand examples of finitely generated, infinite torsion groups. A self-similar group is a group that acts faithfully on an infinite n-ary (typically binary) tree in a way that the self-similarity of the infinite tree is reflected in the group. The most interesting examples are generated by automata, that is, their generators can be encoded by a finite, edge-labeled digraph. A short

introductory read to get a flavor for the topic is [1]. While the project is about groups and their properties, the tools are more combinatorial than algebraic.
Reference: [1] M. Elder. A short introduction to self-similar groups. Austral. Math. Soc. Gaz. , 39:125 - 133, 2012

- **Model Theory of ordered algebraic structures** (Marcus Tressl)
Classical algebraic structures like groups, rings and fields are many times furnished with a natural order: Think of the ring of integers or the real field. Ordered algebraic structures can be thought of as algebraic structures equipped with an order or a partial order that is compatible with the algebraic operations (like addition and multiplication). The precise topic for the project will be decided after consultation with interested students and can be in the areas of ordered fields, lattice ordered groups, lattices or topology. The project may, but does not need to, invoke model theoretic methods. On the other hand, the project will prepare for a subsequent topic in the model theory of the structures studied in the project.
Prerequisites: Basic knowledge in commutative algebra or field theory. A first contact with partially ordered sets is desirable but not necessary. Students interested in a dissertation following this project are required to have some basic knowledge of model theory as for example taught in the model theory course offered in the taught component.
Reference: Stuart A. Steinberg, Lattice-ordered Rings and Modules
- **Model Theory of Valued Fields** (Marcus Tressl)
A valued field is a field equipped with a kind of metric (called 'valuation'). For example, for each prime p, the field of rational numbers has the p-adic valuation, which measures the exponent of a prime p in a given fraction. Another example is the field of formal power series over $\mathbb{C}$, where the valuation of a power series is the exponent n of the first term $a \cdot t^n$ in the series with $a \neq 0$. In the project, the basic theory of valued fields exhibiting many examples will be exposed. The project will build up to the dissertation, which will study the model theory of such fields.
Prerequisites: Basic knowledge in commutative algebra or field theory. Students interested in a dissertation following this project are required to have some basic knowledge of model theory as for example taught in the model theory course offered in the taught component.
Reference: A. Engler, A. Prestel: Valued Fields
- **Non-Hausdorff Topology in Logic and Algebra** (Marcus Tressl)
This is a broad topic requiring interests and some basic formation in the topics mentioned in the title. To see a fundamental connection of these topics, please have a look at Stone Duality for Boolean Algebras. Please get in touch with the project supervisor for further details of this project.
Prerequisites: Basic knowledge in set theoretic topology and either some basic ring theory, or, basic knowledge in predicate logic (as for example taught in the model theory course offered in the taught component).
References:
(a) J. Goubault-Larrecq: Non-Hausdorff Topology and Domain Theory
(b) M. Dickmann, N. Schwartz, M. Tressl: Spectral Spaces
- **Hyperbolic dynamical systems** (Charles Walkden)
A dynamical system consists of a phase space X (which may be an interval, a torus,

a Cantor set, or a more complicated space) and a map $T : X \to X$. Dynamical systems is the study of how points in the phase space X behave as one iterates them under the action of T. A dynamical system is said to be hyperbolic if there is some exponential expansion or contraction in the system: two points that are close together may (locally) move apart or together exponentially fast (this is a particularly strong form of what is popularly called 'chaos'). Hyperbolic dynamical systems form a particularly tractable class of dynamical system. In this project you could survey some examples of hyperbolic systems (the doubling map, the cat map, Smale's horseshoe, etc) and study how they are particular examples of a more general construction.

- **Fractals and iterated function systems** (Charles Walkden)
  Loosely speaking, a fractal is a subset of $\mathbf{R^n}$ that has structure at all scales: no matter how much one 'zooms in', the set remains complicated. Well-known examples of fractals include the Middle Third Cantor Set, the Sierpinksi Gasket, the von Koch Curve, etc. One can attempt to quantify how complicated a fractal is in terms of its fractal dimension, which is often a non-integer. (In fact, there are various different definitions of fractal dimension: the most commonly occurring ones being Hausdorff dimension and box dimension.) Many fractals can be constructed as limit sets for iterated function schemes (IFSs), and one can derive a formula for the dimension of this limit set in terms of quantities that appear in the IFS.
- **Hyperbolic dynamics and hyperbolic geometry** (Charles Walkden)
  Imagine the surface of the Earth. Through any point on the Earth's surface and in any direction there is a unique geodesic; in fact, the geodesic is a great circle (a circle inscribed on the sphere with the same centre and radius). Now imagine, given a starting point and direction, moving along this geodesic at unit speed; this is the geodesic flow on (the unit tangent bundle of) a sphere. In the case of a sphere, you will always return to where you started and facing in the same direction after the same amount of time; dynamically this is not very interesting and is related to the fact the sphere has constant positive curvature. One can construct the geodesic flow on (the unit tangent bundle of) a surface of constant negative curvature which have much more interesting dynamical properties. This project studies such geodesic flows using techniques from hyperbolic geometry and ergodic theory.
- **Mapping class groups of surfaces** (Richard Webb)
  Please contact the project supervisor for more details of this project
- **Outer automorphism groups of free groups** (Richard Webb)
  Please contact the project supervisor for more details of this project