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Nilpotent blocks of quasisimple groups for odd primes [1]

Jianbei An and Charles W. Eaton

Abstract

We investigate the nilpotent blocks of positive defect of the quasisimple groups for odd primes. In particular, it is shown that every nilpotent block of a quasisimple group has abelian defect groups. A conjecture of Puig concerning the recognition of nilpotent blocks is also shown for these groups.

1 Introduction

Let G be a finite group and k an algebraically closed field of odd characteristic p. A block B of kG with defect group D is said to be nilpotent if for each $Q \leq D$ and each block b_Q of $C_G(Q)$ with Brauer correspondent B we have that $N_G(Q, b_Q)/C_G(Q)$ is a p-group, where $N_G(Q, b_Q)$ is the stabilizer of b_Q under conjugation in $N_G(Q)$. In the case of the principal block B_0, D is a Sylow p-subgroup of G and $N_G(Q, b_Q) = N_G(Q)$ for each $Q \leq D$, so that B_0 is nilpotent if and only if G is p-nilpotent (i.e., G has a normal p-complement). Note that every block of defect zero must be nilpotent, and the classification of blocks of defect zero for finite simple groups has been the subject of a separate program of research, culminating in [21]. Hence we give attention here only to blocks with non-central defect groups.

Explicit characterizations of nilpotent blocks are obtained for classical groups, and these are used to prove:

Theorem 1.1 *Let G be a finite quasisimple group and let B be a nilpotent p-block of G with p odd. Then B has abelian defect groups.*

The second main result concerns the conjecture of Puig which states that a block B of G is nilpotent if and only if $l(b_Q) = 1$ for each p-subgroup Q and each block b_Q of $C_G(Q)$ with Brauer correspondent B (where $l(b_Q)$ is the number of irreducible Brauer characters in b_Q). The necessary condition for nilpotency is well-known. The converse is known for blocks with abelian defect groups (see [30]), and is also known to be a consequence of Alperin's weight conjecture (see [33]). We prove that:

Theorem 1.2 *Let G be a finite quasisimple group and let B be a p-block of G with p odd. Then B is nilpotent if and only if $l(b_Q) = 1$ for each p-subgroup Q and each block b_Q of $C_G(Q)$ with $(b_Q)^G = B$.*

The main part of the paper concerns the representation theory of finite groups of Lie type in non-defining characteristic, and makes use of the examination of subpairs of blocks of classical groups given in [17]. The exceptional groups of Lie type are then treated by examination of the centralizer of an element of the centre of a defect group, and the results for the classical groups applied.

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In Section 2 we review the basic notation regarding blocks, give some general results concerning nilpotent blocks, particularly with regard to block domination, and also prove some technical lemmas which will be useful later on. In Section 3 we consider the alternating groups and their covering groups. Here we have able to give a rather complete description of the nilpotent blocks. The covering groups of the sporadic simple groups are treated in Section 4. We give some basic notation used for the classical groups in Section 5. In Section 6 we give a treatment of the general linear and unitary groups, where again we are able to give a full characterization of the nilpotent blocks. In Section 7 we state the set of properties which are central to the study of the nilpotent blocks of the groups of Lie type. These are rather technical conditions, none of which can be satisfied by a nilpotent block with non-abelian defect groups, which amongst other things allow us to use inductive argument when studying the exceptional groups. That these conditions hold for the classical groups is the content of Section 8, and for the exceptional groups is the content of Section 9.

2 Notation and general results

Let G be a finite group and p a prime. Although the classification concerns only blocks with respect to a field of characteristic p, we use methods from ordinary character theory, for example canonical characters, and so must use a p-modular system. Let $\mathcal{O}$ be a local discrete valuation ring, complete with respect to the p-adic valuation, with field of fractions K of characteristic zero and algebraically closed residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p. We assume that $\mathcal{O}$ contains a primitive $|G|$th root of unity. Write $\mathrm{Blk}(G)$ for the set of blocks of $\mathcal{O}G$ and denote by $B_0(G)$ the principal block of G.

Let N be a normal subgroup of G and write $\mathrm{Irr}(G)$ the set of irreducible K-characters of G. For $\theta \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G \mid \theta)$ the subset of $\mathrm{Irr}(G)$ consisting of characters covering θ. We denote by $\mathrm{Irr}(B)$ the set of irreducible characters belonging to B, $k(B) = |\mathrm{Irr}(B)|$, and combine with the above notation freely.

Let B be a p-block of a finite group G. A B-subgroup is a subpair (Q, b_Q), where Q is a p-subgroup of G and b_Q is a block of $QC_G(Q)$ with Brauer correspondent $(b_Q)^G = B$. The B-subgroups with $|Q|$ maximized are called the Sylow B-subgroups, and they are the B-subgroups for which Q is a defect group for B. Recall that the canonical character of b_Q is the unique irreducible character in b_Q with Q in its kernel. This will be a valuable tool when comparing subpairs of a group with those of a normal subgroup.

A useful, and well-known, result is the following:

Proposition 2.1 *Let B be a block of a finite group G. Suppose a defect group D of B is abelian. Then B is nilpotent if and only if $N_G(D, b_D) = C_G(D)$, where (D, b_D) is a Sylow B-subgroup.*

In general, we cannot say very much about the relationship between nilpotency of blocks and nilpotency of covered blocks, and this is a main reason behind the difficulty of the classification of nilpotent blocks of groups of Lie type.

However, we do have the following lemma by [24, Proposition 6.5].

Lemma 2.2 *Let N be a normal subgroup of a finite group G such that G/N is a p-group. Suppose that B is a block of G and that $b \in \mathrm{Blk}(N)$ is covered by B. Then B is nilpotent if and only if b is nilpotent.*

We note that the analogous result does not hold if G/N is not a p-group. There are many examples of non-nilpotent blocks covering nilpotent blocks, but there are also examples of nilpotent blocks covering non-nilpotent blocks, such as the following (which came to light during a conversation with Radha Kessar):

Example 2.3 *Let $G = \mathrm{PGL}(3,7)$, $N = PSL(3,7)$ and $p = 2$, so that $[G{:}N] = 3$. Then N has a unique block b with defect group $D \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and b is not nilpotent. Moreover, b is covered by a nilpotent block B of G.*

Note that $C_N(D) = \mathbb{Z}_6 \times \mathbb{Z}_2$. Let $\mathrm{Irr}^0(C_N(D))$ be the subset of $\mathrm{Irr}(C_N(D))$ consisting of characters of $C_N(D)$ whose kernel contains D. Then $|\mathrm{Irr}^0(C_N(D))| = 3$. In addition, $C_N(D)$ has a unique character (the trivial character) $\xi \in \mathrm{Irr}^0(C_N(D))$ such that $N_{N_N(D)}(\xi) = N_N(D)$, and two characters $\xi \in \mathrm{Irr}^0(C_N(D))$ such that $N_{N_N(D)}(\xi) = C_N(D).3$. It follows that N has exactly one block b with a defect group D and b is non-nilpotent, as $N_N(D, b_D) = C_N(D).3$ for a Sylow b-subgroup (D, b_D).

Moreover, $C_G(D) = \mathbb{Z}_6 \times \mathbb{Z}_6$, $N_G(D)/C_G(D) \cong S_3$ and $|\mathrm{Irr}^0(C_G(D))| = 9$. In addition, $C_G(D)$ has a unique character (the trivial character) $\xi \in \mathrm{Irr}^0(C_G(D))$ such that $N_{N_G(D)}(\xi) = N_G(D)$, and eight characters $\xi \in \mathrm{Irr}^0(C_G(D))$ such that $N_{N_G(D)}(\xi) = C_G(D)$. It follows that G has exactly one block B with a defect group D and B is nilpotent, as $N_G(D, b_D) = C_G(D)$ for a Sylow b-subgroup (D, b_D). Since b is covered by a block of G with a defect group D, it follows that b is covered by B.

Recall that for $N \lhd G$, a block B of G is said to dominate the block $\overline{B}$ of G/N if the inflation to G of an irreducible character in $\overline{B}$ lies in B.

The following lemma follows by [33, Lemma 2].

Lemma 2.4 *Let Z be a central p-subgroup of a finite group G, $B \in \mathrm{Blk}(G)$ and $\overline{B}$ the block of $\overline{G} := G/Z$ dominated by B. Then B is nilpotent if and only if $\overline{B}$ is nilpotent.*

Let Z be a central p'-subgroup of a finite group G, and write $\overline{H} = HZ/Z$, where $H \leq G$. Let $\overline{B} \in \mathrm{Blk}(\overline{G})$. There is a unique block $B \in \mathrm{Blk}(G)$ dominating $\overline{B}$. By [25, Theorem 5.8.8], $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$ and if D is a defect group of B, then $DZ/Z \cong D$ is a defect group of $\overline{B}$.

If Q is a p-subgroup of G, then $C_{\overline{G}}(\overline{Q}) = C_G(Q)/Z$ (since Z is a central p'-subgroup). Let $(\overline{Q}, b_{\overline{Q}})$ be a $\overline{B}$-subgroup. Then $\overline{Q} = QZ/Z$ for a unique p-subgroup Q of G. Since $C_{\overline{G}}(\overline{Q}) = C_G(Q)/Z$, we may consider the unique subpair (Q, b_Q) with b_Q dominating $b_{\overline{Q}}$, which we call the Brauer pair *dominating* $(\overline{Q}, b_{\overline{Q}})$.

We show that (Q, b_Q) must be a B-subgroup, and that dominance of subpairs respects the usual partial order on B-subgroups:

Lemma 2.5 *Let Z be a central p'-subgroup of a finite group G, and let $(\overline{Q}, b_{\overline{Q}})$ and $(\overline{P}, b_{\overline{P}})$ be $\overline{B}$-subgroups, where $\overline{B}$ is the block of $\overline{G}$ dominated by B. Suppose (Q, b_Q) and (P, b_P) are subpairs of G dominating $(\overline{Q}, b_{\overline{Q}})$ and $(\overline{P}, b_{\overline{P}})$, respectively. Then $(\overline{Q}, b_{\overline{Q}}) \leq (\overline{P}, b_{\overline{P}})$ if and only if $(Q, b_Q) \leq (P, b_P)$. In particular, (Q, b_Q) is a B-subgroup.*

PROOF: Let F be a splitting field of characteristic p of G, and μ_Z the map from FG to $F\overline{G}$ defined by $\mu_Z(\sum_{x\in G}\alpha_x x) = \sum_{x\in G}\alpha_x\bar{x}$, where $\bar{x} = \mu_Z(x)$. For $H \leq G$, let $\mathcal{C}\ell(G \mid H)$ be the H-orbits of G under conjugation. Then $\{[C] : C \in \mathcal{C}\ell(G \mid H)\}$ forms a F-basis of the fixed point set $(FG)^H$, where $[C] := \sum_{x\in C} x$.

If H is a p-subgroup, then $(FG)^H = FC_G(H) \oplus I^H(FG)$ as vector spaces, where $I^H(FG) = \sum_{W<H}(FG)^H_W$ is an ideal of $(FG)^H$ and $\{[C] : C \in \mathcal{C}\ell(G \mid H),\ |C| \neq 1\}$ forms an F-basis of $I^H(FG)$. Thus $Br_H([C]) := [C \cap C_G(H)]$ gives the natural algebra homomorphism from $(FG)^H$ onto $FC_G(H)$ with kernel $I^H(FG)$. Similarly,

$$(F\overline{G})^H = F(C_G(H)/Z) \oplus I^H(F\overline{G})$$

and $\mu_Z : I^H(FG) \to I^H(F\overline{G})$ is an isomorphism of algebras. Now

$$Br_{\overline{H}} : (F\overline{G})^H \to F(C_G(H)/Z)$$

and $\mu_Z : FC_G(H) \to F(C_G(H)/Z)$, so $\mu_Z \circ Br_H = Br_{\overline{H}} \circ \mu_Z$.

Suppose $(\overline{Q}, b_{\overline{Q}}) \lhd (\overline{P}, b_{\overline{P}})$, so that $\overline{Q} \lhd \overline{P}$. Since $\overline{Q} = QZ/Z$ and $\overline{P} = PZ/Z$ for p-subgroups Q, P of G and since Q is the only Sylow p-subgroup of QZ, it follows that $Q \lhd P$. Since $PZ = P \times Z$ and $b_{\overline{Q}}$ is $\overline{P}$-invariant, it follows that for any $y \in P$ b_Q^y is a block of $C_G(Q)$ dominating $b_{\overline{Q}}$, so that by the uniqueness, $b_Q^y = b_Q$ and b_Q is P-invariant. Now

$$\mu_Z(Br_P(b_Q)b_P) = \mu_Z(Br_P(b_Q))b_{\overline{P}} = Br_{\overline{P}}(\mu_Z(b_Q))b_{\overline{P}} = Br_{\overline{P}}(b_{\overline{Q}})b_{\overline{P}} = b_{\overline{P}} \neq 0,$$

so that $Br_P(b_Q)b_P \neq 0$ and $Br_P(b_Q)b_P = b_P$. It follows that $(Q, b_Q) \lhd (P, b_P)$. Using induction we have that $(Q, b_Q) \leq (P, b_P)$ if $(\overline{Q}, b_{\overline{Q}}) \leq (\overline{P}, b_{\overline{P}})$.

Suppose $(Q, b_Q) \lhd (P, b_P)$, so that $Q \lhd P$ and $\overline{Q} \lhd \overline{P}$. Since b_Q is P-invariant, it follows that $b_{\overline{Q}}$ is $\overline{P}$-invariant. Since $Br_P(b_Q)b_P = b_P$, it follows that

$$Br_{\overline{P}}(b_{\overline{Q}})b_{\overline{P}} = \mu_Z(Br_P(b_Q)b_P) = \mu_Z(b_P) = b_{\overline{P}},$$

so that $(\overline{Q}, b_{\overline{Q}}) \lhd (\overline{P}, b_{\overline{P}})$. Similarly, if $(Q, b_Q) \leq (P, b_P)$, then $(\overline{Q}, b_{\overline{Q}}) \leq (\overline{P}, b_{\overline{P}})$. □

We obtain as a consequence:

Proposition 2.6 *Let G be a finite group, $Z \leq Z(G)$ and $\overline{G} = G/Z$. Suppose $\overline{B} \in$ Blk$(\overline{G})$ and $B \in$ Blk(G) dominating $\overline{B}$. Then $\overline{B}$ is nilpotent if and only if B is nilpotent.*

PROOF: Write $Z_p = O_p(Z)$, $Z_{p'} = O_{p'}(Z)$, $G_1 = G/Z_{p'}$ and let $B_1 \in \mathrm{Blk}(G_1)$ be the unique block of G_1 dominated by B. Then $\mathrm{Irr}(B_1) = \mathrm{Irr}(B)$, and B_1 dominates $\overline{B}$.

By Lemma 2.4, $\overline{B}$ is nilpotent if and only if B_1 is nilpotent. Hence we suppose $\overline{B} = B_1$ and $Z = Z_{p'}$.

Let $(\overline{D}, b_{\overline{D}})$ be a Sylow $\overline{B}$-subgroup and (D, b_D) the unique B-subgroup dominating $(\overline{D}, b_{\overline{D}})$. Note that (D, b_D) is a Sylow B-subgroup.

Suppose $(\overline{Q}, b_{\overline{Q}})$ is a B-subgroup and (Q, b_Q) is the B-subgroup dominating $(\overline{Q}, b_{\overline{Q}})$. If $\bar{x} \in N_{\overline{G}}(\overline{Q}, b_{\overline{Q}})$, then $\bar{x} = xZ$ for some $x \in G$, and $xZ \subseteq N_G(Q)$. Since $\mu_Z(b_Q) = b_{\overline{Q}}$ and $\mu_Z(b_Q^x) = b_{\overline{Q}}^{\bar{x}} = b_{\overline{Q}}$, it follows that b_Q and b_Q^x both are blocks of $C_G(Q)$ dominating

$b_{\overline{Q}}$ and $b_Q = b_Q^x$ by uniqueness. Thus $x \in N_G(Q, b_Q)$ and $N_{\overline{G}}(\overline{Q}, b_{\overline{Q}}) = N_G(Q, b_Q)/Z$. Since $Z \leq C_G(Q)$ and $C_{\overline{G}}(\overline{Q}) = C_G(Q)/Z$, it follows that

$$N_{\overline{G}}(\overline{Q}, b_{\overline{Q}})/C_{\overline{G}}(\overline{Q})\overline{Q} \cong N_G(Q, b_Q)/C_G(Q)Q.$$

Suppose $\overline{B}$ is not nilpotent, so that there is some $\overline{B}$-subgroup $(\overline{Q}, b_{\overline{Q}})$ such that $N_{\overline{G}}(\overline{Q}, b_{\overline{Q}})/C_{\overline{G}}(\overline{Q})\overline{Q}$ is not a p-group. Thus $N_G(Q, b_Q)/C_G(Q)Q$ is not a p-group and B is not nilpotent.

Suppose B is not nilpotent, so that $N_G(Q, b_Q)/C_G(Q)Q$ is not a p-group for some B-subgroup (Q, b_Q). We may suppose $(Q, b_Q) \leq (D, b_D)$. Thus $\overline{Q} = QZ/Z \leq \overline{D}$, and $(\overline{Q}, b_{\overline{Q}}) \leq (\overline{D}, b_{\overline{D}})$ for a unique $\overline{B}$-subgroup $(\overline{Q}, b_{\overline{Q}})$. Let (Q, b'_Q) be a B-subgroup dominating $(\overline{Q}, b_{\overline{Q}})$. By Lemma 2.5 $(Q, b'_Q) \leq (D, b_D)$, so that by the uniqueness $(Q, b'_Q) = (Q, b_Q)$. Thus $N_{\overline{G}}(\overline{Q}, b_{\overline{Q}})/C_{\overline{G}}(\overline{Q})\overline{Q} \cong N_G(Q, b_Q)/C_G(Q)Q$, and $N_{\overline{G}}(\overline{Q}, b_{\overline{Q}})/C_{\overline{G}}(\overline{Q})\overline{Q}$ is not a p-group. It follows that $\overline{B}$ is not nilpotent. □

When considering groups of Lie type, we will often examine the centralisers of p-elements, which may be written as central products of groups. By a central product $G_1 \circ_Z G_2$ of G_1 and G_2 over $Z \leq Z(G_1) \cap Z(G_2)$, we mean that $G_1 \circ_Z G_2 = G_1 G_2$, where G_1 and G_2 are subgroups of $G_1 \circ_Z G_2$ with $G_1 \cap G_2 = Z$ and $[G_1, G_2] = 1$. When it is clear what Z is, we write $G_1 \circ G_2 = G_1 \circ_Z G_2$. Note that $G_1 \circ G_2 \cong (G_1 \times G_2)/\{(z, z^{-1}) : z \in Z\}$. For $\chi_i \in \mathrm{Irr}(G_i)$ such that χ_1 and χ_2 both cover the same irreducible character of Z, we may define $\chi_1 \circ \chi_2 \in \mathrm{Irr}(G_1 \circ G_2)$ so that $\chi_1 \chi_2 \in \mathrm{Irr}(G_1 \times G_2)$ is the inflation of $\chi_1 \circ \chi_2$. We refer to $\chi_1 \circ \chi_2$ as the central product of χ_1 and χ_2.

We will need the following technical lemma in certain cases in relation to Property 7.1 (a) in Section 7 holds.

Lemma 2.7 *For $i = 1, 2$, let G_i be a finite group, $G_1 \circ G_2$ a central product of G_1 and G_2 over $Z \leq Z(G_1) \cap Z(G_2)$ and N_i a normal subgroup of G_i such that G_i/N_i is abelian, and let $N := N_1 \times N_2 \leq G \leq G_1 \circ G_2$ such that $\pi_i(G) = G_i/Z$, where $\pi_i : (G_1 \circ G_2) \to G_i/Z$ is the canonical projection. Let $\theta_i \in \mathrm{Irr}(N_i)$ such that θ_2 has an extension $\tilde{\theta}_2$ to G_2, and let $\theta = \theta_1 \times \theta_2$ and $\varphi \in \mathrm{Irr}(G \mid \theta)$.*

(i) There exist $\psi_1 \in \mathrm{Irr}(G_1)$ and $\lambda \in \mathrm{Irr}(G_2/N_2)$ covering the same irreducible character of Z, such that the restriction $(\psi_1 \circ (\tilde{\theta}_2\lambda))|_G$ of $\psi_1 \circ (\tilde{\theta}_2\lambda)$ is equal to φ. Moreover, if $\psi \in \mathrm{Irr}(G_1 \circ G_2 \mid \varphi)$, then $\psi|_G = \varphi$.

(ii) If further $Z \cap N_2 = 1$, then λ in (i) may be chosen with ZN_2/N_2 in its kernel, so that it may be regarded as a character of G_2/N_2Z.

(iii) Suppose that $Z \cap N_2 = 1$. If we have $y \in \mathrm{Aut}(G_1 \circ G_2)$ such that y centralizes G_1, stabilizes G, G_2 and $\tilde{\theta}_2$, and $g_2^y \in g_2 N_2 Z$ for any $g_2 \in G_2$, then y stabilizes φ.

Proof: (i) We first claim that we may suppose $Z \leq G$. For since $Z \leq Z(G_1 \circ G_2)$, we have that GZ is a central product over $G \cap Z$. Now $\varphi|_{G \cap Z} = \varphi(1)\alpha$ for some $\alpha \in \mathrm{Irr}(G \cap Z)$. Since Z is abelian, there exists $\tilde{\alpha} \in \mathrm{Irr}(Z)$ extending α. Then $\tilde{\varphi} = \varphi\tilde{\alpha}$ is an extension of φ covering θ. If $\psi_1 \in \mathrm{Irr}(G_1)$ and $\lambda \in \mathrm{Irr}(G_2/N_2)$ such that ψ_1 and λ cover the same irreducible character of Z and $(\psi_1 \circ \tilde{\theta}_2\lambda)|_{GZ} = \tilde{\varphi}$, then $(\psi_1 \circ \tilde{\theta}_2\lambda)|_G = \varphi$, and similarly for the final statement, proving the claim.

Similarly, $(N_1 \times N_2)Z = (N_1 Z) \circ (N_2 Z)$ and φ covers an irreducible character $\xi \in \mathrm{Irr}((N_1 Z) \circ (N_2 Z) \mid \theta)$ with $\xi = \xi_1 \circ \xi_2$ for some $\xi_i \in \mathrm{Irr}(N_i Z \mid \theta_i)$ covering the same irreducible character of Z. Note that $\tilde{\theta}_2|_{N_2Z}$ is also an extension of θ_2 to N_2Z. By Gallagher's theorem, $(\tilde{\theta}_2|_{N_2Z})\beta_2 = \xi_2$ for some $\beta_2 \in \mathrm{Irr}(N_2Z/N_2)$. Since G_2/N_2 is abelian, it follows that β_2 can be viewed as the restriction of a character $\beta \in \mathrm{Irr}(G_2/N_2)$, so $\xi_2 = (\tilde{\theta}_2\beta)|_{N_2Z}$. Write $\tilde{\xi}_2 = \tilde{\theta}_2\beta$, so $\tilde{\xi}_2$ is an extension of ξ_2 to G_2

Let M_1 be a subgroup of G_1 such that ξ_1 has an extension $\tilde{\xi}_1$ to M_1 and M_1 is maximal with this property, that is, either $M_1 = G_1$ or ξ_1 has no extension to H_1 for any $M_1 < H_1 \leq G_1$. Since $G_1/(N_1Z)$ is abelian, it follows that the inertia subgroup $I_{G_1}(\tilde{\xi}_1)$ equals M_1.

Let $M = (M_1 \circ G_2) \cap G \leq G$, $M_2 := G_2$ and $\gamma \in \mathrm{Irr}(M_1 \circ M_2 \mid \xi)$. Then $G/M \cong G_1/M_1$ and $\gamma = \gamma_1 \circ \gamma_2$ for some $\gamma_i \in \mathrm{Irr}(M_i \mid \xi_i)$. Since ξ_i has an extension $\tilde{\xi}_i$ to M_i, it follows that $\gamma_i = \tilde{\xi}_i\lambda_i$ for some $\lambda_i \in \mathrm{Irr}(M_i/N_iZ)$, so that $\gamma|_M$ is an extension of ξ to M. Note that $M \leq I_G(\xi)$ and φ also covers an extension of ξ to M. Replacing γ_i by $\gamma_i\alpha_i$ for some $\alpha_i \in \mathrm{Irr}(M_i/N_iZ)$ if necessary, we may suppose $\varphi \in \mathrm{Irr}(G \mid \gamma|_M)$.

Since $\gamma_1|_{N_1Z} = \xi_1$ and G_2 stabilizes $\tilde{\xi}_2$ (and G_1/N_1Z is abelian), it follows that $I_{G_1}(\gamma_1) = M_1$, $I_G(\gamma) = M$ and $I_{G_1\circ G_2}(\gamma) = M_1 \circ G_2$. Let $\psi_1 = \mathrm{Ind}_{M_1}^{G_1}(\gamma_1)$, so that

$$\zeta := \psi_1 \circ \gamma_2 = \mathrm{Ind}_{M_1\circ G_2}^{G_1\circ G_2}(\gamma_1 \circ \gamma_2).$$

But $\varphi = \mathrm{Ind}_M^G(\gamma|_M)$, so

$$(\zeta|_G, \varphi)_G = (\zeta|_M, \gamma|_M)_M.$$

Since M and $M_1 \circ G_2$ are both normal in $G_1 \circ G_2$ and $M \leq M_1 \circ G_2$, it follows that $(M_1 \circ G_2)\backslash(G_1 \circ G_2)/M = (G_1 \circ G_2)/(M_1 \circ G_2) \cong G_1/M_1 \cong G/M$. Note also $(M_1 \circ G_2)^t \cap M = M$ for any $t \in G/M$. Hence the Mackey decomposition gives us

$$\zeta|_M = \sum_{t\in G/M} ((\gamma_1 \circ \gamma_2)^t|_M) = \sum_{t\in G/M} ((\gamma|_M)^t)$$

and so $(\zeta|_M, \gamma|_M)_M = 1$. Since $\zeta(1) = \varphi(1) = [G{:}M]\gamma(1)$, it follows that $\zeta|_G = \varphi$.

Note that $I_{G_1\circ G_2}(\varphi) = G_1 \circ G_2$ and φ has an extension ζ to $G_1 \circ G_2$. If $\psi \in \mathrm{Irr}(G_1 \circ G_2 \mid \varphi)$, then by Gallagher's theorem, $\psi = \zeta\eta$ for some $\eta \in \mathrm{Irr}((G_1 \circ G_2)/G)$ and so $\psi|_G = \varphi$.

(ii) First note that $ZN_2 = Z \times N_2$ and $(N_1 \times N_2)Z = (N_1 \circ Z) \times N_2$. If $\xi \in \mathrm{Irr}((N_1 \times N_2)Z \mid \theta)$, then $\xi = (\theta_1 \circ \eta) \times \theta_2$ for some $\eta \in \mathrm{Irr}(Z)$. Thus we may suppose $\xi_1 = (\theta_1 \circ \eta) \in \mathrm{Irr}(N_1Z)$ and $\xi_2 = (\theta_2 \times 1_Z) \in \mathrm{Irr}(N_2Z)$, and take $\tilde{\xi}_2 = \tilde{\theta}_2$ as an extension of ξ_2 to G_2. As shown in the proof of part (i) $\varphi = (\psi_1 \circ (\tilde{\theta}_2\lambda))|_G$ for some $\psi_1 \in \mathrm{Irr}(G_1)$ and $\lambda \in \mathrm{Irr}(G_2/N_2Z)$.

(iii) Since y centralizes the factor group G_2/ZN_2, it follows that y stabilizes λ, so does $\tilde{\theta}_2\lambda$. But y centralizes G_1, so y stabilizes $\psi_1 \circ (\tilde{\theta}_2\lambda)$ and hence y stabilizes φ. □

3 The alternating groups

To handle the case $p = 3$ we will need the following. The first lemma will be used in determining non-faithful nilpotent blocks of the double covers of alternating groups. Recall that a partition is self-associate if its Young diagram is symmetric.

Lemma 3.1 *Let n be a positive integer. There is a self-associate 3-core partition $\lambda \vdash n$ if and only if there is a positive integer m such that $n = 3m^2 + 2m$ or $n = 3m^2 - 2m$.*

PROOF: We claim that the self-associate 3-cores, i.e., those Young diagrams which are symmetric about the leading diagonal and have no 3-hooks, are those which arise from partitions of the form

$$(3m, 3m-2, 3m-4, \ldots, 3m-2(m-1), m^2, (m-1)^2, \ldots, 2^2, 1^2)$$

and

$$(3m-2, 3m-4, 3m-6, \ldots, 3m-2m, (m-1)^2, (m-2)^2, \ldots, 2^2, 1^2)$$

for integers $m \geq 1$.

This may be seen directly from methodical construction of the possible Young diagrams. However, we give a formal proof here using [19]. For a fixed t, Garson, Kim and Stanton give a bijection ϕ between the set of t-cores and $\{n_0, n_1, \ldots, n_{t-1} \in \mathbb{Z}^t : n_0 + \cdots + n_{t-1} = 0\}$, defined as follows. We of course only need to consider the case $t = 3$. Let λ be a 3-core. We take the 3-residue diagram, i.e., in the $(i,j)th$ cell of the Young diagram we put the residue of $j-i$ modulo 3 (see [23, p.84]). We also include the $0th$ column (with infinitely many entries), calling this the extended 3-residue diagram. Divide this into regions labelled by the integers as follows: the $(i,j)th$ cell lies in the region r if $3(r-1) \leq j-i < 3r$. Say that a cell is *exposed* if it lies at the end of a row. Define n_i to be the maximal r such that an exposed cell with value i lies in the region r (the inclusion of the $0th$ row ensures the existence of such an r).

It is verified in [19] that ϕ does indeed give a bijection. It is also shown that λ is self-associate if and only if $\phi(\lambda) = (n_0, n_1, n_2) = (-n_2, -n_1, -n_0)$, i.e., if $\phi(\lambda) = (m, 0, -m)$ for some $m \in \mathbb{Z}$.

Suppose first that $m > 0$. Then the end cell on the first row is labelled 0, so the first row has length $\lambda_1 = 3(m-1) + 1 = 3m - 2$. Since regions 0 and $-m$ lie below the leading diagonal, the end cells lying above the diagonal are all labelled 0. Since λ is a 3-core, the difference between adjacent row lengths is at most 2, hence the row lengths decrease in steps of two until the mth row (which has end cell on the leading diagonal). Since λ, is self adjoint, this determines the whole Young diagram and we are done in this case.

Suppose that $m \leq 0$. Then the end cell of the first row is labelled 2, so the first row has length $\lambda_1 = 3m$, and by a similar argument to the above the difference between adjacent row lengths is 2 until the $(m+1)th$ row (which has end cell below the leading diagonal). Again this determines λ, and we are done. □

We now consider the analogue of the above lemma which will be used for faithful blocks. We write $\lambda \succ n$ for a bar partition of n (i.e., a partition with distinct parts). Recall that a bar partition $\lambda \succ n$ is odd or even according as $n - r$ is odd or even, where r is the number of parts in the partition. We refer to [26] for definitions of bars and $\bar{p}$-cores.

Lemma 3.2 *Let n be a positive integer. There is an even $\bar{3}$-core bar partition $\lambda \succ n$ if and only if there is a positive integer m_1 with $m_1 \equiv 0, 1$ mod 4 and $n = (3m_1^2 - m_1)/2$ or a positive integer m_2 with $m_2 \equiv 0, 3$ mod 4 and $n = (3m_2^2 + m_2)/2$.*

Proof: Determining the $\bar{3}$-core partitions is a little more straight-forward than determining 3-core partitions, and the reader can easily verify that the $\bar{3}$-core bar partitions are precisely those of the form

$$\lambda_m^- := (3m-2, 3m-5, \ldots, 3m-2-3i, \ldots, 4, 1)$$

or

$$\lambda_m^+ := (3m-1, 3m-4, \ldots, 3m-1-3i, \ldots, 5, 2).$$

Note that $\lambda_m^- \succ (3r^2-m)/2$ and $\lambda_m^+ \succ (3m^2+m)/2$. Also note that λ_m^- is even if and only if $m \equiv 0, 1 \ mod\ 4$; λ_m^+ is even if and only if $m \equiv 0, 3 \ mod\ 4$. □

Theorem 3.3 *Let n be an integer with $n \geq 5$ and $G = \hat{A}_n$, the double cover of A_n. Let p be an odd prime. If $p \neq 3$, then G does not possess a nilpotent p-block of positive defect. If $p = 3$, then G possesses a non-faithful nilpotent block of positive defect if and only if $n = 3m^2+2m+3$ or $n = 3m^2-2m+3$ for some positive integer m. Also if $p = 3$, then G possesses a faithful nilpotent block of positive defect if and only if $n = (3m_1^2 - m_1 + 6)/2$ for a positive integer m_1 with $m_1 \equiv 0, 1 \ mod\ 4$ or $n = (3m_2^2 + m_2 + 6)/2$ for a positive integer m_2 with $m_2 \equiv 0, 3 \ mod\ 4$. In each case the nilpotent blocks have defect groups of order 3 generated by (the preimage of) a 3-cycle.*

Proof: The properties of $\hat{A}_n$ used here are described in [20, 5.2]. We consider $\hat{A}_n \leq \hat{S}_n$, the double cover of the symmetric group. Write $Z = Z(\hat{S}_n)$ and $\overline{X} = XZ/Z$ whenever $X \leq \hat{S}_n$. For convenience of notation we write $\overline{\hat{S}_n} = S_n$ and $\overline{\hat{A}_n} = A_n$. Since we are taking p odd, for every p-subgroup $Q \leq \hat{S}_n$ we have $C_{S_n}(\overline{Q}) = \overline{C_{\hat{S}_n}(Q)}$ and $N_{S_n}(\overline{Q}) = \overline{N_{\hat{S}_n}(Q)}$. Suppose that B is a nilpotent p-block of $\hat{A}_n$ with non-trivial defect group D. Choose $y \in D$ of order p. Then yZ is a product of say t disjoint p-cycles, fixing the other $n-pt$ points. Then $C_{S_n}(yZ) \cong (\mathbb{Z}_p \wr S_t) \times S_{n-pt}$, and so $C_{\hat{A}_n}(y)$ contains a normal elementary abelian p-group R such that $\overline{R}$ is generated by t disjoint p-cycles. Now R is contained in a conjugate of D, and so in particular D contains an element x for which xZ is a p-cycle. Write $Q = \langle x \rangle$. We have $C_{A_n}(\overline{Q}) \cong \overline{Q} \times A_{n-p}$. By [20, 5.2.6] we have $C_{\hat{A}_n}(Q) \cong Q \times \hat{A}_{n-p}$ (the point here being that the central extension of A_{n-p} does not split). We have $N_{S_n}(\overline{Q}) \cong N_{S_p}(\overline{Q}) \times S_{n-p}$ and $N_{A_n}(\overline{Q}) \cong (N_{A_p}(\overline{Q}) \times A_{n-p})\langle \overline{a} \rangle$ where $\overline{a}^2 = 1$. Note that $[N_{\hat{A}_n}(Q) : C_{\hat{A}_n}(Q)] = p-1$.

The p-blocks of $C_{\hat{A}_n}(Q)$ are in 1-1 correspondence with the p-blocks of $\hat{A}_{n-p}$ and the action of $N_{\hat{A}_n}(Q)$ on these blocks is determined by the action of $\hat{S}_{n-p}$ on the blocks of $\hat{A}_{n-p}$. Hence for each block b_Q of $C_{\hat{A}_n}(Q)$ we have $[N_{\hat{A}_n}(Q, b_Q) : C_{\hat{A}_n}(Q)] = (p-1)/2$ or $p-1$.

If $p > 3$, then this shows that $N_{\hat{A}_n}(Q, b_Q)/C_{\hat{A}_n}(Q)$ is not a p-group, contradicting our choice of B nilpotent.

Now suppose that $p = 3$. We first show that $\overline{D}$ is generated by a 3-cycle. Suppose that D is not cyclic. Then D contains an elementary abelian subgroup of order 9, and in particular contains distinct elements x and y for which xZ and yZ is the product of s and t disjoint 3-cycles respectively (briefly, consider the centralizer of gh, for which ghZ is the product of all the disjoint 3-cycles in gZ and hZ. This has an elementary abelian subgroup contained in a conjugate of D and containing elements whose images in S_n are all the 3-cycles making up ghZ). Then $C_{S_n}(xZ) \cong (\mathbb{Z}_p \wr S_s) \times S_{n-ps}$ and $C_{S_n}(yZ) \cong (\mathbb{Z}_p \wr S_t) \times S_{n-pt}$. Hence D contains elements g and h for which gZ and hZ are each a 3-cycle and these 3-cycles are disjoint. Write $R = \langle g, h \rangle \leq D$. We have $C_{S_n}(\overline{R}) \cong \overline{R} \times S_{n-6}$ and $C_{A_n}(\overline{R}) \cong \overline{R} \times A_{n-6}$. Now $[N_{\hat{S}_n}(R) : C_{\hat{S}_n}(R)] = 8$, and arguing as above we see that $[N_{\hat{A}_n}(R, b_R) : C_{\hat{A}_n}(R)]$ is even for every block b_R of $C_{\hat{A}_n}(R)$, a contradiction.

Hence D is cyclic. Suppose that $|D| > 3$. Then D possesses an element y of order 9. By an argument similar to above we may assume yZ is a 9-cycle. But then y^3Z is a product of three distinct 3-cycles, which as we have seen cannot happen.

Hence D has order three and is generated by an element x where xZ is a single 3-cycle. We have $C_{\hat{A}_n}(D) \cong D \times \hat{A}_{n-3}$. The blocks of $C_{\hat{A}_n}(D)$ with defect group D are in 1-1 correspondence with the blocks of defect zero of $\hat{A}_{n-3}$, and the action of $N_{\hat{A}_n}(D)$ on these blocks is given by the action of $\hat{S}_{n-3}$ on the blocks of defect zero of $\hat{A}_{n-3}$. Hence the nilpotent blocks of $\hat{A}_n$ with defect group D are in 1-1 correspondence with orbits of length two of blocks of defect zero of $\hat{A}_{n-3}$ under the action of $\hat{S}_{n-3}$.

Now blocks of defect zero of $\hat{A}_{n-3}$ are covered by blocks of defect zero of $\hat{S}_{n-3}$. We consider faithful and non-faithful blocks separately. Note that B is faithful if and only if the B-subpairs have kernel intersecting trivially with Z (i.e., if and only if they correspond to faithful blocks of $\hat{A}_{n-3}$).

Suppose that B is non-faithful. Blocks of defect zero correspond to 3-core partitions of $n-3$. By [23, 2.5.7] irreducible characters of S_{n-3} remain irreducible when restricted to A_{n-3} if and only if the corresponding partition is not self-associate. Hence $[N_{\hat{A}_n}(D, b_D) : C_{\hat{A}_n}(D)] = 1$ if and only if the block of defect zero of A_{n-3} corresponding to b_D is labelled by a self-associate partition, and so the result follows in this case from Lemma 3.1.

Suppose that B is faithful. Blocks of defect zero correspond to $\bar{3}$-core bar partitions of $n-3$. By [27, p.212] faithful irreducible characters of $\hat{S}_{n-3}$ remain irreducible when restricted to $\hat{A}_{n-3}$ if and only if the corresponding bar partition is odd. Hence $[N_{\hat{A}_n}(D, b_D) : C_{\hat{A}_n}(D)] = 1$ if and only if the block of defect zero of $\hat{A}_{n-3}$ corresponding to b_D is labelled by an even $\bar{3}$-core bar partition, and so the result follows in this case from Lemma 3.2. □

We have not yet considered all the perfect central extensions of A_6 and A_7. However, by the above theorem, neither yields a nilpotent 3-block with non-central defect group, and further it is easy to check that there are no nilpotent blocks of positive defect for the other odd primes.

It is appropriate here to extend our study to the double covers of the symmetric groups.

Proposition 3.4 *Let $G = \hat{S}_n$ be the double cover of the symmetric group S_n for $n \geq 5$ and let p be an odd prime. If $p \geq 5$, then G does not possess a nilpotent p-block of positive defect. If $p = 3$, then every nilpotent block of positive defect is faithful. These have defect one, and occur if and only if there is a positive integer m_1 with $m_1 \equiv 2, 3$ mod 4 and $n = (3m_1^2 - m_1 + 6)/2$ or a positive integer m_2 such that $m_2 \equiv 1, 2$ mod 4 and $n = (3m_2^2 + m_2 + 6)/2$.*

PROOF: Suppose first that B is a non-faithful block of positive defect. By, for example, [23, 6.2.2] $l(B) = \sum p(w_1)\cdots p(w_{p-1})$, where the sum runs over improper partitions $(w_1, \dots, w_{p-1})$ of the weight w of B and $p(x)$ is the number of partitions of x. But $p - 1 \geq 2$ and $(w, 0, \dots, 0)$ and $(0, w, 0, \dots, 0)$ are improper partitions of w, so $l(B) > 1$ and B cannot be nilpotent.

Now suppose that B is faithful of positive defect. Then by [28, 13.17] $l(B)$ is at least $k((p-1)/2, w)$, the number of $(p-1)/2$-tuples of (possibly empty) partitions with sum w (see [28, 3.11]), where again w is the weight of B. If $w \leq 2$, then $((w), \emptyset, \dots, \emptyset)$ and $((1^w), \emptyset, \dots, \emptyset)$ are such $(p-1)/2$-tuples of partitions, so $l(B) > 1$ and B cannot be nilpotent. Now suppose that $w = 1$. If $p \geq 5$, then $(1, \emptyset, \dots, \emptyset)$ and $(\emptyset, 1, \emptyset, \dots, \emptyset)$ are such $(p-1)/2$-tuples, and again B cannot be nilpotent. We are left with the case $w = 1$ and $p = 3$. By [28, 13.17] $l(B) = 2$ if the $\overline{3}$-core μ of B is even (in the sense that $n - 3 - r$ is even, where r is the number of parts in μ), and $l(B) = 1$ if μ is odd. Note that since B has cyclic defect groups, B is nilpotent if and only if $l(B) = 1$, and so the result follows from Lemma 3.2. □

We now turn our attention to Puig's conjecture.

Lemma 3.5 *Let $G = \hat{S}_n$ be the double cover of S_n, and let B be a block of G with defect group D. If $|D| > p^2$, then $l(B) \geq 3$.*

PROOF: Suppose first that B is a non-faithful block. As above,

$$l(B) = \sum p(w_1)\cdots p(w_{p-1}),$$

where the sum runs over improper partitions $(w_1, \dots, w_{p-1})$ of the weight w of D. If $|D| > p^2$, then $w \geq 2$. But $(w, 0, \dots, 0)$, $(0, w, 0, \dots, 0)$ and $(w-1, 1, 1, 0, \dots, 0)$ are three such improper partitions, so $l(B) \geq 3$.

If B is faithful, then by [28, 13.17] $l(B)$ is at least $k((p-1)/2, w)$, the number of $(p-1)/2$-tuples of (possibly empty) partitions with sum w, where again w is the weight of B. We have $w \geq 2$. Here $((w), \emptyset, \dots, \emptyset)$, $((1^w), \emptyset, \dots, \emptyset)$ and $((w-1, 1), 1, \emptyset, \dots, \emptyset)$ are three such tuples, so $l(B) \geq 3$. □

Corollary 3.6 *Let B a p-block of G for p odd, where G is quasisimple with $G/Z(G) \cong A_n$ for some n. Then B is nilpotent if and only if $l(b_Q) = 1$ for every B-subgroup (Q, b_Q).*

PROOF: If B has abelian defect group D, then this is [30]. So we may assume $|D| > p^2$. Suppose $G \lhd H$, where $H \cong \hat{S}_n$, and let $B_H \in \mathrm{Blk}(H)$ covering B. Then B_H has defect group D, and by Lemma 3.5 $l(B_H) \geq 3$. But $l(B) \geq l(B_H)/2 > 1$, so B is not nilpotent, and of course we can take the B-subgroup $(1, B)$ to show the proposed equivalent condition is also not satisfied.

It remains to consider the exceptional covers, but in these cases it is easy to check that every block with non-central defect groups has more than one irreducible Brauer character. □

4 Sporadic groups

In this section we determine the nilpotent blocks with non-central defect groups of quasisimple groups G where $G/Z(G)$ is one of the 26 sporadic simple groups. Note that due to Lemma 2.4 it suffices to consider the case $Z(G)$ is a p'-group.

In order to provide a reasonably unified treatment of the classification of nilpotent blocks of the sporadic groups, we work from [20, Table 5.3]. However, in all cases the number of irreducible Brauer characters in the blocks are known, which would lead to a shorter but less illuminating proof. To avoid an overly long proof we do use these results in showing that Puig's conjecture holds.

We use [20, Table 5.3] and apply the following simple results to demonstrate the non-existence of such blocks in many cases:

Lemma 4.1 *Suppose that D is a defect group for a nilpotent block of a finite group G. Let $x \in D$ have order p and write $Q = \langle x \rangle$ and $R = O_p(C_G(Q))$. Then*
(i) there is no p-regular $g \in N_G(Q) - C_G(Q)$ which fixes every block of $C_G(Q)$;
(ii) there is no p-regular $g \in N_G(R) - C_G(R)$ which fixes every block of $C_G(R)$.

Proof: Note that R is contained in every defect group of every block of $C_G(Q)$. Hence $R \leq D$. The result then follows from the definition of a nilpotent block. □

Lemma 4.2 *Let Q be a p-subgroup of G. If $|N_G(Q)/C_G(Q)|$ is prime to p and, for every n, is strictly greater than the number of p-blocks of $C_G(Q)$ of dimension n (or is greater than or equal to n in the case n is the dimension of the principal block of $C_G(Q)$), then Q cannot be a subgroup of a defect group of a nilpotent block of G.*

Proof: In this case every p-block of $C_G(Q)$ must be fixed by a p-regular element of $N_G(Q) - C_G(Q)$, and we apply Lemma 4.1. □

Lemma 4.3 *Let B be a nilpotent block with defect group D, and let $1 \neq Q \leq Z(D)$. Then $C_G(Q)/Q$ possesses a nilpotent block with defect group D/Q.*

Proof: Let $b_D \in \mathrm{Blk}(DC_G(D))$ with $b_D^G = B$. Now $DC_G(D) \leq C_G(Q)$, and $b = b_D^{C_G(Q)}$ is nilpotent. D is the unique defect group of b_D and $b^G = B$, so D is a defect group of b. There is a one-to-one correspondence between the blocks of $C_G(Q)$ with defect group D and the blocks of $C_G(Q)/Q$ with defect group D/Q. Let $\overline{b}$ be the correspondent of b. By Lemma 2.4 $\overline{b}$ is nilpotent. □

Write $Z = Z(G)$ and $\overline{G} = G/Z$. Note that when Z is a p'-group, for every p-subgroup Q of G we have $\overline{C_G(Q)} = C_{\overline{G}}(\overline{Q})$ and $\overline{N_G(Q)} = N_{\overline{G}}(\overline{Q})$.

Throughout our notation for the conjugacy classes of $\overline{G}$ follows that of [20].

Proposition 4.4 *Let G be a quasisimple group such that $\overline{G}$ is a sporadic simple group, with $|\overline{G}|_p = p$. Let B be a p-block of maximal defect of G. Then B is not nilpotent.*

PROOF: Let $D \in \mathrm{Syl}_p(G)$. Note that D is abelian. If $C_{\overline{G}}(\overline{D}) \leq \overline{D}$, then $C_G(D) \leq DZ$ and every p-block of $C_G(D)$ is $N_G(D)$-stable. But p does not divide $[N_G(D) : DC_G(D)]$, whilst by Burnside's transfer theorem we cannot have $N_G(D) = C_G(D)$, so a block with defect group D cannot be nilpotent. Hence, using [20, Table 5.3], we may rule out all but the following cases: $p = 3$ and $\overline{G} = J_1$; $p = 5$ and $\overline{G} = M_{24}, J_1, J_3, J_4$; $p = 7$ and $\overline{G} = M_{24}, J_4, Co_3$, Co_2, Suz, Ly, Ru, Fi_{22}, Fi_{23}, HN; $p = 11$ and $\overline{G} = Co_3$, Co_1, Ly, Fi_{22}, Fi_{23}, Fi'_{24}, HN, F_2; $p = 13$ and $\overline{G} = Co_1$, Ru, Fi_{23}, Fi'_{24}, Th, F_2; $p = 17$ and $\overline{G} = F_2$, F_1; $p = 19$ and $G = F_2$, F_1; $p = 23$ and $\overline{G} = F_2$, F_1; $p = 29$ and $\overline{G} = F_1$; $p = 31$ and $\overline{G} = F_1$.

Applying Lemma 4.2 with $Q = D$ to these cases eliminates all but the case $p = 3$ and $G = J_1$. Here $C_{\overline{G}}(\overline{D}) \cong D \times D_{10}$ and $N_{\overline{G}}(\overline{D}) \cong S_3 \times D_{10}$, and it is clear that $N_G(D)$ fixes every block of $C_G(D)$. □

Theorem 4.5 *Let B be a nilpotent p-block with non-central defect group D of a quasisimple group G such that $\overline{G}$ is a sporadic simple group. Then $|D| = 3$ and G is one of M_{23}, J_4, Ly. In each of these cases G does indeed possess a nilpotent block with defect group D.*

PROOF: We need only consider the case p^2 divides $|\overline{G}|$.

Suppose that D is a non-central defect group of a nilpotent p-block B. We assume that Z is a p'-group. Choose $x \in D$ of order p, and write $Q = \langle x \rangle$. In each case $N_{\overline{G}}(\overline{Q})$ is given by [20, Table 5.3], and $C_{\overline{G}}(\overline{Q})$ may be deduced using [14].

Let $P = O_p(N_G(Q))$. Then $P \leq D$.

We eliminate each possibility for the conjugacy class containing x in turn using a succession of methods until we are left with the three cases listed. For each of these we then verify the existence of a nilpotent block with defect group Q.

Suppose that $N_{\overline{G}}(\overline{Q}) \cong \overline{H}_1 \times \overline{H}_2$ and $C_{\overline{G}}(\overline{Q}) \cong \overline{Q} \times \overline{H}_2$ for some H_1, H_2 such that $Q \lhd H_1$, and $\overline{H}_1/\overline{Q}$ not a p-group. Then every p-block of $C_G(Q)$ is fixed by $N_G(Q)$ and $N_G(Q)/C_G(Q)$ is not a p-group, so B cannot be nilpotent. In this way we eliminate the following pairs $(\overline{G}, C)$, where C is the conjugacy class in $\overline{G}$ containing xZ: $(M_{11}, 3A)$, $(M_{12}, 3B)$, $(M_{24}, 3B)$, $(J_2, 3B)$, $(J_2, 5)$, $(Co_3, 3C)$, $(Co_3, 5B)$, $(Co_2, 3B)$, $(Co_2, 5B)$, $(Co_1, 3D)$, $(HS, 3A)$, $(HS, 5B)$, $(He, 3B)$, $(He, 7A)$, $(He, 7B)$, $(Ru, 5B)$, $(Fi_{22}, 3A)$, $(Fi_{22}, 5A)$, $(Fi_{23}, 3A)$, $(Fi_{23}, 5A)$, $(Fi'_{24}, 7A)$, $(F_2, 3A)$, $(F_2, 5A)$, $(F_1, 3C)$.

Suppose that $C_{\overline{G}}(\overline{P})$ is a p-group and $N_G(P)/C_G(P)$ is not a p-group. Then every p-block of $C_G(P)$ is $N_G(P)$-stable, and B cannot be nilpotent. In this way we may eliminate the pairs $(M_{12}, 3A)$, $(J_3, 3B)$, $(J_4, 11)$, $(Co_3, 3A)$, $(Co_3, 3B)$, $(Co_3, 5A)$, $(Co_2, 3A)$, $(Co_2, 5A)$, $(Co_1, 3C)$, $(Co_1, 5C)$, $(HS, 5A)$, $(HS, 5C)$, $(McL, 3)$, $(McL, 5)$, $(Suz, 3B)$, $(He, 7C)$, $(He, 7D)$, $(He, 7E)$, $(Ly, 3B)$, $(Ly, 5)$, $(Ru, 5A)$, $(O'N, 7)$, $(Fi_{22}, 3B)$, $(Fi_{22}, 3C)$, $(Fi_{22}, 3D)$, $(Fi_{23}, 3B)$, $(Fi_{23}, 3C)$, $(Fi'_{24}, 3B)$ [since in this case no involution in G centralizes a subgroup of the form 3^{1+10}], $(Fi'_{24}, 3C)$ [since in this case no involution in $\overline{G}$ centralizes a subgroup of the form C_3^7], $(Fi'_{24}, 7B)$, $(HN, 3B)$, $(HN, 5B)$, $(HN, 5C)$, $(HN, 5D)$, $(HN, 5E)$, $(Th, 3B)$, $(Th, 3C)$, $(Th, 5A)$, $(F_2, 3B)$ [since no involution in $\overline{G}$ centralizes a subgroup of the form 3^{1+8}], $(F_2, 5B)$, $(F_1, 3B)$, $(F_1, 5B)$, $(F_1, 7B)$, $(F_1, 13B)$.

Suppose that $N_{\overline{G}}(\overline{Q}) \cong (\overline{H}_1 \times \overline{H}_2)n$, where n is an integer, and $C_{\overline{G}}(\overline{Q}) \leq \overline{H}_1 \times \overline{H}_2$ for some H_1 and H_2 such that $\overline{Q}$ is a proper normal self-centralizing Sylow p-subgroup of $\overline{H}_1$. Then $H_1 \leq N_G(Q)$ fixes every p-block of $C_G(Q)$ and $N_G(Q)/C_G(Q)$ is not a p-group. It follows that B cannot be nilpotent. In this way we may eliminate the pairs $(Co_1, 5A)$, $(Co_1, 5B)$, $(Co_1, 7)$, $(Suz, 3C)$, $(Suz, 5)$, $(He, 5A)$, $(O'N, 3A)$, $(Fi'_{24}, 3E)$, $(Fi'_{24}, 5A)$, $(HN, 5A)$, $(Th, 7A)$, $(F_2, 7A)$, $(F_1, 5A)$, $(F_1, 7A)$, $(F_1, 11A)$, $(F_1, 13A)$.

Suppose that $N_G(Q) \cong (H_1 \times H_2)n$ for some H_1 and H_2, where n is an integer which is not a power of p, $Q \leq H_1$, and $H_i n$ (with the appropriate action) fixes every p-block of H_i for $i = 1, 2$. Then $N_G(Q)$ fixes every p-block of $C_G(Q)$ and $N_G(Q)/C_G(Q)$ is not a p-group. It follows that B cannot be nilpotent. In this way we may eliminate the pairs $(M_{22}, 3A)$, $(M_{24}, 3A)$, $(J_2, 3A)$, $(J_3, 3A)$, $(Co_1, 3A)$, $(Co_1, 3B)$, $(Suz, 3A)$, $(He, 3A)$, $(Ru, 3A)$, $(Fi'_{24}, 3A)$, $(HN, 3A)$, $(Th, 3A)$, $(F_1, 3A)$.

The only cases left unaccounted for are Fi_{23} and Fi'_{24}, where in each case xZ belongs to the class labelled $3D$. Suppose $\overline{G} = Fi_{23}$ or Fi'_{24} and $xZ \in 3D$. In this case $Z = 1$ (since $p = 3$). We have already seen that a nilpotent 3-block of G cannot contain elements of order three outside of $3D$. Note that x is conjugate to x^{-1} (to see this consider the orders of the centralizers). Irreducible characters in such a block must vanish on $3A$, $3B$ and $3C$. This happens for only one irreducible character, and this lies in a block of defect zero.

If $\overline{G} = M_{23}$ and $p = 3$, then $Z = 1$ and $N_G(Q) \cong (\mathbb{Z}_3 \times A_5) \cdot 2$, $C_G(Q) \cong \mathbb{Z}_3 \times A_5$. Note that M_{23} possesses just one 3-block of maximal defect, which cannot then be nilpotent. Hence we may assume $D = Q$, and so if b_Q is a block of $C_G(Q)$ with $b_Q^G = B$, then b_Q has defect group Q. Now $C_G(Q)$ has two blocks with defect group Q. The action of $N_G(Q)$ on the blocks of $C_G(Q)$ is given by the action of S_5 on the blocks of A_5, so the two blocks with defect group Q are fused by $N_G(Q)$. Hence $[N_G(Q, b_Q) : C_G(Q)] = 1$, and b_Q^G is nilpotent.

If $\overline{G} = J_4$ and $p = 3$, then $Z = 1$ and $N_G(Q) \cong (6M_{22}) \cdot 2$ and $C_G(Q) \cong (6M_{22})$. By [14] $2M_{22}$ possesses precisely two 3-blocks of defect zero fused by $2M_{22}2$ (the rest are fixed). These correspond to two 3-blocks of $C_G(Q)$ with defect group Q fused by $N_G(Q)$. Hence G possesses a nilpotent block with defect group Q (the Brauer correspondent of the above blocks of $C_G(Q)$).

If $\overline{G} = Ly$ and $x \in 3A$, then $Z = 1$ and G possesses a nilpotent block with defect group Q, since $N_G(Q) \cong (3McL) \cdot 2$, $C_G(Q) \cong 3McL$, and McL possesses precisely two 3-blocks of defect zero which are fused in $McL \cdot 2$ (all other 3-blocks of McL are fixed by $McL \cdot 2$). These correspond to two 3-blocks of $C_G(Q)$ with defect group Q fused by $N_G(Q)$. Hence G possesses a nilpotent block with defect group Q (the Brauer correspondent of the above blocks of $C_G(Q)$).

Note that we have shown in particular that whenever p divides the Schur multiplier of a sporadic simple group, there is no nilpotent block of positive defect of the quotient group (by the Sylow p-subgroup of the centre). □

We conclude:

Proposition 4.6 *Let G be a quasisimple group such that $G/Z(G)$ is a sporadic simple group and let p be an odd prime. If B is a nilpotent block of G, then B has defect groups of order at most three.*

We now address Puig's conjecture.

Proposition 4.7 *Let G be a quasisimple group such that $G/Z(G)$ is a sporadic simple group and let p be an odd prime. Let B be a p-block of G. If B has positive defect, then $l(B) > 1$. In particular, B is nilpotent if and only if $l(b_Q) = 1$ for every B-subgroup (Q, b_Q).*

Proof: We may assume that $Z(G)$ is a p'-group. Let D be a defect group of B. If D is cyclic, then the result follows from the theory of blocks with cyclic defect groups. In the following table we list all the numbers of irreducible Brauer characters in blocks with non-cyclic defect groups, along with a reference. A '*' will be used to denote a faithful block in a group with non-trivial centre. The result then follows from examination of the table.

$G/Z(G)$	$\|D\|$	$l(B)$	reference
M_{11}	3^2	7	[18]
M_{12}	$3^3/3^3$	$8/8^*$	[18]
M_{22}	$3^2/3^2/3^2$	$5/5^*/5^*$	[18]
M_{23}	3^2	7	[18]
M_{24}	3^3	7	[18]
J_2	$3^3/3^3/5^2/5^2$	$8/8^*/6/6^*$	[18]
J_3	3^5	8	[18]
J_4	$3^3/3^3/3^2/11^3$	9/9/5/40	[8]/[10]
HS	$3^2/3^2/3^2/5^3/5^3$	$7/7/5^*/10/10^*$	[18]
McL	$3^6/5^3/5^3$	$10/12/12^*$	[18]
Suz	$3^7/3^2/3^7$	$13/5/10^*$	[18]
Ly	$3^7/5^6$	21/35	
He	$3^3/3^2/5^2/7^3$	7/7/14/10	[18]
Ru	$3^3/3^3/5^3/5^3$	$9/9^*/18/18^*$	[18]
$O'N$	$3^4/3^2/7^3/7^3$	$14/5/19/19^*$	[18]
Co_3	$3^7/5^3$	20/18	[18]
Co_2	$3^6/5^3$	23/16	[18]
Co_1	$3^9/3^3/3^2/5^4/5^2/7^2$	29/7/5/29/12/21	[7]
Fi_{22}	$3^9/3^9/5^2/5^2/5^2/5^2$	$22/18^*/16/16^*/16^*/16^*$	[18]
Fi_{23}	$3^{13}/5^2/5^2$	32/16/16	[5]
Fi'_{24}	$3^{16}/3^2/5^2/5^2/5^2/5^2/5^2$	$25/4/16/16/14/16^*/16^*$	[4]
	$5^2/5^2/7^3/7^3/7^3$	$14^*/14^*/22/22^*/22^*$	[4]
Th	$3^{10}/5^3/7^2$	10/30/24	
HN	$3^6/3^2/5^6$	20/7/16	[6]
$F_2 = B$	$3^{13}/3^2/3^2/3^2/3^{13}$	$71/7/7/5/31^*$	[9]
	$5^6/5^2/5^2/5^6/7^2/7^2/7^2/7^2$	$51/16/16/33^*/24/24/21/24^*$	[9]
$F_1 = M$	$3^{20}/3^3/5^9/5^2/7^6/7^2/11^2/13^3$	83/7/91/16/70/24/45/52	[18]

Table 1: Numbers of irreducible Brauer characters in blocks with non-cyclic, non-central defect groups of sporadic groups

If $G = Ly$ and $p = 3$, then by [14] G has thirty 3-regular conjugacy classes. By [31], aside from the principal block, G has five 3-blocks of defect zero and two 3-blocks of defect one. Since we have shown that neither of these blocks of defect one is nilpotent, it follows that they each have two irreducible Brauer characters. Hence the principal block has 21 irreducible Brauer characters for $p = 3$. A similar computation for $p = 5$ shows that the principal 5-block of Ly has 35 irreducible Brauer characters (and this is the unique 5-block with non-cyclic defect groups).

If $G = Th$ and $p = 3$, then by [14] G has sixteen 3-regular conjugacy classes. By [32] G has four 3-blocks of defect zero and one 3-block of defect one (which we have seen cannot be nilpotent, so has two irreducible Brauer characters. Hence the principal 3-block possesses ten irreducible Brauer characters. A similar computation for $p = 5$ shows that the principal 5-block of Th has 30 irreducible Brauer characters (and this is the unique 5-block with non-cyclic defect groups). For $p = 7$, by [32] G has fourteen blocks of defect zero, a block of defect one (with six irreducible Brauer characters, by consideration of the inertial quotient) and the principal block, which must then have 24 irreducible Brauer characters.

If $G/Z(G) = Fi'_{24}$, then the result may be found in [4, 4.2] when $p = 3$, and when $p = 5$ or 7 for non-faithful blocks in the case $|Z(G)| = 3$. Suppose $|Z(G)| = 3$, and consider faithful blocks B with a defect group D covering a block, say c of $Z(G)$. Suppose first $p = 5$. We have $D = 5^2$ and from [4, p.141] $k(B) = 20$. Note that G has only one conjugacy class of elements of order 5. If $x \in D \setminus \{1\}$ and $b \in \mathrm{Blk}(C_G(x))$ with $b^G = B$, then $C_G(x) = 3 \times 5 \times A_9$ and $b = c \times B_0(5) \times b'$ for some $b' \in \mathrm{Blk}(A_9)$ with $D(b') = 5$. As shown in [4, p.114] A_9 has three such blocks $b'_0 = B_0(A_9), b'_1, b'_2$ and $l(b'_0) = l(b'_1) = 4$, $l(b'_2) = 2$. The canonical characters of the root blocks of b'_2 and b'_1 are linear and degree 3 characters of $C_{A_9}(5) = 5 \times A_4$, respectively. Since $N_G(D) = 3.(5^2{:}4A_4 \times A_4).2$ and a Sylow 3-subgroup of $N_G(D)$ is isomorphic to 3_+^{1+2}, it follows that $c \times B_0(5) \times b'_0$ and $c \times B_0(5) \times b'_2$ induce the same block B of G and so $l(B) = 20 - 4 - 2 = 14$. Also $c \times B_0(5) \times b'_1$ induces another block B of G and $l(B) = 20 - 4 = 16$. If $p = 7$, then by [4, p.141], $k(B) = k(B_0(Fi'_{24}))$ and $C_G(x) = 3 \times C_{Fi'_{24}}(x)$ for any $x \in D \setminus \{1\}$. Thus $l(B) = l(B_0(Fi_{24})) = 22$. □

5 Notation for classical groups and their blocks

Let V be a linear, unitary, non-degenerate orthogonal or symplectic space over the field $\mathbb{F}_q$, where $q = r^a$ for some prime $r \neq p$. We will follow the notation of [3], [11], [16] and [17].

If V is orthogonal (and q is odd), then there is a choice of equivalence classes of quadratic forms. Write $\eta(V)$ for the type of V as defined in [17, p.124], so $\eta(V) = \eta = +$ or $-$. Write $\eta(V) = +$ if V is linear and $\eta(V) = -$ if V is unitary. If V is non-degenerate orthogonal or symplectic, then denote by $I(V)$ the group of isometries on V and let $I_0(V) = I(V) \cap \mathrm{SL}(V)$.

If V is symplectic, then $I(V) = I_0(V) = \mathrm{Sp}_{2n}(q)$.

If V is a $(2n+1)$-dimensional orthogonal space, then $I(V) = \langle -1_V \rangle \times I_0(V)$ with $I_0(V) = \mathrm{SO}_{2n+1}(q)$.

If V is a $2n$-dimensional orthogonal space, then $I(V) = O^\eta(V) = O^\eta_{2n}(q)$ and $I_0(V) = \mathrm{SO}^\eta_{2n}(q)$.

If V is a $2n$-dimensional non-degenerate orthogonal or symplectic space, then denote by $J_0(V)$ the conformal isometries of V with square determinant. If V is orthogonal of dimensional at least two, then write $D_0(V)$ for the special Clifford group of V (cf. [17]).

Denote by $\mathrm{GL}^+(V)$ the general linear group $\mathrm{GL}(V)$ and $\mathrm{GL}^-(V)$ the unitary group $\mathrm{U}(V)$.

Let $G = \mathrm{GL}^\eta(V)$ or $I(V)$. Write $\mathcal{F}_q = \mathcal{F}_q(G)$ for the set of polynomials (with coefficients in $\mathbb{F}_q$) serving as elementary divisors for semisimple elements of G (cf. [3, p.6]). For $\Gamma \in \mathcal{F}_q(G)$, let d_Γ be the degree of Γ, and δ_Γ be the reduced degree defined as in [3], [16] and [17]. So $\delta_\Gamma = d_\Gamma$ or $\delta_\Gamma = \frac{1}{2}d_\Gamma$ according as d_Γ is even or odd (note that if V is symplectic or orthogonal, Γ must have even degree unless $\Gamma = X \pm 1$).

If $G = \mathrm{GL}(V)$, then let $\epsilon_\Gamma = 1$. Otherwise ϵ_Γ is given by [3, p.6]. Let e_Γ be the multiplicative order of $\epsilon_\Gamma q^{\delta_\Gamma}$ modulo p. Thus we may write $e_\Gamma \delta_\Gamma = ep^{\alpha_\Gamma}\delta'_\Gamma$ for some α_Γ and δ'_Γ with $p \nmid \delta'_\Gamma$, where $e = e_{X-1}$.

Given a semisimple element $s \in G$, there is a unique orthogonal decomposition $V = \sum_{\Gamma\in\mathcal{F}_q} V_\Gamma(s)$, with $s = \prod_{\Gamma\in\mathcal{F}_q} s(\Gamma)$, where the $V_\Gamma(s)$ are nondegenerate subspaces of V and $s(\Gamma) \in \mathrm{GL}(V_\Gamma(s))$, $\mathrm{U}(V_\Gamma(s))$ or $I(V_\Gamma(s))$ (depending on G) has minimal polynomial Γ. This is called the primary decomposition of s. Write $m_\Gamma(s)$ for the multiplicity of Γ in $s(\Gamma)$. We have $C_G(s) = \prod_{\Gamma\in\mathcal{F}_q} C_\Gamma(s)$, where $C_\Gamma(s) = I(V_\Gamma(s))$ or $\mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma})$ as appropriate.

6 Blocks of linear and unitary groups

Suppose $G = \mathrm{GL}^\eta_n(q) = \mathrm{GL}^\eta(V)$ and p is odd and distinct to r, and let B be a p-block of G with a defect group D and label (s, κ). Then we may write

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+, \quad s = s_0 \times s_+, \tag{6.1}$$

where $V_0 = C_V(D)$, $V_+ = [D, V]$, $s_0 \in G_0 = \mathrm{GL}^\eta(V_0)$ and $s_+ \in G_+ := \mathrm{GL}^\eta(V_+)$. For convenience we denote $\mathrm{GL}^\eta(V)$ by $G(V)$ and $\mathrm{SL}^\eta(V)$ by $S(V)$.

Theorem 6.1 *Let $G = \mathrm{GL}^\eta(V) = \mathrm{GL}^\eta(n, q)$ and suppose p is odd with $p \nmid q$. Then the following are equivalent.*

(a) B is a nilpotent block of G.

(b) $m_\Gamma(s_+) = e_\Gamma = 1$ for all $\Gamma \in \mathcal{F}_q$ which are elementary divisors of s_+.

(c) κ_Γ is a e_Γ-core of $m_\Gamma(s)$ whenever $e \nmid \delta_\Gamma$, and $m_\Gamma(s) \leq 1$ whenever $e \mid \delta_\Gamma$, where $\kappa_\Gamma = \emptyset$ is viewed as an e_Γ-core of $0 = m_\Gamma(s)$.

(d) Let (D, b_D) be a Sylow B-subgroup and θ the canonical character of b_D. Then $C_G(D) = G_0 \times C_+$ and $\theta = \theta_0 \times \theta_+$, where $C_+ := C_{G_+}(D_+)$ is regular in G_+, θ_0 is an irreducible character of defect 0 of G_0 labelled by (s_0, κ) and $\theta_+ = \pm R^{C_+}_{T_+}(s_+)$ with $T_+ = C_{G_+}(s_+)$ a torus of both G_+ and C_+, and $D_+ = O_p(T_+)$. Here $R^{C_+}_{T_+}(s_+)$ is the Deligne-Lusztig generalized character.

In particular, if B is nilpotent, then D is abelian.

Proof: Let $s_+ = \prod_\Gamma s(\Gamma)$ be a primary decomposition, so that $V_+ = \bigoplus_\Gamma V_\Gamma$ with V_Γ the underlying space of $s(\Gamma)$. Write m_Γ for $m_\Gamma(s_+)$. Then

$$C_{G_+}(s_+) = \prod_\Gamma C_\Gamma, \tag{6.2}$$

where $C_\Gamma \cong \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma, q^{\delta_\Gamma})$. We may suppose $D_+ \in \mathrm{Syl}_p(C_{G_+}(s_+))$, so that

$$D_+ = \prod_\Gamma D_\Gamma, \quad D_\Gamma \in \mathrm{Syl}_p(C_\Gamma). \tag{6.3}$$

So D is a direct product of wreath product p-groups.

Let Γ be an elementary divisor of s_+. Since $C_{V_\Gamma}(D_\Gamma) = 0$, it follows that p divides $q^{\delta_\Gamma m_\Gamma} - \epsilon_\Gamma$ and so $e_\Gamma \mid m_\Gamma$. Hence we may write $m_\Gamma = e_\Gamma w_\Gamma$ for some w_Γ. Let $A(D)$ be the subgroup of D generated by all the abelian normal subgroups of D. By [1, Theorem 2], $A(D)$ is the base subgroup of D. Write $R = A(D)$. Then

$$R = D_0 \times \prod_\Gamma (R_\Gamma)^{w_\Gamma}, \quad C_G(R) = G_0 \times \prod_\Gamma (K_\Gamma)^{w_\Gamma}, \quad K_\Gamma \cong \mathrm{GL}^\epsilon(\delta'_\Gamma, q^{ep^{\alpha_\Gamma}}) \tag{6.4}$$

where $\epsilon = \epsilon_{X-1}$ and $R_\Gamma = O_p(Z(K_\Gamma))$ is cyclic and $(R_\Gamma)^{w_\Gamma}$ is a diagonal subgroup of $\mathrm{GL}^{\epsilon_\Gamma}(w_\Gamma, q^{\delta_\Gamma e_\Gamma}) \leq C_\Gamma$. Thus $C_{G_\Gamma}((R_\Gamma)^{w_\Gamma}) = (K_\Gamma)^{w_\Gamma}$, $C_G(R)$ is regular in G,

$$N_{G_\Gamma}((R_\Gamma)^{w_\Gamma}) = K_\Gamma \wr \mathbf{S}(w_\Gamma),$$

and we may suppose $s \in C_G(R)$, where $G_\Gamma := G(V_\Gamma)$ and $\mathbf{S}(m)$ is the symmetric group on m letters.

Suppose $w_\Delta \geq 2$ for some Δ. Then there is $P(D_\Delta) \leq (R_\Delta)^{w_\Delta}$ such that

$$C_{G_\Delta}(P(D_\Delta)) = (K_\Delta)^{w_\Delta - 2} \times \mathrm{GL}^\epsilon(2\delta'_\Delta, q^{ep^{\alpha_\Delta}}) \quad \text{and} \quad P(D_\Delta) = O_p(C_{G_\Delta}(P(D_\Delta))).$$

Thus $C_{C_\Delta}(P(D_\Delta)) = \mathrm{GL}^{\epsilon_\Delta}(1, q^{\delta_\Delta e_\Delta})^{w_\Delta - 2} \times \mathrm{GL}^{\epsilon_\Delta}(2, q^{\delta_\Delta e_\Delta})$ and

$$N_{C_{G_\Delta}(P(D_\Delta))}((R_\Delta)^{w_\Delta}) = (K_\Delta)^{w_\Delta - 2} \times K_\Delta \wr \mathbf{S}(2).$$

There is an element y_Δ of $(N_{C_\Delta}((R_\Delta)^{w_\Delta}) \cap C_{G_\Delta}(P(D_\Delta))) \setminus C_{C_\Delta}((R_\Delta)^{w_\Delta})$ which swaps exactly two factors K_Δ in $C_{G_\Delta}((R_\Delta)^{w_\Delta})$, $|y_\Delta| = 4$, $y_\Delta^2 \in C_{C_\Delta}((R_\Delta)^{w_\Delta})$ and $\det(y_\Delta) = 1$.

Writing $y_\Gamma := 1$ when $\Gamma \neq \Delta$, define

$$y := 1_{V_0} \times \prod_\Gamma y_\Gamma \quad \text{and} \quad P(D) := D_0 \times \prod_{\Gamma \neq \Delta} (R_\Gamma)^{w_\Gamma} \times P(D_\Delta). \tag{6.5}$$

Then $y \in (N_{C_G(P(D))}(R) \cap C_G(s)) \setminus C_G(R)$ and $y^2 \in C_G(R)$. Let (R, b_R) be a B-subgroup, so that $D(b_R) = R$ and we may suppose $b_R \subseteq \mathcal{E}_p(C_G(R), (s))$. Since $y \in C_G(s)$, it follows that $y \in N_G(R, b_R) \setminus C_G(R)$, so B is not nilpotent, a contradiction. Thus $m_\Gamma = e_\Gamma$ for all Γ and D is abelian with each D_Γ cyclic.

For each Γ,

$$N_{\mathrm{GL}^{\epsilon_\Gamma}(e_\Gamma, q^{\delta_\Gamma})}(D_\Gamma) = \langle \tau_\Gamma, C_{\mathrm{GL}^{\epsilon_\Gamma}(e_\Gamma, q^{\delta_\Gamma})}(D_\Gamma) \rangle, \quad C_{\mathrm{GL}^{\epsilon_\Gamma}(e_\Gamma, q^{\delta_\Gamma})}(D_\Gamma) = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma \delta_\Gamma}),$$

where $\tau_\Gamma \in \mathrm{GL}^{\epsilon_\Gamma}(e_\Gamma, q^{\delta_\Gamma})$ has order e_Γ modulo $\mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma\delta_\Gamma})$. If $\tau = 1_{V_0} \times \prod_\Gamma \tau_\Gamma$, then $\tau \in N_G(D) \cap C_G(s)$ and so $\tau \in N_G(D, b_D)$, where b_D is the block of $C_G(D)$ labelled by (s, κ).

Since e_Γ and p are coprime, it follows that τ is a p'-element, $e_\Gamma = 1$ and $C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$. In particular, $C_{G_+}(s_+)$ is a torus and $e \mid \delta_\Gamma$.

Conversely, if $m_\Gamma(s_+) = e_\Gamma = 1$, then $C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(1, q^{\delta_\Gamma})$ and so $N_{C_\Gamma}(D_\Gamma) = C_{C_\Gamma}(D_\Gamma) = C_\Gamma$. Thus D and $C_G(D)$ are abelian, and

$$N_{C_G(s)}(D) = C_{C_G(s)}(D) = C_G(s).$$

Now the canonical character of b_D is labelled by $(s, 1)$ and is stable in $N_G(D, b_D)$. Let $x \in N_G(D, b_D)$. Then s^x and s are $C_G(D)$-conjugate elements of the abelian group $C_G(D)$, and so $s^x = s$. Hence $x \in C_G(s) \leq C_G(D)$, and we have shown $N_G(D, b_D) = C_G(D)$. By Proposition 2.1, B is nilpotent.

Note that $e_\Gamma = 1$ if and only if $e \mid \delta_\Gamma$. Since D is a Sylow p-subgroup of $C_G(s)$, it follows that D_0 is a Sylow p-subgroup of $C_{G_0}(s_0)$. But $m_\Gamma(s) = m_\Gamma(s_0) + m_\Gamma(s_+)$, so $m_\Gamma(s_+) = e_\Gamma = 1$ if and only if $m_\Gamma(s_0) = 0, m_\Gamma(s) = 1$ and $e \mid \delta_\Gamma$. In addition, $m_\Gamma(s) \neq 0$ and $m_\Gamma(s_+) = 0$ if and only if $m_\Gamma(s_0) = m_\Gamma(s) \neq 0$ and $e \nmid \delta_\Gamma$. This happens by [16, Theorem (5D)] if and only if κ_Γ is an e_Γ-core of $m_\Gamma(s)$ and $e \nmid \delta_\Gamma$. Thus (b) and (c) are equivalent.

Let (D, b_D) be a Sylow B-subgroup and θ the canonical character of b_D. If (b) holds, then $C_G(D)$ is regular and by [11, Theorem 3.2], (D, b_D) is labelled by (D, s, κ). Follow the notation of (d). By [16, p. 135], θ_0 is an irreducible character of G_0 labelled by (s_0, κ), θ_+ is labelled by $(s_+, -)$ and so $\theta_+ = \pm R_{T_+}^{C_+}(s_+)$. Conversely, if (d) holds, then $m_\Gamma(s_+) = e_\Gamma = 1$ as $C_{G_+}(s_+)$ is a torus of G_+.

□

For integers c and m, we write $p^c \| m$ when $p^c \mid m$ and $p^{c+1} \nmid m$.

Remark 6.2 *In the notation of the proof above, we may suppose the element $\tau \in C_{G_+}(s_+)$ has determinant 1 whenever $e_\Gamma \geq 2$ for some Γ.*

Proof We may suppose $q = q^{\delta_\Gamma}$, so that $e_\Gamma = e$. Let $\mathbf{T}$ be the diagonal maximal torus of $\mathbf{G} = \mathrm{GL}(\mathbb{F} \otimes V)$, σ the Frobenius map of $\mathbf{G}$ such that $C_\mathbf{G}(\sigma) = \mathrm{GL}^\epsilon(e, q)$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$.

Choose matrices P_{ij} (with $i \neq j$) of $\mathbf{G}$ such that P_{ij} acts on $\mathbf{T}$ as the permutation swapping the (i, i) and (j, j) entries of $\mathbf{T}$, the entries of P_{ij} are 1 or -1, $\det(P_{ij}) = 1$ and P_{ij} is fixed by σ. If W is generated by the matrices P_{ij}, then $N_\mathbf{G}(\mathbf{T}) = \mathbf{T}W$ and $W\mathbf{T}/\mathbf{T} \cong \mathbf{S}(e)$.

Note $\omega^\sigma = \omega$ for each $\omega \in W$. Let $\omega_0 \in W$ such that

$$C_\mathbf{T}(\sigma\omega_0) \cong \mathrm{GL}^\epsilon(1, q^e),$$

so that ω_0 acts on $\mathbf{T}$ as the cycle $(1, 2, \cdots, e)$. Now $C_\mathbf{T}(\sigma\omega_0)$ is conjugate in $\mathbf{G}$ to the Coxeter torus $\mathrm{GL}^\epsilon(1, q^e)$ of $C_\mathbf{G}(\sigma)$ and ω_0 normalizes $C_\mathbf{T}(\sigma\omega_0)$. Thus there is an element $\beta \in \mathrm{SL}^\eta(e,q)$ such that β normalizes the torus $\mathrm{GL}^\epsilon(1,q^e)$ and β has order e modulo $\mathrm{GL}^\epsilon(1,q^e)$. Since $N_{\mathrm{GL}^\epsilon(e,q)}(\mathrm{GL}^\epsilon(1,q^e))/\mathrm{GL}^\epsilon(1,q^e) \cong \mathbb{Z}_e$, it follows that

$$N_{\mathrm{GL}^\epsilon(e,q)}(\mathrm{GL}^\epsilon(1,q^e)) = \langle \beta, \mathrm{GL}^\epsilon(1,q^e)\rangle$$

and we may suppose $\tau = \beta$.

7 A set of technical conditions

In order to investigate nilpotent blocks of exceptional groups of Lie type it is not sufficient just to find the nilpotent blocks of classical groups. We need in addition some somewhat technical properties which we will identify in classical groups and their extensions by diagonal automorphisms which relate to nilpotency (in particular, they will be used to examine centralizers of elements of defect groups of nilpotent blocks).

These properties also ensure that Puig's conjecture holds for the groups under consideration.

We state these properties in this section, along with some general results which will be needed in proving that they hold for classical groups.

Let G be a finite group, Q a p-subgroup of G, and $B \in \mathrm{Blk}(G)$. If p is odd, we denote by $A(Q)$ the subgroup of Q generated by all the abelian normal subgroups of Q. Recall that a B-subgroup (R, b_R) is called *self-centralizing* if $Z(R)$ is a defect group of $b_R \in \mathrm{Blk}(C_G(R))$.

We will prove for some finite groups of Lie type that one of (a)-(d) of the following holds. A feature of these properties is that none can be satisfied by a nilpotent block with non-abelian defect groups.

Property 7.1 *Let K be a normal subgroup of a finite group H, and let $B \in \mathrm{Blk}(K)$ and $B_H \in \mathrm{Blk}(H)$ such that B_H covers B.*

(*a*) *There exist B-subgroups $(P,g) \leq (R,b)$, where R is abelian, with abelian defect groups $D(g)$ and $D(b)$ respectively such that $D(g) = D(b)$, and an element $y \in N_{C_K(P)}(R)$ such that $y^4 = 1$, $[y,x] \notin Z(K)$ for some $x \in R$ and $y^2 \in C_K(R)$, and such that $\theta^y = \theta$, where θ is the canonical character of b. There exist B_H-subgroups $(P, g_H) \leq (R, b_H)$ where g_H covers g, b_H covers b, such that $b_H^y = b_H$ and $D(b_H) = D(g_H)$ is abelian for defect groups $D(b_H)$ and $D(g_H)$ of b_H and g_H respectively.*

(*a**) *Property (a) above holds, and there exist subgroups $N_i \lhd M_i$ of H, and characters $\theta_i \in \mathrm{Irr}(N_i)$ for $i = 1, 2$ such that M_i/N_i is abelian,*

$$Z(K) \leq N_1 \times N_2 \leq C_K(R) \leq C_H(R) \leq M_1 \circ M_2,$$

θ covers $\theta_1 \times \theta_2$, $Z_0 \cap N_2 = 1$, θ_2 has a y-stable extension to M_2 and $[y,x] = 1$ or $[y,x] \in Z_0N_2$ according as $x \in M_1$ or M_2, where $Z_0 \leq Z(M_1) \cap Z(M_2)$ such that $M_1 \circ M_2$ is a central product over Z_0

(*b*) *$D(B) \cong 3_+^{1+2}$, $l(B) \geq 2$, and either $D(B_H) = D(B)$ or $D(B_H) \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$.*

(*c*) $D(B) = 3^2$, $l(B) \geq 2$, *and either* $D(B_H) = D(B)$ *or* $D(B_H) \cong 3_+^{1+2}$.

(*d*) *Both* $D(B)$ *and* $D(B_H)$ *are abelian.*

Remark 7.2 (*i*) *Suppose that* $(P, g) \leq (R, b)$ *are* B*-subgroups with abelian defect groups* $D(g)$ *and* $D(b)$, *and* R *is abelian. By [15, Lemma 4.1], there exists a* B_H*-subgroup* (P, g_H) *such that* g_H *covers* g. *Since* $R \leq D(b)$ *is abelian and* $(P, g) \leq (R, b)$, *it follows that* (R, b) *is a* g*-subgroup and by [15, Lemma 4.1] again, there exists a* g_H*-subgroup* (R, b_H) *such that* b_H *covers* b. *Thus* (R, b_H) *is a* B_H*-subgroup and* $(P, g_H) \leq (R, b_H)$.

(*ii*) *Note that in the notation of the proof of Theorem 6.1,* (R, b_R) *is self-centralizing.*

We also observe that there is some redundancy in (*a*) when (*a*∗) holds:

Remark 7.3 *In the notation of Property* 7.1 (*a*) *and* (*a*∗) *suppose* $N_1 \times N_2 \leq E \leq M_1 \circ M_2$ *such that* y *normalizes* E *and suppose* $\varphi \in \mathrm{Irr}(E \mid \theta_1 \times \theta_2)$. *Then* $\varphi^y = \varphi$.

Proof: Since φ has an extension $\tilde{\varphi}$ to EZ_0, it follows that we may suppose $Z_0 \leq E$. Let $\pi_i : (M_1 \circ M_2) \to M_i/Z_0$ be the canonical projection and let $E_i/Z_0 = \pi_i(E)$ for some $E_i \leq M_i$. Then

$$N_1 \times N_2 \leq E \leq E_1 \circ E_2 \leq M_1 \circ M_2.$$

Let ζ be the y-stable extension of θ_2 to M_2, and set $\tilde{\theta}_2 = \zeta|_{E_2}$. Then $\tilde{\theta}_2$ is an extension of θ_2 to E_2 which is stabilized by y. Since $[y, x] = 1$ or N_2Z_0 according as $x \in M_1$ or $x \in M_2$, it follows that y centralizes E_1 and $[y, x] \in N_2Z_0$ for any $x \in E_2$. By Lemma 2.7, $\varphi^y = \varphi$. □

Applying Remark 7.3 to the canonical characters of b and b_H, we see that in the notation of (*a*), $\theta^y = \theta$ and $b_H^y = b_H$ are automatically satisfied if the other parts of (*a*∗) hold.

Before stating the key consequence of Property 7.1, we need the following:

Lemma 7.4 *Let* B *be a block of a finite group* G *and suppose there are* B*-subgroups* $(P, b_P) \leq (R, b_R)$ *such that* R *is abelian,* b_P *has abelian defect groups and there is* p*-regular element* $y \in N_{C_G(P)}(R) \setminus C_G(R)$ *such that* $b_R^y = b_R$. *Then there is a* B*-subgroup* (Q, b_Q) *such that* $l(b_Q) > 1$.

Proof: For convenience write $L = C_G(P)$. Note first that b_P is not nilpotent, since $yC_G(R) \in N_L(R, b_R)/C_G(R)$ is not a p-element and (R, b_R) is a b_P-subgroup. Hence by Lemma 2.4 the unique block $\overline{b_P}$ of $\overline{L} := L/P$ dominated by b_P is not nilpotent either. But $\overline{b_P}$ has abelian defect groups, so by [30, Theorem 3] there is a $\overline{b_P}$-subgroup $(\overline{Q}, b_{\overline{Q}})$, where $\overline{Q} = Q/P$ for some $Q \geq P$, such that $l(b_{\overline{Q}}) > 1$.

Note that $\overline{C_L(Q)} \lhd C_{\overline{L}}(\overline{Q})$ and $C_{\overline{L}}(\overline{Q})/\overline{C_L(Q)}$ is a p-group. By [33, Lemma 1] there is a b_P-subgroup (Q, b_Q) such that $b_{\overline{Q}}$ is the unique block of $C_{\overline{L}}(\overline{Q})$ covering the block

$b'_{\overline{Q}}$ of $\overline{C_L(Q)}$ dominated by b_Q. Note that $l(b_Q) = l(b'_{\overline{Q}}) = l(b_{\overline{Q}}) > 1$. But $P \leq Q$, so (Q, b_Q) is also a B-subgroup, and we are done. □

We make the key observation, and see also that the conjecture of Puig is a consequence of any of Properties 7.1 (a)-(d).

Corollary 7.5 *Suppose one of Property 7.1 (a) − (d) holds for a block B. If B has non-abelian defect groups, then there is a B-subgroup (Q, b_Q) such that $l(b_Q) > 1$, and hence B is not nilpotent. In particular, B is nilpotent if and only if $l(b_Q) = 1$ for every B-subgroup (Q, b_Q).*

Proof: If (a) holds, then the result follows immediately from Lemma 7.4, since B is not nilpotent and there is a B-subgroup (Q, b_Q) such that $l(b_Q) > 1$. If (b) or (c) holds, then $l(B) > 1$, so B is not nilpotent, and we may take $(Q, b_Q) = (1, B)$. If (d) holds, then this is [30, Theorem 3]. □

We prove a lemma which will be useful in establishing the given properties. Let H be a finite group, $K \lhd H$, $Z \leq Z(H) \cap K$ and $\overline{K} := K/Z \leq \overline{H} := H/Z$. Let $\overline{B} \in \mathrm{Blk}(\overline{K})$ and $B \in \mathrm{Blk}(K)$ dominating $\overline{B}$, and (Q, b_Q) a B-subgroup. Let $\gamma : H \to \overline{H}$ be the natural homomorphism, and write $\overline{X} = \gamma(X)$ for any $X \subseteq H$,

If Z is a p'-group, then $(\overline{Q}, \bar{b}_Q)$ is defined in Section 2 and it is a $\overline{B}$-subgroup. Suppose Z is a p-group. Then $\gamma^{-1}(C_{\overline{K}}(\overline{Q})) \leq N_K(QZ)$ and $\gamma^{-1}(C_{\overline{K}}(\overline{Q}))/C_K(Q)$ is a p-group. Thus $\gamma^{-1}(C_{\overline{K}}(\overline{Q}))$ has a unique block $\hat{b}_Q$ covering b_Q and we denote $\bar{b}_Q$ the block of $C_{\overline{K}}(\overline{Q})$ corresponding to $\hat{b}_Q$, so that by [33, Lemma 1], $(\overline{Q}, \bar{b}_Q)$ is also a $\overline{B}$-subgroup.

In general, since $\overline{K} \cong (K/O_p(Z))/(Z/O_p(Z))$ and $Z/O_p(Z) \leq Z(K/O_p(Z))$, it follows that $(\overline{Q}, \bar{b}_Q)$ is defined and is a $\overline{B}$-subgroup.

Lemma 7.6 *Let H be a finite group, $K \lhd H$, $Z \leq Z(H) \cap K$. Define $\overline{K} := K/Z$ and $\overline{H} := H/Z$. Let $\overline{B} \in \mathrm{Blk}(\overline{K})$ and $B \in \mathrm{Blk}(K)$ dominating $\overline{B}$. Suppose B-subgroups $(P, g) \leq (R, b)$ satisfy Property* 7.1 *(a∗), and suppose $Z(\overline{K}) = Z(K)/Z$. In addition, if $Z = O_p(Z)$, then suppose, moreover that $C_H(P)/Z = C_{\overline{H}}(\overline{P})$ and $C_H(R)/Z = C_{\overline{H}}(\overline{R})$. Then the $\overline{B}$-subgroups $(\overline{P}, \bar{g}) \leq (\overline{R}, \bar{b})$ satisfies Property* 7.1 *(a∗).*

Proof: Let $\overline{B}_{\overline{H}} \in \mathrm{Blk}(\overline{H})$ covering $\overline{B}$, and $B_H \in \mathrm{Blk}(H)$ dominating $\overline{B}_{\overline{H}}$ and $\chi \in \mathrm{Irr}(\overline{B}_{\overline{H}})$, so that χ covers some $\psi \in \mathrm{Irr}(\overline{B})$. But $\mathrm{Irr}(\overline{B}_{\overline{H}}) \subseteq \mathrm{Irr}(B_H)$ and $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$, so B_H covers B.

Let f be the unique block of Z covered by B. Then each character χ in $\mathrm{Irr}(B)$ covers a character in $\mathrm{Irr}(f)$. Since $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$, it follows that f is the principal block. Since (P, g) is a B-subgroup and $Z \leq Z(K)$ and since B covers f, it follows that g covers f, and similarly, b covers f. The same applies to B_H and to B_H-subgroups.

Since $C_K(PZ) = C_K(P)$, it follows that we may suppose $O_p(Z) \leq P$. Let $y \in N_{C_K(P)}(R)$ such that $y^4 = 1$, $[y, x] \notin Z(K)$ for some $x \in R$, $y^2 \in C_K(R)$, and suppose N_i and M_i are subgroups of H, and $\theta_i \in \mathrm{Irr}(N_i)$ for $i = 1, 2$, such that M_i/N_i is abelian,

$$Z \leq Z(K) \leq N_1 \times N_2 \leq C_K(R) \leq C_H(R) \leq M_1 \circ M_2,$$

θ covers $\theta_1 \times \theta_2$, $Z_0 \cap N_2 = 1$, θ_2 has a y-stable extension $\tilde{\theta}_2$ to M_2 and $[y, x] = 1$ or in Z_0N_2 according as $x \in M_1$ or M_2, where $Z_0 \leq Z(M_1) \cap Z(M_2)$ such that $M_1 \circ M_2$ is the central product over Z_0 and θ is the canonical character of b.

It suffices to consider the cases p-group and a p'-group separately. Then by [33, Lemma 1 (iii)] and Lemma 2.5, $(\overline{P}, \bar{g}) \leq (\overline{R}, \bar{b})$ are $\overline{B}$-subgroups. If Z is a p'-group, then $C_K(R)/Z = C_{\overline{K}}(\overline{R})$. If Z is a p-group, then $C_H(R)/Z = C_{\overline{H}}(\overline{R})$, $\gamma^{-1}(C_{\overline{H}}(\overline{R})) = C_H(R)$ and so $\gamma^{-1}(C_{\overline{K}}(\overline{R})) = C_K(R)$. Thus in either case $C_K(R)/Z = C_{\overline{K}}(\overline{R})$ and $D(b)Z/Z = D(\bar{b})$. Similarly, $D(\bar{g}) = D(g)Z/Z$, and $D(\bar{g}) = D(\bar{b})$ is abelian as $D(g) = D(b)$ is abelian.

Let $\bar{y} = \gamma(y)$, so that $\bar{y}^4 = 1$, $\bar{y} \in C_{\overline{K}}(\overline{P}) \cap N_{\overline{K}}(\overline{R})$, $\bar{y}^2 \in C_{\overline{K}}(\overline{R})$. Since $[y, x] \notin Z(K)$ for some $x \in R$ and since $Z(\overline{K}) = Z(K)/Z$, it follows that $[\bar{y}, \bar{x}] \notin Z(\overline{K})$ and in particular, $\bar{y} \notin C_{\overline{K}}(\overline{R})$.

Let $\overline{N}_i = \gamma(N_i)$ and $\overline{M}_i = \gamma(M_i)$. Then $\overline{N}_i \lhd \overline{M}_i$ and $\overline{M}_i/\overline{N}_i$ is abelian such that

$$Z(\overline{K}) \leq \overline{N}_1 \times \overline{N}_2 \leq C_{\overline{K}}(\overline{R}) \leq C_{\overline{H}}(\overline{R}) \leq \overline{M}_1 \circ \overline{M}_2,$$

where $\overline{M}_1 \circ \overline{M}_2$ is the central product over $\overline{Z}_0 = \gamma(Z_0)$.

Since $Z \leq Z(K) \leq N_1 \times N_2$ and θ is the canonical character of b, it follows that θ is the lift of the canonical character $\bar{\theta}$ of $\bar{b}$. Similarly, since θ covers $\theta_1 \times \theta_2$, it follows that $Z \leq \ker(\theta_1 \times \theta_2)$ and $\theta_1 \times \theta_2$ is the lift of $\bar{\theta}_1 \times \bar{\theta}_2$ for some $\bar{\theta}_i \in \mathrm{Irr}(\overline{N}_i)$. In particular, $\bar{\theta} \in \mathrm{Irr}(C_{\overline{K}}(\overline{R}) \mid \bar{\theta}_1 \times \bar{\theta}_2)$. If $\varphi \in \mathrm{Irr}(M_1 \circ M_2 \mid \theta_1 \times \theta_2)$, then by Lemma 2.7 $\varphi = \psi \circ (\tilde{\theta}_2 \lambda)$ for some $\psi \in \mathrm{Irr}(M_1 \mid \theta_1)$ and $\lambda \in \mathrm{Irr}(M_2/Z_0N_2)$. Since $[y, x] \in N_2Z_0$ for all $x \in M_2$, it follows that $\lambda^y = \lambda$ and so $\tilde{\theta}_2\lambda$ is y-invariant. But $Z \leq \ker(\varphi)$, so $Z \cap M_2 \leq \ker(\tilde{\theta}_2\lambda)$, and $\tilde{\theta}_2\lambda$ can be viewed as a character of $\mathrm{Irr}(\overline{M}_2)$, which is a $\bar{y}$-invariant extension of $\bar{\theta}_2$ to $\overline{M}_2$. Thus Property 7.1 ($a*$) holds for $\overline{B}$.

Similarly, if $(P, g_H) \leq (R, b_H)$ are B_H-blocks such that $D(g_H) = D(b_H)$ is abelian, and g_H and b_H cover g and b, respectively, then there exist $\overline{B}_{\overline{H}}$-subgroups $(\overline{P}, \bar{g}_{\overline{H}}) \leq (\overline{R}, \bar{b}_{\overline{H}})$ such that $\bar{g}_{\overline{H}}$ is dominated by g_H and $\bar{b}_{\overline{H}}$ is dominated by b_H and $D(\bar{g}_{\overline{H}}) = D(g_H)Z/Z$ and $D(\bar{b}_{\overline{H}}) = D(b_H)Z/Z$. Thus $D(\bar{g}_{\overline{H}}) = D(\bar{b}_{\overline{H}}) = D(b_H)Z/Z$ is abelian.

Since g_H covers g, it follows that the canonical character θ_H of g_H covers the canonical character θ of g. But θ_H is the lift of the canonical character of $\bar{g}_{\overline{H}}$ and θ is the lift of the canonical character of $\bar{g}$, so $\bar{g}_{\overline{H}}$ covers $\bar{g}$. Similarly, $\bar{b}_{\overline{H}}$ covers $\bar{b}$. This proves that Property 7.1 (a) holds for $(\overline{P}, \bar{g}) \leq (\overline{R}, \bar{b})$. □

8 Classical groups

Suppose p is odd. In this section we demonstrate that every nilpotent block of a classical group of has abelian defect groups.

Proposition 8.1 *Let $K := \mathrm{SL}_n^\eta(q) \leq H \leq G := \mathrm{GL}_n^\eta(q) = \mathrm{GL}^\eta(V)$, $Z \leq Z(K)$, $B \in \mathrm{Blk}(K)$, $B_H \in \mathrm{Blk}(H)$ covering B. Let $B_G \in \mathrm{Blk}(G)$ be a weakly regular cover of B_H. Write $R := A(D(B_G)) \cap K$. Then Property* 7.1 *($a*$) holds for some B-subgroups $(P, g) \leq (R, b)$ with $C_H(P)/Z = C_{\overline{H}}(\overline{P})$ and $C_H(R)/Z = C_{\overline{H}}(\overline{R})$, or Property* 7.1 *(b) or (d) holds, where $\overline{X} = XZ/Z$ for any $X \leq G$. Moreover, if Property* 7.1 *(b) holds, then $n = 3d$ with $\gcd(6, d) = 1$ and $3 \| (q - \eta)$.*

PROOF: Suppose B_G is labelled by (s,κ). Since B_H covers B, it follows that $D(B) = D(B_H) \cap K$ for some defect group $D(B_H)$. There exists a defect group $D(B_G)$ such that $D(B_H) = D(B_G) \cap H$, so

$$D(B) = D(B_H) \cap K = D(B_G) \cap K \quad \text{and} \quad D(B_H) = D(B_G) \cap H.$$

We may suppose $D(B_G) \in \mathrm{Syl}_p(C_G(s))$.

Suppose the decompositions $V = V_0 \perp V_+, D(B_G) = D_0 \times D_+$ and $s = s_0 \times s_+$ are given as in (6.1). Set $R_G = A(D(B_G))$. Then R_G and $C_G(R_G)$ are given by (6.4) with R replaced by R_G.

In the notation of the proof of Theorem 6.1, suppose each $w_\Gamma \leq (p-1)$. Then $D(B_G)$ is abelian, and both $D(B)$ and $D(B_H)$ are abelian. Thus Property 7.1 (d) holds. So we suppose that $w_\Delta \geq p$ for some Δ. There exists $y \in C_G(s) \cap K$ such that $y \in N_G(R_G) \setminus C_G(R_G)$, $|y| = 4$, $y|_{V_0} = 1_{V_0}$, $y|_{V_\Gamma} = 1_{V_\Gamma}$ for all $\Gamma \neq \Delta$, and y swaps exactly two factors K_Δ of $C_G(R_G)$. Let $R_H := R_G \cap H$ and $R = R_G \cap K$.

Let $P_G := P(D(B_G))$ be defined by (6.5), so that $P_G \leq R_G$ and we may suppose $y \in C_G(P_G) \cap K$. Let $P := P_G \cap K$ and $P_H = P_G \cap H$. Since $w_\Delta \geq p$, it follows that $|\Omega_1(P_G)| \geq p^{p-1}$, and P is cyclic if and only if $p = 3$, $w_\Delta = 3$, $w_\Gamma = 0$ for all $\Gamma \neq \Delta$ and $P_G \not\leq K$.

We claim that $C_G(P) \neq C_G(P_G)$ if and only if $V_0 = 0$, $p = 3 = w_\Delta$, $w_\Gamma = 0$ when $\Gamma \neq \Delta$, $\eta = \epsilon$, $\alpha_\Delta = 0$, $e_\Delta = 1$ and $3 \| (q-\epsilon)$. In particular, $D(B) = 3_+^{1+2}$ in this case.

Indeed, if $w_\Delta > 3$, then P is noncyclic and so $C_G(P) = C_G(P_G)$. Thus $w_\Delta = 3$, and so $p = 3$. If $w_\Gamma \neq 0$, then P is also noncyclic and hence $C_G(P) = C_G(P_G)$. Suppose $p = 3 = w_\Delta$ and $w_\Gamma = 0$, so that $|\Omega_1(P_G)| = 3^2$. Define $c \geq 1$ by $3^c \| (q^{e3^{\alpha_\Delta}} - \epsilon)$, and choose $\beta \in \mathbb{F}^\times_{q^{2e3^{\alpha_\Delta}}}$ with $|\beta| = 3^c$. Note that $x_\beta := 1_{V_0} \times \mathrm{diag}\{\beta^{-2}, \beta, \beta\} \in P$ and so if $c \geq 2$, then $C_G(P) = C_G(P_G)$. Thus $3 \| (q^{e3^{\alpha_\Delta}} - \epsilon)$ and so $\alpha_\Delta = 0$. Note that $e = 1$ or 2 as $p = 3$. If $e = 2$, then we may suppose $\beta \in \mathrm{SL}_2(q)$, since $\mathrm{SL}_2(q)$ contains a maximal torus $\mathbb{Z}_{q-\epsilon}$. Thus $\det(\beta) = 1$, $P = P_G$ and $C_G(P) = C_G(P_G)$, a contradiction. So $e = 1$ and $3 \| (q-\epsilon)$. But $Z(G(V_0)) \times Z(G(V_+)) \leq Z(C_G(P))$, so $\eta = \epsilon$. Since $3 \mid (q-\eta)$ and $1 = D_0 = O_3(G(V_0))$, it follows that $V_0 = 0$. Similarly, since $Z(G) = \mathbb{Z}_{q-\eta} \leq Z(C_G(s)) = \mathbb{Z}_{q^{d_\Delta} - \epsilon_\Delta}$, it follows that $e_\Delta = 1$ and the claim holds.

Suppose $V_0 = 0$, $p = 3 = w_\Delta$, $w_\Gamma = 0$ when $\Gamma \neq \Delta$, $\eta = \epsilon$, $\alpha_\Delta = 0$, $e_\Delta = 1$ and $3 \| (q-\epsilon)$. Then

$$C_G(P) = \mathrm{GL}^\epsilon(3d_\Delta, q) = G, \quad C_G(P_G) = \mathrm{GL}^\epsilon(d_\Delta, q) \times \mathrm{GL}^\epsilon(2d_\Delta, q),$$

so $P = O_3(Z(G)) = O_3(Z(K)) = \mathbb{Z}_3$ and $P_G = O_3(Z(C_G(P_G))) = \mathbb{Z}_3^2$. In addition, $C_G(s) \cong \mathrm{GL}^{\epsilon_\Delta}(3, q^{\delta_\Delta})$ and $D(B_G) \in \mathrm{Syl}_3(\mathrm{GL}^{\epsilon_\Delta}(3, q^{\delta_\Delta}))$. Since $\alpha_\Delta = 0$, it follows that $3 \| (q^{\delta_\Delta} - \epsilon_\Delta)$, $D(B_G) = \mathbb{Z}_3 \wr \mathbb{Z}_3$ and $|D(B)| = 3^3$. But $3_+^{1+2} \in \mathrm{Syl}_3(\mathrm{SL}^{\epsilon_\Delta}(3, q^{\delta_\Delta}))$, so $D(B) \cong 3_+^{1+2}$.

Write $D_G = D(B_G)$, $D_H = D(B_H)$ and $D = D(B)$. Since $[D_G{:}D] = 3$, it follows that $D_H = D$ or $D_H = D_G$. Note that $D_G = \langle R_G, \sigma \rangle$ for some permutation σ of order 3. So $D = \langle R, \sigma \rangle$, $C_G(D_G) = C_G(D) \cong \mathrm{GL}^\epsilon(d_\Delta, q)$, $C_G(C_G(D)) = \mathrm{GL}^\epsilon(3, q)$ and $\langle D_G, y \rangle \leq C_G(C_G(D))$. Thus y centralizes $C_G(D)$ and so $y \in N_K(D, b_D)$, where (D, b_D) is a Sylow B-subgroup. Since $Z(D) \leq Z(K)$, it follows that B dominates a block $\overline{B} \in \mathrm{Blk}(K/Z(D))$, $D(\overline{B}) = D/Z(D) \cong 3^2$ and $\bar{y}$ stabilizes the $\overline{B}$-subgroup

$(D/Z(D), \bar{b}_D)$ with $|\bar{y}| = 4$, where $\bar{y} = yZ(D)$. In particular, $\overline{B}$ is non-nilpotent. But $D(\overline{B})$ is abelian, so $l(\bar{b}_{\overline{Q}}) \geq 2$ for some $\overline{B}$-subgroup $(\overline{Q}, \bar{b}_{\overline{Q}})$. If $\overline{Q} \cong \mathbb{Z}_3$, then $C_{K/Z(D)}(\overline{Q}) = C_K(Q)D/Z(D) = (C_K(Q)/Z(D)).3$, where $Q = \langle Z(K), w\rangle$ for some $w \in D\backslash Z(D)$ of order 3. Let B_Q be a block of $C_K(Q)D$ dominating $\bar{b}_{\overline{Q}}$ and $b_Q \in \mathrm{Blk}(C_K(Q))$ covered by B_Q. Then $D(b_Q) = Q$ and the canonical character θ_Q of b_Q is the only irreducible Brauer character of b_Q. If ϕ is any irreducible Brauer character of B_Q, then ϕ covers θ_Q. But $C_K(Q)$ contains a representative set of the conjugacy $3'$-classes of $C_K(Q)D$, so $l(B_Q) = 1$. In particular, $l(\bar{b}_{\overline{Q}}) = 1$. Similarly, if $\overline{Q} = D/Z(D)$, then $l(\bar{b}_{\overline{Q}}) = 1$. Thus if $l(\bar{b}_{\overline{Q}}) \geq 2$, then $\overline{Q} = 1$ and $\bar{b}_{\overline{Q}} = \overline{B}$. It follows that $l(B) \geq l(\overline{B}) \geq 2$, and hence Property 7.1 (b) holds.

Since p is odd, it follows that $C_G(R) = C_G(R_H) = C_G(R_G)$. Let $x \in G$ such that for any $u \in R$, there exists $z \in Z(G)$ such that $x^{-1}ux = uz$. Then $x^{-1}ux = cu$ for some $c \in O_p(\mathbb{F}_{q^2}^\times)$ and so $x \in N_G(R) = N_G(R_G)$. If λ is an eigenvalue of u in some algebraic closure of $\mathbb{F}_{q^2}$ and $m_{X-\lambda}(u)$ is the multiplicity, then $c\lambda$ is also an eigenvalue of u with the same multiplicity. In particular, $m_{X-1}(u) = m_{X-c}(u)$. It follows that if we choose $u \in R$ such that $m_{X-1}(u) \neq m_{X-c}(u)$ for any $c \in \mathbb{F}_{q^2}^\times$, then $x^{-1}ux = u$. Since $|\Omega_1(R)| \geq p^{p-1}$, it follows that $x \in C_G(R)$ and so $C_G(R)/Z = C_{G/Z}(R/Z)$ for any $Z \leq Z(G)$, except when $p = 3 = w_\Delta$, $\alpha_\Delta = 0$ and $3 \| (q^e - \epsilon)$, in which case $D(B) = 3_+^{1+2}$. If $3 \nmid (q - \eta)$, then $e = 2$ and as shown above $R = R_G$, in which case we still have $x \in C_G(R)$. If $3 \mid (q - \eta)$, then $3 \| (q - \eta)$, $P \cong \mathbb{Z}_3$ and $C_G(P) \neq C_G(P_G)$, which is discussed above.

Similarly, $C_G(P)/Z = C_{G/Z}(P/Z)$ for any $Z \leq Z(G)$ when $C_G(P) = C_G(P_G)$ (note in this case that for any $u \in P$, we have $m_{X-\lambda}(u) \geq 2$ for some eigenvalue λ).

Suppose $C_G(P) = C_G(P_H) = C_G(P_G)$ and $C_G(R) = C_G(R_G) = C_G(R_H)$. Thus $C_G(R)$ is regular in G and $s \in C_G(R)$. Let (P, g) and (R, b) be B-subgroups such that $(P, g) \leq (R, b)$.

Let $(P, g_H) \leq (R, b_H)$ be B_H-subgroups such that g_H covers g and b_H covers b (see the remark after the definition of Property 7.1), and $(P, g_G) \leq (R, b_G)$ B_G-subgroups such that g_G covers g_H and b_G covers b_H. By [12, Theorem 3.2], we may suppose $b_G \subseteq \mathcal{E}_p(C_G(R), (s))$ and so $b_G^y = b_G$ as $y \in C_G(s) \cap K$. Now

$$b_G = b(0) \times \prod_{\Gamma} b(\Gamma)^{w_\Gamma},$$

where $b(0)$ is a block of G_0 labelled by (s_0, κ) with defect 0, and $b(\Gamma) = \mathcal{E}_p(K_\Gamma, (t_\Gamma))$ with t_Γ the restriction of s to K_Γ. Note that if U_Γ is the underlying space of K_Γ and view t_Γ as an element of $G(U_\Gamma)$, then we have $m_\Gamma(t_\Gamma) = e_\Gamma$. Thus for any generator $x_\Gamma \in R_\Gamma$

$$T_\Gamma := C_{K_\Gamma}(t_\Gamma) = C_{G(U_\Gamma)}(x_\Gamma t_\Gamma) \cong \mathrm{GL}^{\epsilon_\Gamma}(1, q^{e_\Gamma \delta_\Gamma})$$

is a Coxeter torus of both K_Γ and $G(U_\Gamma)$, and $R_\Gamma = O_p(T_\Gamma)$ is a defect group of $b(\Gamma)$. In particular, R_G is a defect group of b_G. We may suppose $D(b) = D(b_G) \cap C_K(R)$ and $D(b_H) = D(b_G) \cap H$, so that $D(b) = R$ and $D(b_H) = R_H$.

Since $(P, g_G) \leq (R, b_G)$ and $D(b_G) \leq C_G(P)$, it follows that $D(g_G) = D(b_G) = R_G$ and so $D(g) = D(b_G) \cap C_K(P) = D(b_G) \cap K = R$ is abelian. Similarly, $D(g_H) = R_H = D(b_H)$.

Let θ, θ_H and θ_G be canonical characters of b, b_H and b_G, respectively. Then θ_G covers θ_H and θ_H covers θ. Now

$$\theta_G = \theta(0) \times \theta(+), \quad \theta(+) := \prod_{\Gamma} \theta(\Gamma)^{w_\Gamma}, \quad \theta(\Gamma) := \pm R_{T_\Gamma}^{K_\Gamma}(t_\Gamma)$$

where $\mathrm{Irr}(b(0)) = \{\theta(0)\}$. If $H_\Gamma := K_\Gamma \cap S(U_\Gamma)$, then there exists an element $x_\Gamma \in K_\Gamma$ permutes all the irreducible constituents of the restriction $\theta(\Gamma)|_{H_\Gamma}$. Since $w_\Delta \geq p$, it follows that $\theta(+)|_{C_{S(V_+)}(R_+)}$ is irreducible, where $R_+ = S(V_+) \cap (\prod_\Gamma (R_\Gamma)^{w_\Gamma})$.

Let $K_0 = S(V_0)$ and $K_+ = S(V_+)$, so that

$$C_K(R) = \langle K_0 \times C_{K_+}(R_+), u_K \rangle, \quad u_K = u_0 \times u_+$$

with $u_0 \in G_0 \setminus K_0$ and $u_+ \in C_{G_+}(R_+) \setminus C_{K_+}(R_+)$. Note that $G_0 = \langle K_0, u_0 \rangle$ and $C_{G_+}(R_+) = \langle C_{K_+}(R_+), u_+ \rangle$.

Let θ_0 and θ_+ be irreducible constituents of $\theta|_{K_0}$ and $\theta|_{C_{K_+}(R_+)}$, respectively. Then $\theta \in \mathrm{Irr}(C_K(R) \mid \theta_0 \times \theta_+)$ and θ_G covers $\theta_0 \times \theta_+$. But $\theta(+)|_{C_{K_+}(R_+)}$ is irreducible, so $\theta_+ = \theta(+)|_{C_{K_+}(R_+)}$ and θ_+ has an extension $\theta(+)$ to $C_{G_+}(R_+)$. Applying Lemma 2.7 to

$$K_0 \times C_{K_+}(R_+) \leq C_K(R) \leq C_G(R) = G_0 \times C_{G_+}(R_+)$$

(with $Z_0 = 1$), we have that $\theta_G|_{C_K(R)} = \theta$. In particular, $\theta_G|_{C_K(R)}$ is irreducible and so is $\theta_G|_{C_H(R)}$. But θ_G covers θ_H, so $\theta_G|_{C_H(R)} = \theta_H$. Thus $\theta^y = \theta$, $\theta_H^y = \theta_H$, $b^y = b$ and $b_H^y = b_H$. Note that $\theta_H|_{C_K(R)} = \theta$. Thus Property 7.1 $(a*)$ holds. □

Let V be a non-degenerate orthogonal or symplectic space, $G = I_0(V)$ and let G^* be the dual group of G. Then

$$\mathrm{Sp}_{2n}(q)^* = \mathrm{SO}_{2n+1}(q), \quad \mathrm{SO}_{2n+1}(q)^* = \mathrm{Sp}_{2n}(q), \quad \mathrm{SO}_{2n}^\eta(q)^* = \mathrm{SO}_{2n}^\eta(q).$$

If B is a block of $I_0(V)$, then there exists a semisimple p'-element $s \in I_0(V)^*$ such that

$$B \subseteq \mathcal{E}_p(I_0(V), (s)).$$

Let (D, b_D) be a Sylow B-subgroup of $I_0(V)$. Then V and D have corresponding decompositions

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+. \tag{8.1}$$

We have $V_0 = C_V(D)$, $V_+ = [\ V, D\ ]$, $D_0 = \{1_{V_0}\}$ and $D_+ \leq I_0(V_+)$. Let $G_0 := I_0(V_0)$, $G_+ := I_0(V_+)$, $C_+ := C_{I_0(V_+)}(D_+)$ and let V^* be the underlying space of $I_0(V)^*$.

Let $z \in D$ be a primitive element. Then $z \in Z(D)$ with $|z| = p$ (cf. [17, p.176]). Thus

$$z = z_0 \times z_+, \quad L := C_G(z) = L_0 \times L_+, \quad L_0 = G_0, \quad L_+ := \mathrm{GL}^\epsilon(m, q^e), \tag{8.2}$$

where $z_0 = 1_{V_0}$, $z_+ \leq D_+$ and $\dim V_+ = 2em$. Then L is a regular subgroup of G and we may suppose $s \in L^* \leq G^*$. In particular,

$$V^* = U_0 \perp U_+ \quad \text{and} \quad s = s_0 \times s_+, \tag{8.3}$$

where $U_0 = V_0^*$, $s_0 \in L_0^* = I_0(U_0)$, U_+ is the underlying space of L_+^* and $s_+ \in L_+^* \leq I_0(U_+)$.

Let $C_{I(U_+)}(s_+) = \prod_\Gamma C_\Gamma$ and let U_Γ be the underlying vector space of C_Γ, so that

$$C_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s_+), q^{\delta_\Gamma}) \quad \text{or} \quad I(U_\Gamma) \tag{8.4}$$

according as $\Gamma \neq X \pm 1$ or $\Gamma = X \pm 1$.

Proposition 8.2 *Let $K := \Omega_{2n}^\eta(q) := \Omega^\eta(V) \leq H \leq J_0(V)$, $B_K \in \mathrm{Blk}(K)$ and $B_H \in \mathrm{Blk}(H)$ covering B_K. Write $R := A(D(B_K))$. Then either Property* 7.1 *(a∗) holds for some B_K-subgroups $(P, g) \leq (R, b)$ or Property* 7.1 *(d) holds, where P is some subgroup of R.*

Proof: Let $G := \mathrm{SO}_{2n}^\eta(q) := \mathrm{SO}(V)$ and $B \in \mathrm{Blk}(G)$ covering B_K. Then $D(B_K) = D \cap K$ for some defect group $D := D(B)$. Since G is self dual, we have $V = V^*, U_0 = V_0, U_+ = V_+$ in (8.3).

(1). Since $|G{:}K| = 2$, it follows that $D = D(B_K)$ and $D = D(B_H) \cap G$ for some $D(B_H)$. In the notation above we have

$$C_K(z) \leq L = L_0 \times L_+.$$

Let $K_0 = \Omega(V_0)$, $K_+ = \Omega(V_+)$ and $M_+ := \mathrm{SL}^\epsilon(m, q^e) \leq L_+ \cap K_+$, so that

$$M_+ \leq C_{K_+}(z_+) \leq L_+, \quad C_K(z) = \langle K_0 \times C_{K_+}(z_+), t_0 \times t_+ \rangle \tag{8.5}$$

and $[L_+{:}C_{K_+}(z_+)] \leq 2$, where $t_0 \in L_0 \setminus K_0$ and $t_+ \in L_+$. Let (z, B_z) be a major subsection of B_K. Then B_z covers a block $B_0 \times B_+$ of $K_0 \times C_{K_+}(z_+)$ with $B_0 \in \mathrm{Blk}(K_0)$ and $B_+ \in \mathrm{Blk}(C_{K_+}(z_+))$ such that $D(B_z) = D(B_0 \times B_+) = D$. Note since $[L_+{:}C_{K_+}(z_+)] \leq 2$, it follows that $D(B_+) = D_+ \not\cong 3_+^{1+2}$.

By [15, Lemma 4.1], there exists a B-subgroup (z, B_L) such that B_L covers B_z. Thus (z, B_L) is a major subsection of B.

Let $R := A(D)$, so that $z \in R$ and

$$R = D_0 \times \prod_\Gamma (R_\Gamma)^{w_\Gamma}, \quad C_G(R) = \mathrm{SO}(V_0) \times \prod_\Gamma (K_\Gamma)^{w_\Gamma},$$

where $R_\Gamma = O_p(K_\Gamma)$, $m_\Gamma(s_+) = w_\Gamma e_\Gamma$ or $2w_\Gamma e_\Gamma$ according as $\Gamma \neq X \pm 1$ or $\Gamma = X \pm 1$, $K_\Gamma \cong \mathrm{GL}^\epsilon(\delta'_\Gamma, q^{ep^{\alpha_\Gamma}})$. Thus $R = O_p(C_G(R))$, $C_G(R)$ is a regular subgroup of G and we may suppose $s \in C_G(R)$. Set $R_+ = \prod_\Gamma (R_\Gamma)^{w_\Gamma}$, so that $R_+ \leq C_{K_+}(z_+)$ and

$$K_0 \times C_{K_+}(R_+) \leq C_K(R) \leq C_G(R) = L_0 \times C_{L_+}(R_+).$$

Let (R_+, b_+) be a B_+-subgroup, so that

$$(B_0 \times b_+)^{K_0 \times C_{K_+}(z_+)} = B_0 \times B_+$$

and $b_+ \in \mathrm{Blk}(C_{K_+}(R_+))$ as $C_{K_+}(R_+) \leq C_{K_+}(z_+)$. Now $K_0 \times C_{K_+}(z_+)$ is normal in $C_K(z)$, $R \leq K_0 \times C_{K_+}(z_+)$ and B_z covers $B_0 \times B_+$. It follows by [15, Lemma 4.1] that there exists a B_z-subgroup (R, b) such that b covers $B_0 \times b_+$, so that (R, b) is a B_K-subgroup.

Similarly, there exists a B_L-subgroup (R, b_G) such that b_G covers b. Thus $b_G \subseteq \mathcal{E}_p(C_G(R), (s))$, $R = D(b_G)$, and so $R = D(b) = D(B_0 \times b_+)$.

Since $L = L_0 \times L_+$ with $L_0 = G_0 = \mathrm{SO}(V_0)$, it follows that $B_L = B_{L_0} \times B_{L_+}$ with $B_{L_0} \in \mathrm{Blk}(G_0)$ and $B_{L_+} \in \mathrm{Blk}(L_+)$. But B_L covers B_z and B_z covers $B_0 \times B_+$, so B_{L_0} covers B_0 and B_{L_+} covers B_+. In particular, $D_+ = D(B_+) = D(B_{L_+})$. Similarly, $b_G = b_{G_0} \times b_{G_+}$ with $b_{G_0} = B_{L_0} \in \mathrm{Blk}(G_0)$ and $b_{G_+} \in \mathrm{Blk}(C_{L_+}(R_+))$, and b_{G_+} covers b_+.

Suppose D is non-abelian, so that $w_\Delta \geq p$ for some Δ. Let $P := D_0 \times P_+ \leq R$ and $(P, g) \leq (R, b)$, where $P_+ := P(D_+)$ is given by (6.5). By Proposition 8.1, there exists B_+-subgroup (P_+, g_+) such that $(P_+, g_+) \leq (R_+, b_+)$ and $D(g_+) = D(b_+) = R_+$. By the remark after Property 7.1, we may choose (P, g) such that g covering $B_0 \times g_+$, so $D(g) = D_0 \times D(g_+) = D_0 \times D(b_+) = D(b) = R$.

In addition, there exists $y \in C_{L_+}(s_+) \cap M_+$ such that $y \in N_{C_{L_+}(P_+)}(R_+) \setminus C_{L_+}(R_+)$, $|y| = 4$, $y^2 \in C_{L_+}(R_+)$, $y|_{V_0} = 1_{V_0}$, $y|_{V_\Gamma} = 1_{V_\Gamma}$ for all $\Gamma \neq \Delta$ and y swaps exactly two factors K_Δ of $C_G(R)$. Since $y \in C_{L_+}(s_+)$, it follows that $(b_{G_+})^y = b_{G_+}$.

Let θ_G and θ be canonical characters of b_G and b, respectively. Then θ_G covers θ,

$$\theta_G = \theta_{G_0} \times \theta_{G_+}$$

with $\mathrm{Irr}(b_{G_0}) = \{\theta_{G_0}\}$ and θ_{G_+} the canonical character of b_{G_+}. If $\mathrm{Irr}(B_0) = \{\theta_0\}$ and θ_+ is the canonical character of b_+, then θ_{G_+} covers θ_+. Since $C_{K_+}(R_+) = C_{C_{K_+}(z_+)}(R_+)$, it follows by the proof of Proposition 8.1 that $\theta_+ = \theta_{G_+}|_{C_{K_+}(R_+)}$ and $\theta_+^y = \theta_+$.

Now

$$C_K(R) = \langle K_0 \times C_{K_+}(R_+), u_K \rangle, \quad u_K = u_0 \times u_+ \tag{8.6}$$

with $[C_K(R){:}K_0 \times C_{K_+}(R_+)] \leq 2$, where $u_0 \in G_0$ and $u_+ \in C_{L_+}(R_+)$. If $C_K(R) = K_0 \times C_{K_+}(R_+)$, then $\theta = \theta_0 \times \theta_+$ and $\theta^y = \theta$. If $[C_K(R){:}K_0 \times C_{K_+}(R_+)] = 2$, then $L_0 = \langle K_0, u_0 \rangle$ and $C_{L_+}(R_+) = \langle C_{K_+}(R_+), u_+ \rangle$. Now $\theta \in \mathrm{Irr}(C_K(R) \mid \theta_0 \times \theta_+)$ and $\theta_G \in \mathrm{Irr}(C_G(R) \mid \theta)$ and θ_+ has an extension θ_{G_+} to $C_{L_+}(R_+)$, so by Lemma 2.7, $\theta_G|_{C_K(R)} = \theta$ and hence $\theta^y = \theta$ and $y \in N_K(R, b)$.

(2). Suppose $K \leq H \leq J_0(V)$. Let $(P, g_H) \leq (R, b_H)$ be B_H-subgroups such that g_H covers g and b_H covering b. Since $J_0(V)/Z(J_0(V))K$ is a 2-group and p is odd, it follows that $D(g_H) = O_p(Z(H))D(g) = O_p(Z(H))D(b) = D(b_H)$ and both are abelian, and the canonical character θ_H of b_H covers θ. Now

$$K_0 \times C_{K_+}(R_+) \leq C_K(R) \leq C_H(R) \leq C_{J_0(V)}(R) \leq J_0(V_0) \times C_{J_0(V_+)}(R_+).$$

By [17, (1A)], $C_{J_0(V_+)}(z_+) = \langle L_+, \tau_+ \rangle$ with $[\tau_+, L_+] = 1$ and so $C_{J_0(V_+)}(R_+)$ is a central product $C_{L_+}(R_+) \circ \langle \tau_+ \rangle$. In particular, θ_{G_+} has an extension $\tilde{\theta}_{G_+}$ to $C_{J_0(V_+)}(R_+)$, and $\tilde{\theta}_{G_+}$ is also an extension of θ_+. Moreover, y stabilizes $\tilde{\theta}_{G_+}$, since $\tilde{\theta}_{G_+}$ is a central produce $\theta_{G_+} \circ \beta$ for some $\beta \in \mathrm{Irr}(\langle \tau_+ \rangle)$ and $\theta_{G_+}^y = \theta_{G_+}$. Since $y \in M_+ \leq K_+$ and y normalizes K_+, it follows that $[y, x] \in C_{K_+}(R_+) = C_{L_+}(R_+) \cap K_+$ for any $x \in C_{L_+}(R_+)$. But $C_{J_0(V_+)}(R_+) = C_{L_+}(R_+) \circ \langle \tau_+ \rangle$, so $[y, x] \in C_{K_+}(R_+)$ for all $x \in C_{J_0(V_+)}(R_+)$. It follows by Remark 7.3 that y stabilizes θ_H and so $b_H^y = b_H$. Thus Property 7.1 $(a*)$ holds.

(3). Suppose $w_\Gamma < p$ for any Γ with $m_\Gamma(s_+) \neq 0$. Then $D = D(B_K)$ is abelian, and so $D(B_H) = DO_p(Z(H))$ is abelian. Thus Property 7.1 (d) holds. □

Proposition 8.3 *Let* $K := \Omega_{2n+1}(q) = \Omega(V)$ *or* $K := \mathrm{Sp}_{2n}(q) = \mathrm{Sp}(V)$, *and*

$$K \leq H \leq J_0(V),$$

$B_K \in \mathrm{Blk}(K)$ *and* $B_H \in \mathrm{Blk}(H)$ *covering* B_K, *where* $H = \mathrm{SO}(V)$ *when* $K = \Omega(V)$. *Write* $R := A(D(B_K))$. *Then either Property* 7.1 $(a*)$ *holds for some* B_K*-subgroups* $(P, g) \leq (R, b)$ *or Property* 7.1 (d) *holds, where* P *is some subgroup of* R.

Proof: Suppose V is orthogonal. Replacing G by H in the proof (1) of Proposition 8.2 with some obvious modifications, we have that Property 7.1 $(a*)$ holds for $(P, g) \leq (R, b)$. Suppose V is symplectic, so that H/K is cyclic. Applying the proofs (1) and (2) of Proposition 8.2 with some obvious modifications, we have that Property 7.1 $(a*)$ holds for $(P, g) \leq (R, b)$.

If $D(B_K)$ is abelian, then $D(B_K) = D(B_H) \cap K$ for some $D(B_H)$. Since the outer-diagonal group of K is order 2, it follows that $D(B_H) \leq KZ(H)$ and so $D(B_H) = D(B_K)O_p(Z(H))$ is abelian. □

Theorem 8.4 *Let* $G = I_0(V)$, $B \in \mathrm{Blk}(G)$, *and* (D, b_D) *a Sylow* B*-subgroup. Follow the notation in* (8.1), (8.2) *and* (8.3). *Then the following are equivalent:*

(a) *B is nilpotent.*

(b) $C_{I_0(V)}(D) = G_0 \times C_{I_0(V_+)}(D_+)$ *is a regular subgroup of* $I_0(V)$ *and* $s \in C_{I_0(V)}(D)^* \leq L^*$ *satisfies the following conditions.*

(i) *Suppose* $I_0(V) = \mathrm{Sp}_{2n}(q)$ *or* $\mathrm{SO}_{2n+1}(q)$. *Then*

$$m_\Gamma(s_+) = \begin{cases} 0 \ \ or \ \ 1 & if \ \ \Gamma \neq X \pm 1 \ and \ e \mid \delta_\Gamma, \\ 0 & otherwise. \end{cases}$$

(ii) *Suppose* $I_0(V) = \mathrm{SO}^\eta_{2n}(q)$. *Then*

$$\begin{cases} m_\Gamma(s_+) = 0 \ \ or \ \ 1 & if \ \ \Gamma \neq X \pm 1 \ and \ e \mid \delta_\Gamma, \\ m_{X-1}(s_+) + m_{X+1}(s_+) = 0 \ \ or \ \ 2 & if \ \ p \mid (q - \epsilon), \\ m_\Gamma(s_+) = 0 & otherwise, \end{cases}$$

where ϵ *is the type of the underlying space of* $(s_+)_{X\pm1}$ *when* $m_{X-1}(s_+) + m_{X+1}(s_+) = 2$.

(c) $C_{I_0(V)}(D) = G_0 \times C_{I_0(V_+)}(D_+)$ *is a regular subgroup of* $I_0(V)$ *and* $s \in C_{I_0(V)}(D)^* \leq L^*$ *such that* $T^*_+ := C_{I_0(U_+)}(s_+)$ *is a maximal torus of* $I_0(U_+)$. *In particular, if* $\theta = \theta_0 \times \theta_+$ *is the canonical character of* b_D *with* $\theta_0 \in \mathrm{Irr}(G_0)$ *and* $\theta_+ \in \mathrm{Irr}(C_+)$, *then* θ_0 *has defect zero and* $\theta_+ = \pm R^{C_+}_{T_+}(s_+)$, *where* $C_+ = C_{I_0(V_+)}(D_+)$ *and* $T_+ \leq C_+$ *is a dual of* T^*_+.

PROOF: Suppose B is nilpotent. By Propositions 8.2 and 8.3, D is abelian, so $m_\Gamma(s_+) = e_\Gamma$ or 0 when $\Gamma \neq X \pm 1$ and $m_\Gamma(s_+) = 2e$ or 0 when $\Gamma = X \pm 1$.

Suppose $m_\Gamma(s_+) = e_\Gamma \geq 2$ with $\Gamma \neq X \pm 1$. As shown in the proof of Theorem 6.1 there exists a p'-element $\tau_\Gamma \in C_\Gamma^*$ of order e_Γ normalizing $D_\Gamma := D \cap C_\Gamma^*$, so that there exists a p'-element of order e_Γ normalizing the Sylow B-subgroup (D, b_D), a contradiction, so $e_\Gamma \leq 1$ and $e \mid \delta_\Gamma$.

Suppose $\Gamma = X \pm 1$, so that $m_\Gamma(s_+) = 2e$ or 0 and $C_\Gamma = I(U_\Gamma)$.

Suppose, moreover that $G = \mathrm{SO}_{2n}^\eta(q)$, so that $G = G^*$. Let $\Delta = X \pm 1$ and suppose $m_\Delta(s_+) = 2e$. By [17, (1.14)],

$$|N_{I(U_\Delta)}(\langle z_\Delta \rangle)/C_{I(U_\Delta)}(z_\Delta)| = 2e.$$

If $e \geq 2$, then there exists $y_\Delta \in N_{I_0(U_\Delta)}(\langle z_\Delta \rangle) \setminus C_{I_0(U_\Delta)}(z_\Delta)$ of order e, so that y_Δ normalizes the Sylow subgroup D_Δ of $C_{I_0(U_\Delta)}(z_\Delta)$. Let $y_\Gamma = 1 \in I_0(U_\Gamma)$ and $y = 1_{V_0} \times \prod_\Gamma y_\Gamma$. Then

$$y \in (N_G(D) \cap C_G(s)) \setminus C_G(D)$$

and y normalizes (D, b_D). Since $|y| = e \neq 1$, it follows that B is not nilpotent, which is impossible. Thus $e = 1$, $I_0(U_\Delta) = \mathrm{SO}^\epsilon(2, q)$ with $\epsilon = \eta(U_\Delta)$, so $p \mid (q - \epsilon)$.

Similarly, suppose $m_{X-1}(s_+) = m_{X+1}(s_+) = 2$. Since $\Omega_4^-(q) = P\mathrm{SL}_2(q^2)$, $\Omega_4^+(q) = \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and $\mathrm{SO}_4^\pm(q) = \Omega_4^\pm(q).2$, it follows that there exists an element $w \in (N_G(D) \cap C_G(s)) \setminus C_G(D)$ of order 2 such that $w \in N_G(D, b_D)$, a contradiction.

If $G = \mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$, then by [17, (1.14)] again, there exists an element $w \in N_G(D) \setminus C_G(D)$ of order $2e$ such that w normalizes (D, b_D), which is impossible. Thus $m_{X\pm1}(s_+) = 0$ and (b) holds.

Suppose (b) holds. Then $T_+^* := C_{C_+^*}(s_+)$ is a maximal torus of $I_0(U_+)$ and so D is abelian. Since $C_G(D)$ is regular in G, we may suppose $s \in C_G(D)^*$ and so

$$b_D \subseteq \mathcal{E}_p(C_G(D), (s)).$$

Thus $\theta_+ = \pm R_{T_+}^{C_+}(s_+)$ and $\theta = \theta_0 \times \theta_+$ is the canonical character of b_D, where $\theta_0 \in \mathrm{Irr}(G_0)$ has defect 0. In particular, $N_G(D, \theta) = C_G(D)$ and B is nilpotent. □

Proposition 8.5 *Let $K := \mathrm{Spin}^\eta(V) \lhd H$ such that H/K is abelian, $C_H(K) \leq Z(H)$ and $H/Z(H) \leq \mathrm{SO}(V)$ or $J_0(V)/Z(J_0(V))$ according as $\dim V$ is odd or even. Let $B \in \mathrm{Blk}(K)$, $B_H \in \mathrm{Blk}(H)$ covering B, and $Z \leq Z(K)$ such that $K_c := K/Z = \Omega^\eta(V)$, so that $|Z| = \gcd(2, q - \eta)$. Write $R := A(D(B))$. Then either Property* 7.1 *(a∗) holds for some B_K-subgroups $(P, g) \leq (R, b)$ or Property* 7.1 *(d) holds, where P is some subgroup of R.*

PROOF: Let $D := D(B)$, $G := \mathrm{SO}^\eta(V)$ and $Z_+ \leq Z(D_0(V))$ such that $G = D_0(V)/Z_+$, so that $Z = Z_+ \cap K$ and $Z_+ \cong \mathbb{Z}_{q-1}$.

We may suppose $D = DZ/Z \leq K_c$. Thus D is of defect type in K_c, where a p-subgroup Q of K_c is of defect type if Q is a Sylow p-subgroup of a centralizer $C_{K_c}(t)$ of a semisimple p'-element t. So D is of defect type of G, and D has a primary element $z \in Z(D)$ (see [17, Section 5]). Thus we have the corresponding decompositions

$V = V_0 \perp V_+, D = D_0 \times D_+$, $z = z_0 \times z_+, C_G(z) = L_0 \times L_+$ given by (8.1) and (8.2). Since K is universal, it follows by [20, Theorem 4.2.2] and [17, (2E)] that

$$C := C_K(z) = L_C T_C, \quad L_C = L_1 \times \mathrm{SL}^\epsilon(m, q^e), \quad L_1 = \mathrm{Spin}(V_0),$$

where T_C is an abelian r'-group inducing inner-diagonal automorphisms on $\mathrm{SL}^\epsilon(m, q^e)$ and L_1. Here for simplicity, we identify z with its preimage (with the same order) in K. Since p is odd and $D \le C$, it follows that $D_+ \cap L_1 = 1$ and so $L_1 L_2 = L_1 \times L_2$, where $L_2 = \mathrm{SL}^\epsilon(m, q^e) D_+ \le L_+$. Let $L = L_1 \times L_2 \le C$, so that $C = LT_C$. Let (z, B_z) be a major subsection of B, and $B_L = B_1 \times B_2$ a block of L covered by B_z, where $B_1 \in \mathrm{Blk}(L_1)$ and $B_2 \in \mathrm{Blk}(L_2)$. We may suppose $D(B_2) = D_+ \cap L_2$, so that $D(B_2) = D_+$.

Suppose B_2 satisfies Property 7.1 ($a*$). Let $R_2 := A(D_+) \le D(B_2)$, $R := R_1 \times R_2$, $P_2 := P(D_+) \le R_2$, $P = P_1 \times P_2 \le R$ and let $(P, g_L) \le (R, b_L)$ be B_L-subgroups, where $P_1 = R_1 = D_0$. So $g_L = g_1 \times g_2$, $b_L = b_1 \times b_2$ with $g_1 = b_1 = B_1$, $g_2^{L_2} = b_2^{L_2} = B_2$ and $D(b_2) = R_2 = D(g_2)$. In addition, there exists $y \in (N_{L_2}(R_2) \cap C_{L_2}(P_2)) \setminus C_{L_2}(R_2)$ such that $y^4 = 1$, $y^2 \in C_{L_2}(R_2)$, and $b_2^y = b_2$. Thus $b_L^y = b_L$ and $D(b_L) = R = D(g_L)$.

(1). For $t \in T_C$ write $t = t_1 t_2$ such that $[t_1, t_2] = 1$ and t_2 induces inner-diagonal automorphism on L_2. Let $J_i = \langle L_i, t_i : t = t_1 t_2 \in T_C \rangle$, so that $C \lhd J := J_1 \times J_2$ and $L_2 \le J_2 \le L_+$. Let $B_J \in \mathrm{Blk}(J)$ be a weakly regular cover of B_z, $(P, g) \le (R, b)$ B_z-subgroups such that g covers g_L and b covers b_L, and $(P, g_J) \le (R, b_J)$ B_J-subgroups such that g_J covers g and b_J covers b, so that g_J covers g_L and b_J covers b_L.

If $g_J = g_{J_1} \times g_{J_2}$ and $b_J = b_{J_1} \times b_{J_2}$ for some $g_{J_i} \in \mathrm{Blk}(C_{J_i}(P_i))$ and $b_{J_i} \in \mathrm{Blk}(C_{J_i}(R_i))$, then g_{J_2} covers g_2 and b_{J_2} covers b_2 and by Proposition 8.1 and its proof (1), $D(g_{J_2}) = D(b_{J_2})$ is abelian $\theta_{J_2}|_{C_{L_2}(R_2)} = \theta_2$ and $\theta_{J_2}^y = \theta_{J_2}$, where θ_{J_i} and θ_i are canonical characters of b_{J_i} and b_i, respectively. Thus $D(g_J) = D(b_J)$ is abelian. But $P \le R \le D(g_J)$ and $D(g_J)$ is abelian, so

$$D(g) = D(g_J) \cap C_K(P) = D(g_J) \cap K = D(b_J) \cap K = D(b_J) \cap C_K(R) = D(b),$$

which is also abelian. In addition, as shown in the proof (1) of Proposition 8.1 θ_{J_2} has an extension to $C_{L_+}(R_2)$.

Let θ_J and θ be canonical characters of b_J and b, respectively, so that $\theta_J = \theta_{J_1} \times \theta_{J_2}$ covers θ, $\theta_J^y = \theta_J$ and $\theta \in \mathrm{Irr}(C_K(R) \mid \theta_1 \times \theta_2)$. Applying Lemma 2.7 to

$$L_1 \times C_{L_2}(R_2) \le C_K(R) \le J_1 \times C_{J_2}(R_2)$$

we have that $\theta_J|_{C_K(R)} = \theta$, so $\theta^y = \theta$ and $b^y = b$.

(2). If q is even, then the outer-diagonal group of K is trivial, so we may suppose q is odd. Let $(P, g_H) \le (R, b_H)$ be B_H-subgroups such that g_H covers g and b_H covers b. Then b_H is a block of $C_H(R)$ and $R = D(b_H) \cap C_K(R)$ for some $D(b_H)$. Since H induces inner-diagonal automorphisms and since the outer-diagonal group of K is a 2-group, it follows that $D(b_H) \le RZ(H)$ and $D(b_H) = RO_p(Z(H))$ is abelian. Similarly, $R = D(b) = D(g) = D(g_H) \cap C_K(P)$ and $D(g_H) = RO_p(Z(H)) = D(b_H)$.

Now $z \in K \le D_0(V)$, so by [17, (2E)], $C_{D_0(V)}(z) = D_0(V_0) \circ \tilde{L}_+$, $C_{D_0(V)}(z)/Z_+ = C_G(z)$ and $C = C_{D_0(V)}(z) \cap K$, where $\tilde{L}_+$ is a central extension of L_+ by Z_+.

To show that $b_H^y = b_H$ we may suppose $H/KZ(H) = J_0(V)/K_c Z(J_0(V)) = \mathrm{Outdiag}(K)$, so that $H/KZ(H)$ is a 2-group. Let $t \in C_H(z) \setminus C_K(z)$, so that $t^4 \in$

$C_K(z)Z(H)$. In the notation of [20, Table 4.5.2], t induces an element of $C^* := C_{Inndiag(K)}(zZ(K))$ (note here C^* is not the dual group of C). But $C^*/C^{\circ *}$ is a p-group, so t induce an element of $C^{\circ *}$ and hence $t \in C_K(z)Z(H)$. Thus

$$C_H(z) = \langle C_K(z), x_H, t_H\rangle, \quad x_H = x_1x_2, \quad t_H = t_1t_2$$

where $x_1 \in D_0(V_0)$ and $x_2 \in \tilde{L}_+$, and t_H centralizes $\tilde{L}_+$. So

$$\langle C_K(z), x_H\rangle \leq \langle J_1, x_1\rangle \circ \langle J_2, x_2\rangle \leq D_0(V_0) \circ \tilde{L}_+.$$

Let $H_1 = \langle J_1, x_1, t_H\rangle$ and $H_2 = \langle J_2, x_2\rangle$, so that

$$L_1 \times L_2 \leq C_K(z) \leq C_H(z) \leq H_1 \circ H_2.$$

It follows that

$$L_1 \times C_{L_2}(R_2) \leq C_K(R) \leq C_H(R) \leq H_1 \circ C_{H_2}(R_2). \tag{8.26}$$

By [20, Table 4.5.2],

$$\tilde{L}_+ = (Z \times L_2 \circ \mathbb{Z}_{q-\epsilon})\langle x_+\rangle$$

where x_+ induces outer-diagonal automorphism of order dividing $\gcd(m, q-\epsilon)$ on L_2. Thus $C_{H_2}(R_2) = (Z \times C_{L_2}(R_2) \circ \mathbb{Z}_{q-\epsilon})\langle y_2\rangle$ for some $y_2 \in \tilde{L}_+$ inducing outer-diagonal automorphism on L_2. View θ_2 as character of $Z \times C_{L_2}(R_2)$ with $Z \leq \ker(\theta_2)$. Now $C_{L_2}(R_2) \leq C_{H_2}(R_2)/Z \leq C_{L_+}(R_2)$. As shown in the proof (1) of Proposition 8.1 θ_2 has an extension $\tilde{\theta}_2$ to $C_{H_2}(R_2)/Z$ such that $\tilde{\theta}_2^y = \tilde{\theta}_2$. By Remark 7.3 $\theta_H^y = \theta_H$ and $b_H^y = b_H$. Thus the Property 7.1 $(a*)$ holds for (R, b).

Suppose B_2 satisfies Property 7.1 (d), so that $D(B_2)$ is abelian. Thus D_+ and so $D(B)$ are abelian. Since $D(B) = D(B_H) \cap K$ for some $D(B_H)$, it follows that $D(B_H) \leq KZ(H)$ and $D(B_H) = D(B)O_p(Z(H))$ which is abelian. □

Theorem 8.6 *Let K be a finite quasi-simple group of classical type over a field $\mathbb{F}_q$ and $B \in \mathrm{Blk}(K)$, and let $K \lhd H$ such that H/K is abelian, $C_H(K) \leq Z(H)$, H induces inner-diagonal automorphisms on K and $B_H \in \mathrm{Blk}(H)$ covering B. If $p \mid q$, then either $D(B) = D(B_H)$ is cyclic or $l(B) \geq 2$. Suppose $p \nmid q$ and p is odd. Then one of Properties 7.1 $(a*), (b), (c)$ and (d) holds. In addition, if Property 7.1 (b) or (c) holds, then $p = 3$, $K = \mathrm{SL}^\eta(3d, q)/Z$ for some $Z \leq Z(\mathrm{SL}^\eta(3d, q))$ with $\gcd(6, d) = 1$ and $3 \| (q - \eta)$.*

Proof: We will follow the notation of [20]. In particular, K_u denotes the universal group with the same type as K. If $p|q$ and $D(B)$ is noncyclic, then $D(B)$ is a Sylow subgroup of K and $l(B) = l(B_0)$ with principal $B_0 := B_0(K) \in \mathrm{Blk}(K)$. But B_0 dominates the principal block $\overline{B}$ of $K/Z(K) = K_a$ and $l(\overline{B}) + 1$ is the number of p'-conjugacy classes of K_a, so $l(B_0) \geq l(\overline{B}) \geq 2$. Suppose $p \nmid q$.

If $K = A_n^\eta(q)$, then set $\widehat{K} = K_u = \mathrm{SL}_{n+1}^\eta(q) \leq \mathrm{GL}_{n+1}^\eta(q)$, so that $K = \widehat{K}/Z$ for some $Z \leq Z(\mathrm{GL}_{n+1}^\eta(q)) \cap \widehat{K}$. We may take $\widehat{H} \leq \mathrm{GL}_{n+1}^\eta(q)$ such that $H = \widehat{H}/Z$.

If $K = B_n(q) = K_a = \Omega_{2n+1}(q)$, then set $\widehat{K} = \Omega_{2n+1}(q) \leq \widehat{H} \leq \mathrm{SO}_{2n+1}(q)$ such that $H = \widehat{H}/Z$. If $K = B_n(q) = K_u = \mathrm{Spin}_{2n+1}(q) = \mathrm{Spin}(V)$, then take $K = \widehat{K} \lhd \widehat{H} = H$ such that $H/Z(K) \leq \mathrm{SO}(V)$.

If $K = C_n(q)$, then we may take $\widehat{K} = \mathrm{Sp}_{2n}(q) = \mathrm{Sp}(V) \leq \hat{H} \leq J_0(V)$ such that $H = \widehat{H}/Z$.

Suppose $K = D_n^\eta(q)$ with $(n, \eta) = (2k+1, \pm)$ or $(2k, -)$. If $K = \Omega_{2n}^\eta(q) = \Omega(V)$, then $K = \widehat{K} \lhd \widehat{H} = H \leq J_0(V)$. If $K = P\Omega_{2n}^\eta(q) = P\Omega(V)$, then take $\widehat{K} = \Omega_{2n}^\epsilon(q) \leq \widehat{H} \leq J_0(V)$ such that $H = \widehat{H}/Z$. If $K = \mathrm{Spin}_{2n}^\eta(q) = \mathrm{Spin}(V)$, then take $K = \widehat{K} \lhd \widehat{H} = H$ such that $H/Z(K) \leq J_0(V)$.

Suppose $K = D_{2k}^+(q)$ with q even. Then $K = H$ and we may take $\widehat{K} = \widehat{H} = H$. Suppose $K = D_{2k}^+(q)$ with q odd, so that $Z(K_u) = \{1, z, z_s, z_c\}$ and $K_u/Z = \Omega_{4k}^+(q)$, where $Z = \langle z \rangle$. If $K = \Omega_{4k}^+(q) = \Omega(V)$, then take $\widehat{K} = K \leq \widehat{H} \leq J_0(V)$. If $K = P\Omega_{4k}^+(q) = P\Omega(V)$, then take $\widehat{K} = \Omega(V) \leq \widehat{H} \leq J_0(V)$ such that $\widehat{H}/Z = H$. If $K = \mathrm{Spin}_{4k}^+(q)/Z'$ for $Z' = \langle z_s \rangle$ or $\langle z_c \rangle$, then we may take $\widehat{K} = \mathrm{Spin}_{4k}^+(q) = \mathrm{Spin}(V) \leq \widehat{H} \leq D_0(V)$ such that $H = \widehat{H}/Z'$. If $K = \mathrm{Spin}_{4k}^+(q) = \mathrm{Spin}(V)$, then take $\widehat{K} = K$ and $\widehat{H} = H$.

Let $\widehat{B} \in \mathrm{Blk}(\widehat{K})$ dominating B and $\widehat{B}_H \in \mathrm{Blk}(\widehat{H})$ dominating B_H, so that $\widehat{B}_H$ covers $\widehat{B}$. By Propositions 8.1, 8.2, 8.3 and 8.5, one of Properties 7.1 ($a*$), (b) of (d) holds for $\widehat{B}$.

If Property 7.1 ($a*$) holds for $\widehat{B}$, then there exist $\widehat{B}$-subgroups $(\widehat{P}, \hat{g}) \leq (\widehat{R}, \hat{b})$ satisfying Property 7.1 ($a*$). By Lemma 7.6, Property 7.1 ($a*$) holds for some B-subgroups $(P, g) \leq (R, b)$.

Suppose Property 7.1 (b) holds for $\widehat{B}$. By Proposition 8.1, $\widehat{K} = \mathrm{SL}_{n+1}^\eta(q)$, $D(\widehat{B}) = 3_+^{1+2}$ and $n + 1 = 3d$ with $\gcd(6, d) = 1$ and $3 \| (q - \eta)$. In particular, $Z(D(\widehat{B})) = O_3(Z(\widehat{K}))$ and we may suppose $D(B) \leq D(\widehat{B})Z/Z$. If $O_3(Z(\widehat{K})) \leq Z$, then $D(B) = \mathbb{Z}_3^2$, $D(B_H) = 3^2$ or $\mathbb{Z}_3 \wr \mathbb{Z}_3/O_3(Z(\widehat{K})) \cong 3_+^{1+2}$ and (c) holds. If Z is a $3'$-group, then $D(B) = D(\widehat{B})$ and (b) holds.

If Property 7.1 (d) holds for $\widehat{B}$, then $D(\widehat{B})$ and $D(\widehat{B}_H)$ are both abelian. Since $Z \leq Z(\widehat{H}) \cap \widehat{K}$, it follows that $D(B) = D(\widehat{B})Z/Z$ and $D(B_H) = D(\widehat{B}_H)Z/Z$, and so $D(B)$ and $D(B_H)$ are both abelian. □

9 Exceptional groups

Suppose p is odd. We will follow the notation of [20]. In this section we demonstrate that every nilpotent block of an exceptional group of Lie type has abelian defect groups. We first prove a simple lemma.

Lemma 9.1 *Let J_i be a finite group and P_i a p-subgroup of J_i such that $C_{J_i/Z_i}(P_i/Z_i) = C_{J_i}(P_i)/Z_i$ for $i = 1, 2$, where $Z_i = O_p(Z(J_i))$. Let $J = J_1 \times J_2$, $P = P_1 \times P_2$ and $Z \leq O_p(Z(J))$. Then*

$$C_J(P)/Z = C_{J/Z}(P/Z).$$

PROOF: Let $Z_+ = Z_1 \times Z_2 = O_p(Z(J))$. Then

$$C_{J/Z_+}(P/Z_+) = C_{J_1/Z_1}(P_1/Z_1) \times C_{J_1/Z_1}(P_1/Z_1) = C_{J_1}(P_1)/Z_1 \times C_{J_2}(P_2)/Z_2 = C_J(P)/Z_+.$$

It is clear that $C_J(P)/Z \leq C_{J/Z}(P/Z)$. If $xZ \in C_{J/Z}(P/Z)$ for some $x \in J$, then $xZ_+ \in C_{J/Z_+}(P/Z_+)$ and so $xZ_+ = (x_1, x_2)Z_+$ for some $x_i \in C_{J_i}(P_i)$. In particular, $x = (x_1, x_2)x_+$ for some $x_+ \in Z_+$. Since $Z_+ \leq C_J(P)$, it follows that $x \in C_J(P)$ and hence $C_{J/Z}(P/Z) = C_J(P)/Z$. □

The lemma will be applied to a central product $J_1 \circ J_2$ as $J_1 \circ J_2 = (J_1 \times J_2)/Z_0$ for some $Z_0 \leq Z(J_1) \cap Z(J_2)$.

Theorem 9.2 *Let K be a finite quasi-simple group of exceptional type over a field $\mathbb{F}_q$, let $B \in \mathrm{Blk}(K)$, and let $K \lhd H$ such that $C_H(K) \leq Z(H)$, H/K is cyclic, and H induces inner-diagonal automorphisms on K. Let $B_H \in \mathrm{Blk}(H)$ be a block covering B. Choose (as we may) defect groups $D(B)$ and $D(B_H)$ of B and B_H respectively such that $D(B) = D(B_H) \cap K$. If $p|q$, then either $D(B) = D(B_H)$ is cyclic or $l(B) \geq 2$. If $p \nmid q$ and p is odd, then one of the Properties* 7.1 *(a), (b) and (d) holds.*

PROOF: If $p|q$, then a proof similar to that of Theorem 8.6 shows that either $D(B) = D(B_H)$ is cyclic or $l(B) \geq 2$.

Suppose $p \nmid q$. Let K_u be the universal group, so that $K = K_u/Z$ for some $Z \leq Z(K_u)$. Since $Z(K_u)$ is cyclic of order 1, 2 or 3, it follows that H induces the trivial action on $Z(K_u)$.

Before beginning the proof proper we introduce some notation.

Write $D := D(B)$. If $Z(K) \neq \Omega_1(Z(D))$, then take $z \in Z(D) \setminus Z(K)$ with $|z| = p$. If $Z(K) = \Omega_1(Z(D))$ (so in particular $p = 3$), then take $z \in D$ such that $|z| = p^2$ and $zZ(K) \in Z(D/Z(K))$. Let (z, B_z) be a B-subsection, which we choose to be major in the case $z \in Z(D)$ (that such a major subsection exists is [2, 4.15]). In the case that $z \in Z(D)$, we may further choose (z, B_z) so that B_z (a block of $C := C_K(z)$) has defect group D. By [20, Theorem 4.2.2] $C = O^{r'}(C)T$, where $O^{r'}(C)$ is a central product

$$O^{r'}(C) = L_1 \circ L_2 \circ \cdots \circ L_\ell$$

with each $L_i \in \mathcal{L}ie(r)$, and T is an abelian r'-group inducing inner-diagonal automorphisms on each L_i. In general, it may be the case that $z \notin O^{r'}(C)$. We introduce some more notation as follows to allow for this inconvenience: If $Z(C) \leq O^{r'}(C)$, then define $s := \ell$ and $L := O^{r'}(C)$. If $Z(C) \not\leq O^{r'}(C)$, then define $s = \ell + 1$, $L_s = Z(C)$ and

$$L := L_1 \circ L_2 \circ \cdots \circ L_s. \tag{9.1}$$

In all cases $C = LT$, $z \in L$ and $L \lhd C$. Let B_L be a block of L covered by B_z. There are uniquely defined blocks $B_i \in \mathrm{Blk}(L_i)$ such that if $\chi \in \mathrm{Irr}(B_L)$ with $\chi = \chi_1 \circ \cdots \circ \chi_s$ for some $\chi_i \in \mathrm{Irr}(L_i)$, then $\chi_i \in \mathrm{Irr}(B_i)$. We write

$$B_L = B_1 \circ B_2 \circ \cdots \circ B_s.$$

Each element $t \in T$ has the form $t_1 t_2 \cdots t_s t'$, where t' centralizes L and t_i induces an inner-diagonal automorphism on L_i and $[L_i, t_j] = 1$ for $i \neq j$. Let $T' = \langle t' : t = t_1 t_2 \cdots t_s \circ t' \in T \rangle$, and

$$J_i := \langle L_i, t_i : t = t_1 t_2 \cdots t_s t' \in T \rangle, \text{ and } J := J_1 \circ J_2 \circ \cdots \circ J_s \circ T'. \tag{9.2}$$

Then $LT \lhd J$ and T' is abelian. Let B_J be a block of J covering B_z, so that B_J covers B_L. Thus

$$B_J = B_{J_1} \circ B_{J_2} \circ \cdots \circ B_{J_s} \circ B_{T'},$$

where $B_{J_i} \in \mathrm{Blk}(J_i)$ covering B_i and $B_{T'} \in \mathrm{Blk}(T')$. Note that if $C_{J_i}(L_i) \leq Z(L_i)$ for all i, then the central product J is over a subgroup of $Z(L)$.

Case 1. Suppose each B_i satisfies Property 7.1 (d). Then each $D(B_{J_i})$ is abelian and so is $D(B_J)$. Thus $D(B_z) = D(B_J) \cap C$ is abelian.

Case 2. Suppose L is a direct product of L_i's, $C_{J_i}(L_i) \leq Z(L_i)$ for all i and some B_j satisfies Property 7.1 $(a*)$. Without loss of generality, take $j = 1$. In addition, suppose each L_i is classical and universal (or $L_s = Z(C)$). Thus

$$L = L_1 \times \cdots \times L_s \leq C \lhd J = J_1 \times \cdots \times J_s \times T'. \tag{9.3}$$

We now define R_i. If $L_i = \mathrm{SL}^\eta(V_i)$, then denote $G_i = \mathrm{GL}^\eta(V_i)$ and let $B_{G_i} \in \mathrm{Blk}(G_i)$ be a weakly regular cover of B_{J_i} and $R_i := A(D(B_{G_i})) \cap L_i$. If L_i is not linear and unitary, then set $R_i = A(D(B_i))$. In addition, let (R_i, b_i) be a B_i-subgroup, and note that (R_i, b_i) is a Sylow B_i-subgroup when $D(B_i)$ is abelian. Let

$$R := R_1 \times \cdots \times R_s \leq L, \quad b_L := b_1 \times \cdots \times b_s,$$

so that (R, b_L) is a B_L-subgroup. Since $z \in O_p(Z(L))$ and R is abelian, it follows that $z \in R$ and $R \leq C$. By Propositions 8.1, 8.2, 8.3 and 8.5, each defect group $D(b_i)$ of b_i is abelian. Let (R, b_z) be a B_z-subgroup such that b_z covers b_L, and (R, b_J) be a B_J-subgroup such that b_J covers b_z, so that b_J covers b_L and

$$b_J = b_{J_1} \times \cdots \times b_{J_s} \times B_{T'} \tag{9.4}$$

where each b_{J_i} covers b_i. By Propositions 8.1, 8.2, 8.3 and 8.5 again, each defect group $D(b_{J_i})$ is abelian, so defect groups $D(b_J)$ and $D(b_z)$ of b_J and b_z respectively are both abelian, since we may suppose $D(b_z) = D(b_J) \cap C$. Note in the proof above that (R, b_z) can be any B_z-subgroup such that b_z covers b_L. Later we will choose a special such B_z-subgroup.

Suppose further that B_1 satisfies Property 7.1 $(a*)$ in Propositions 8.1, 8.2, 8.3 or 8.5 for B_1-subgroups $(P_1, g_1) \leq (R_1, b_1)$. Let

$$L_+ = \prod_{i=2}^{s} L_i, \quad R_+ := \prod_{i=2}^{s} R_i, \quad P = P_1 \times R_+, \quad g_L = g_1 \times \left(\prod_{i=2}^{s} b_i \right), \quad J_+ = \left(\prod_{i=2}^{s} J_i \right) \times T',$$

so that $C_L(R) = C_{L_1}(R_1) \times C_{L_+}(R_+) \leq C_C(R) \leq C_{J_1}(R_1) \times C_{J_+}(R_+)$, and $(P, g_L) \leq (R, b_L)$. Since B_z covers B_L, it follows that there exist B_z-subgroups $(P, g_z) \leq (R, b_z)$

such that g_z covers g_L and b_z covers b_L. Let $(P, g_J) \le (R, b_J)$ be B_J-subgroups such that g_J covers g_z and b_J covers b_z. Thus g_J covers g_L, b_J covers b_L and $g_J = g_{J_1} \times \cdots \times g_{J_s} \times B_{T'}$. In particular, $g_{J_i} = b_{J_i}$ for $i \ge 2$, where b_{J_i} are given in (9.4). By Propositions 8.1, 8.2, 8.3 and 8.5 again, each $D(b_{J_i})$ is abelian and $D(g_{J_1}) = D(b_{J_1})$, and hence $D(g_J) = D(b_J)$ is abelian and $D(g_z) = D(g_J) \cap C = D(b_J) \cap C = D(b_z)$.

Let θ_i be the canonical character of b_i and $\theta_+ = \prod_{i\ge 2} \theta_i$, so that $\theta := \theta_1 \times \theta_+$ is the canonical character of b_L and the canonical character θ_z of b_z covers θ.

Since $(P_1, g_1) \le (R_1, b_1)$ satisfy Property 7.1 $(a*)$, it follows that there exists $y \in N_{C_{L_1}(P_1)}(R_1, b_1) \setminus C_{L_1}(R_1)$ such that $y^4 = 1, y^2 \in C_{L_1}(R_1)$ and $[y, x] \notin Z(L_1)$ for some $x \in R_1$. Moreover, there exist subgroups $N_i \lhd M_i$ of J_1, and character $\phi_i \in \mathrm{Irr}(N_i)$ for $i = 1, 2$ such that M_i/N_i is abelian,

$$Z(L_1) \le N_1 \times N_2 \le C_{L_1}(R_1) \le C_{J_1}(R_1) \le M_1 \circ M_2,$$

θ_1 covers $\phi_1 \times \phi_2$, $Z \cap N_2 = 1$, ϕ_2 has a y-stable extension $\tilde{\phi}_2$ to M_2 and $[y, x] = 1$ or in ZN_2 according as $x \in M_1$ or M_2, where $Z \le Z(M_1) \cap Z(M_2)$ such that $M_1 \circ M_2$ is the central product over Z.

Let $N_2' = N_1 \times C_{L_+}(R_+)$, $M_2' := M_1 \times C_{J_+}(R_+)$, and $\phi_2' = \phi_1 \times \theta_+$. Then M_2/N_2 and M_2'/N_2' are abelian,

$$Z(C) \le N_2' \times N_2 \le C_C(R) \le C_J(R) \le M_2' \circ M_2,$$

ϕ_2 has an extension $\tilde{\phi}_2$ to M_2 which is y-invariant, $[y, x] = 1$ for any $x \in M_2'$, $[y, x] \in ZN_2$ for any $x \in M_2$, $M_2 \circ M_2'$ is a central product over Z and $\theta_z \in \mathrm{Irr}(C_C(R) \mid \phi_2' \times \phi_2)$. By Remark 7.3, $\theta_z^y = \theta_z$, and so $b_z^y = b_z$. If $[y, x] \in Z(C)$ for all $x \in R$, then $[y, x] \in Z(C) \cap L_1 = Z(L_1)$ for all $x \in R_1$, which is impossible. Thus $[y, x] \notin Z(C)$ for some $x \in R$ and Property 7.1 $(a*)$ holds for $(P, g_z) \le (R, b_z)$ (with $H := K$).

Case 3. Suppose that $K := {}^2B(2^{2a+1}), {}^2G_2(3^{2a+1}), {}^2F_4(2^{2a+1}), G_2(q), {}^3D_4(q)$, $F_4(q)$ or $E_6^{-\epsilon}(q)$ with $q \equiv \epsilon \ mod\ 3$, and $B \in \mathrm{Blk}(K)$. Then B satisfies one of Property 7.1 $(a*), (b)$ or (d).

In each case $K = K_u$ and z induces an inner automorphism on K, so it follows that each L_i is a classical group (or possibly L_s is abelian). Hence by the results of Section 8 each B_i satisfies one of Property 7.1 $(a*)$, (b), (c) or (d).

Case 3.1. Suppose B_i satisfies either Property 7.1 (b) or (c) for some i. Without loss of generality, take $i = 1$. By Theorem 8.6, $p = 3$, $L_1 = \mathrm{SL}^{\epsilon_1}(3d_1, q_1)/Z$ for some $Z \le Z(\mathrm{SL}^{\epsilon_1}(3d_1, q_1))$, $\gcd(6, d_1) = 1$ and $3 \| (q_1 - \epsilon_1)$. By [20, Table 4.7.3A], $(q_1, \epsilon_1) = (q, \epsilon)$ or $(q^2, 1)$ and (K, C) are given in Table 2, where $L_\epsilon := \mathrm{SL}_3^\epsilon(q)$.

Case 3.1.1. If $K = G_2(q)$ or ${}^2F_4(2^{2m+1})$, then $s = 1$ and $L = C$ and $B_z = B_L = B_0(L)$, so $B = B_0(K)$ with $D(B) = 3_+^{1+2}$. In particular, $l(B) \ge 2$.

Case 3.1.2. Let $K = {}^3D_4(q)$, so that $C = \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)} \times H_\epsilon$, where $H_\epsilon = \langle L_\epsilon, x\rangle$ with x inducing outer-diagonal automorphism of order 3 on L_ϵ. So $D(B_z) = \mathbb{Z}_3 \wr \mathbb{Z}_3 \in \mathrm{Syl}_3(C)$ and we may suppose $D(B_z) \in \mathrm{Syl}_3(G_\epsilon)$, where $G_\epsilon = \mathrm{GL}_3^\epsilon(q)$ contains H_ϵ. Let $R_\epsilon = A(D(B_z)) = (\mathbb{Z}_3)^3$ and $P_\epsilon = \langle Z(L_\epsilon), \mathrm{diag}\{1, w, w\}\rangle \le R_\epsilon$ such that $|w| = 3$ in $\mathbb{F}_{q^2}^\times$. Then $C_{G_\epsilon}(P_\epsilon) = \mathbb{Z}_{q-\epsilon} \times \mathrm{GL}_2^\epsilon(q)$ and $C_{G_\epsilon}(R_\epsilon) = (\mathbb{Z}_{q-\epsilon})^3$. Thus

$$\mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)} \times C_{H_\epsilon}(R_\epsilon) = C_C(R_\epsilon) \le C_C(P_\epsilon) = \mathbb{Z}_{\frac{1}{3}(q^2+\epsilon q+1)} \times C_{H_\epsilon}(P_\epsilon).$$

K	C	K	C
${}^3D_4(q)$	$(\mathbb{Z}_{q^2+\epsilon q+1} \circ L_\epsilon).3$	$G_2(q)$	L_ϵ
${}^2F_4(2^{2m+1})$	$\mathrm{SU}_3(2^{2m+1})$	$F_4(q)$	$(L_\epsilon \circ L_\epsilon).(3{:}3)$
$E_6^{-\epsilon}(q)$	$(L_\epsilon \circ \mathrm{SL}_3(q^2)).(3{:}3)$	$E_6^\epsilon(q)_u$	$(L_\epsilon \times L_\epsilon \circ L_\epsilon).(3{:}3{:}3)$
$E_7(q)_u$	$(L_\epsilon \circ \mathrm{SL}_6^\epsilon(q)).(3{:}3)$	$E_8(q)$	$(E_6^\epsilon(q)_u \circ L_\epsilon).(3{:}3)$

Table 2: Possible (K, C) with some B_i satisfying Property 7.1 (b) or (c)

As shown in the proof of Proposition 8.1 the B_z-subgroups $(P_\epsilon, g) \leq (R_\epsilon, b)$ satisfy Property 7.1 $(a*)$.

Note that in the notation above $C_{G_\epsilon}(P_\epsilon)/Z = C_{G_\epsilon/Z}(P_\epsilon/Z)$ and $C_{G_\epsilon}(R_\epsilon)/Z = C_{G_\epsilon/Z}(R_\epsilon/Z)$ for any $Z \leq Z(L_\epsilon)$. Let $b_\epsilon \in \mathrm{Blk}(H_\epsilon)$ and $B_\epsilon \in \mathrm{Blk}(G_\epsilon)$ covering b_ϵ, so that $D(b_\epsilon) = D(B_\epsilon) \cap H_\epsilon = D(B_\epsilon)$. Thus $D(b_\epsilon) \in \mathrm{Syl}_3(C_{G_\epsilon}(t))$ for some semisimple $3'$-element t. In particular, $D(b_\epsilon)$ is either abelian with $|D(b_\epsilon)| \geq 9$ and $D(b_\epsilon) \not\leq L_\epsilon$ or $D(b_\epsilon) = \mathbb{Z}_3 \wr \mathbb{Z}_3$. In the former case, $C_{G_\epsilon}(D(b_\epsilon))/Z = C_{G_\epsilon/Z}(D(b_\epsilon)/Z)$ for any $Z \leq Z(L_\epsilon)$.

Case 3.1.3. Suppose $K = E_6^{-\epsilon}(q)$ or $F_4(q)$ and $L_1 = L_\epsilon$, so that $C = \langle L_1 \circ L_2, x\rangle$, where $L_2 = L_\epsilon$ or $\mathrm{SL}_3(q^2)$, and $x = x_1x_2$ such that each x_i induces outer-diagonal automorphism of order 3 on L_i. Let $J_i = \langle L_i, x_i\rangle$ and $B_{J_2} \in \mathrm{Blk}(J_2)$ covering B_2. Let $R_1 = R_\epsilon \leq J_1$, $P_1 = P_\epsilon \leq R_1$ and $P_2 = R_2 = A(B_{J_2})$, so that by the remark of Case 3.1.2 above, $C_{J_i}(P_i)/Z = C_{J_i/Z}(P_i/Z)$ and $C_{J_i}(R_i)/Z = C_{J_i/Z}(R_i/Z)$ for any $Z \leq Z(L_i)$. By Lemmas 9.1 and 7.6, we may suppose

$$L = L_1 \times L_2 \leq C \lhd J := J_1 \times J_2$$

Let $R = (R_1 \times R_2) \cap C$, $P = (P_1 \times P_2) \cap C$ and let $(P, g) \leq (R, b)$ be B_z-subgroups, so that $\pi_1(P) = P_\epsilon$ and $\pi_1(R) = R_\epsilon$, where π_i is the natural projection form J to J_i. A proof similar to that of Case 2 shows that $(P, g) \leq (R, b)$ satisfy Property 7.1 $(a*)$. If $K = E_6^{-\epsilon}(q)$ and $L_1 = \mathrm{SL}_3(q^2)$, then $L_2 = L_\epsilon$ and a similar proof shows that Property 7.1 $(a*)$ holds for some B-subgroups $(P, g) \leq (R, b)$.

Note that $C_J(P)/Z = C_{J/Z}(P/Z)$ and $C_J(R)/Z = C_{J/Z}(R/Z)$ for any $Z \leq O_3(Z(L))$.

Case 3.2. Suppose that each B_i satisfies either Property 7.1 $(a*)$ or (d). By Case 1, we may suppose, moreover that B_1 satisfies Property 7.1 $(a*)$. In particular, a Sylow p-subgroup of L_1 is nonabelian.

Case 3.2.1. Suppose $p \geq 5$, so that z is of parabolic type. By [20, Theorem 4.2.2 (f)], $O^{r'}(C)$ is a direct product and each L_i is universal. In addition, if a Sylow p-subgroup of L_1 is nonabelian, then $\ell = 1$ or 2 and each L_i is universal.

Suppose $\ell = 1$, so that $s = 1$ or 2. Since B_1 satisfies Property 7.1 $(a*)$ and $L_s = Z(C)$ when $s = 2$, it follows by Lemma 9.1 that we may suppose $L = L_1 \times L_s$ and $C_{J_i}(L_i) \leq Z(L_i)$.

Suppose $\ell = 2$, so that $s = 2$ or 3. Since L_1 has a nonabelian Sylow p-subgroup, it follows that L_2 has an abelian Sylow p-subgroup and, moreover $L_2 \cap O_p(Z(C)) = 1$. Since the central product $L_1 \circ L_2 \circ L_s$ is over a subgroup of $Z(C)$, it follows that each p-subgroup of L_2 satisfies the assumption of Lemma 9.1. Since B_1 satisfies Property

7.1 ($a*$) and $L_s = Z(C)$ when $s = 3$, it follows by Lemma 9.1 that we may suppose $L = L_1 \times L_s \times L_s$ and $C_{J_i}(L_i) \le Z(L_i)$.

By Case 2, B satisfies Property 7.1 ($a*$).

Case 3.2.2. Suppose $p = 3$, so that C is given by [20, Table 4.7.3a]. Thus either $\ell = 1$ or $\ell = 2$ with C given by Table 2. In addition, each L_i is also universal for $1 \le i \le \ell$ and $C_{J_i}(L_i) \le Z(L_i)$.

A proof similar to that of Case 3.2.1 shows that we may suppose (9.3) holds and by Case 2, B satisfies Property 7.1 ($a*$).

Case 4. Let $3 \mid (q - \epsilon)$, $K = K_u = 3.E_6^\epsilon(q) \le E := 3.E_6^\epsilon(q).3$, $B \in \mathrm{Blk}(K)$ and $B_E \in \mathrm{Blk}(E)$ covering B. Either Property 7.1 ($a*$) holds for some B-subgroups $(P, g) \le (R, b)$ with $C_E(P)/Z = C_{E/Z}(P/Z)$ and $C_E(R)/Z = C_{E/Z}(R/Z)$ for $Z \le O_p(K)$, or Property 7.1 (d) holds for B.

Let $D := D(B)$ and $m^* := \gcd(m, q - \epsilon)$.

If some B_i satisfies either Property 7.1 (b) or (c), then $p = 3$ and C is given by Table 2. A proof similar to that of Case 3.2.2 shows that Property 7.1 (a*) holds for some B-subgroups $(P, g) \le (R, b)$.

Suppose some B_i satisfies Property 7.1 ($a*$). Since z is parabolic or equal-rank type and z induces an inner automorphism on K, it follows that each L_i is classical. We first show that there exist B-subgroups $(P, g) \le (R, b)$ and y satisfying the Property 7.1 ($a*$) with $H := K$.

If $p \ge 5$, then z is parabiloc. A proof similar to that of Case 3.2.1 shows that the Property 7.1 ($a*$) holds for B-subgroups $(P, g) \le (R, g)$.

Suppose $p = 3$. By [20, Table 4.7.3A],

$$C_K(z) = \langle \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q), 3{:}3{:}3\rangle, \quad (\mathrm{SL}_6^\epsilon(q) \circ_{2^*} (q - \epsilon)).2^*,$$

$\mathrm{Spin}_8^+(q) \circ_{2^*} ((q - \epsilon) \times (q - \epsilon)).(2^* \times 2^*)$, $\mathrm{Spin}_{10}^\epsilon(q) \circ (q - \epsilon).(2^* \times 2_\epsilon^*)$ (when $q \equiv \epsilon \pmod 9$) with $2_\epsilon^* = 1$ or 2^* according as $\epsilon = -$ or $+$, or $(\mathrm{SL}_2(q) \times \mathrm{SL}_5^\epsilon(q)) \circ (q - \epsilon).2^*$ (when $q \equiv \epsilon \pmod 9$).

Thus $\ell = 1, 2$ or 3. If $\ell = 1$ or 2, then a proof similar to that of Cases 3.2.1 and 3.2.2 shows that the Property 7.1 ($a*$) holds for B-subgroups $(P, g) \le (R, g)$. If $\ell = 3$, then each $L_i = \mathrm{SL}_3^\epsilon(q)$ and $C_{J_i}(L_i) \le Z(L_i)$. A proof similar to that of Cases 3.1.3 shows that we may suppose (9.3) holds and by Case 2, the Property 7.1 ($a*$) holds for B-subgroups $(P, g) \le (R, g)$.

If all B_i satisfies Property 7.1 (d), then by Case 1, $D(B_z)$ is abelian.

Now we prove the rest of Property 7.1 ($a*$) with $H = E$. Suppose $(P, g_E) \le (R, b_E)$ are B_E-subgroups such that g_E covers g and b_E covering b.

Case 4.1. If $p \ge 5$, then $D(b) = D(b_E)$, $D(g) = D(g_E)$, and so $D(g_E) = D(b_E)$ is abelian. Now $C_E(z) = C$ or $\langle C, x\rangle$ for some $x \in E \setminus K$. If $C_E(z) = C$, then $C_E(R) = C_C(R)$. Apply the proof of Case 2 we have that

$$Z(C) \le N_2' \times N_2 \le C_C(R) \le C_{C_E(z)}(R) \le M_2' \circ M_2. \tag{9.5}$$

Suppose $C_E(z) = \langle LT, x\rangle$ for some $x \in E \setminus K$, so that x induces inner-diagonal automorphism on each L_i. Thus $x = x_1 x_2 \cdots x_s x'$. Replacing J_i by $\langle J_i, x_i\rangle$ and T' by

$\langle T', x'\rangle$ in the proof Case 2 with some obvious modifications, we have that (9.5) still holds. Thus Property 7.1 $(a*)$ holds for B-subgroups $(P, g) \le (R, b)$ (with $H := E$).

Case 4.2. Suppose $p = 3$. By [20, Table 4.7.3A],

$$C_E(z) = \langle \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q), 3{:}3{:}1, 1{:}3{:}3\rangle, \quad (\mathrm{SL}_6^\epsilon(q) \circ_{2^*} (q-\epsilon)).(3 \times 2^*),$$

$\mathrm{Spin}_8^+(q) \circ_{2^*} ((q-\epsilon) \times (q-\epsilon)).(2^* \times 2^* \times 3)$, $\mathrm{Spin}_{10}^\epsilon(q) \circ (q-\epsilon).(3 \times 2^* \times 2^*_\epsilon)$ (when $q \equiv \epsilon \pmod 9$) with $2^*_\epsilon = 1$ or 2^* according as $\epsilon = -$ or $+$, or $(\mathrm{SL}_2(q) \times \mathrm{SL}_5^\epsilon(q)) \circ (q-\epsilon).(2^* \times 3)$ (when $q \equiv \epsilon \pmod 9$).

Suppose $C_E(z) = \langle \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q), t, x\rangle$, so that $L = \mathrm{SL}_3^\epsilon(q) \times \mathrm{SL}_3^\epsilon(q) \circ \mathrm{SL}_3^\epsilon(q)$, $T = \langle t\rangle \le K$ with t induces 3:3:3 on L, and $x \in E \setminus K$ induces 1:3:3 on L. Let $L_i = \mathrm{SL}_3^\epsilon(q) \le G_i := \mathrm{GL}_3^\epsilon(q)$, $t = t_1 \times t_2 \times t_3$, $x = x_1 \times x_2 \times x_3$ with t_i, x_i act on L_i and centralizes L_j when $i \ne j$. In addition, let $H_i = \langle L_i, t_i, x_i\rangle$, so that $H_i \le G_i$. Let $S = \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon} \le \mathrm{SL}_3^\epsilon(q)$ be a maximal torus, and $S \times S \circ_3 S \le L$. Since $C_{\mathrm{GL}_3^\epsilon(q)}(S) = \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon}$ is a maximal torus, it follows that $A := C_E(S \times S \circ_3 S)$ is abelian such that $A \cap K \cong \mathbb{Z}_{q-\epsilon}^6$ is a maximal torus of K and $A/(A \cap K) = \mathbb{Z}_3$. In particular, we may suppose $t, x \in A$ and $C_E(z) = LA$ with abelian A and $L \lhd C_E(z)$.

Similarly, if $C_E(z) = (\mathrm{SL}_6^\epsilon(q) \times (q-\epsilon)).6^*$, $\mathrm{Spin}_8^+(q) \circ_2 ((q-\epsilon) \times (q-\epsilon)).(2^* \times 6^*)$, $\mathrm{Spin}_{10}^\epsilon(q) \circ (q-\epsilon).(6^* \times 2^*_\epsilon)$ or $(\mathrm{SL}_2(q) \times \mathrm{SL}_5^\epsilon(q)) \circ (q-\epsilon).6^*$, then $A \le C_E(z)$ and so $C_E(z) = LA$ with abelian A and $L \lhd C_E(z)$, and A induces inner-diagonal automorphisms on each L_i.

A proof similar to that of Case 2 with LT replaced by LA and some modifications shows that $D(g_E) = D(b_E)$ is abelian, and hence Property 7.1 $(a*)$ holds.

Case 4.3. Suppose $p = 3$ and $Z = Z(K)$. If $C_E(z)/Z = C_{E/Z}(zZ)$, then $C_E(P)/Z = C_{E/Z}(P/Z)$ and $C_E(R)/Z = C_{E/Z}(R/Z)$. Suppose $C_E(z)/Z \ne C_{E/Z}(zZ)$. By [20, Table 4.7.3A], either $L = L_1 \circ L_2$ with $L_1 = \mathrm{Spin}_8^+(q)$ and $L_2 = Z(C) = \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon}$ or $L = L_\epsilon \times L_\epsilon \circ L_\epsilon \le K$. In the former case,

$$C_{E/Z}(zZ) = \langle C_E(z)/Z, wZ\rangle,$$

where $w \in E$ such that $w = \gamma{:}\omega$, that is, w acts on L_1 as a graph automorphism of order 3 and $(h_1, h_2)^\omega = (h_2, (h_1h_2)^{-1})$ for any $(h_1, h_2) \in L_2$. Now $O_3(L_2) = O_3(\mathbb{Z}_{q-\epsilon}) \times O_3(\mathbb{Z}_{q-\epsilon}) \le P$ and $Z(K) = \{(x, x) : x \in \Omega_1(O_3(\mathbb{Z}_{q-\epsilon}))\} \le Z(C)$. Suppose $h \in E$ such that for any $u \in P$ we have $h^{-1}uh = cu$ for some $c \in Z(K)$ and suppose $h \notin C_E(z)$. Then we may suppose $h = tw$ for some $t \in C_E(z)$, and so $(1, h_2)^h = (1, h_2)^\omega = (h_2, 1)$ for any $h_2 \in O_3(\mathbb{Z}_{q-\epsilon}) \setminus \{1\}$. But $(h_2, 1) \ne c(1, h_2)$ for any $c \in Z(K)$, which is a contradiction. Thus $h \in C_E(z)$ and so $C_E(P)/Z = C_{E/Z}(P/Z)$. Similarly, $C_E(R)/Z = C_{E/Z}(R/Z)$. If $L = L_\epsilon \times L_\epsilon \circ L_\epsilon$, then $L/Z = L_\epsilon \circ L_\epsilon \circ L_\epsilon$ and $C_{E/Z}(zZ) = \langle L/Z, tZ, xZ, wZ\rangle$, where t, x are given above and $w \in E \setminus K$ permutes transitively the three components L_ϵ of L. The proof in this case is similar. Suppose $h \in E$ such that for all $u \in P$ $h^{-1}uh = cu$ for some $c \in Z$, so that $hZ \in C_{E/Z}(zZ)$. Since $|\Omega_1(P)| \ge 3^4$ and $C_J(P)/Z = C_{J/Z}(P/Z)$, it follows that $h \in C_E(z) = \langle L, x, t\rangle$ and hence $C_E(P)/Z = C_{J/Z}(P/Z)$. Similarly, $C_E(R)/Z = C_{E/Z}(R/Z)$.

Case 4.4. Now we prove the rest of Property 7.1 (d). Suppose Property 7.1 (d) holds for each B_i and suppose $D_E \cap K = D$ for some $D_E = D(B_E)$, so that $D(B_{J_i})$ is abelian and so is $D(B_z) = D(B_J) \cap C$. If B_z is a major subsection, then $D = D(B_z)$

is abelian. If $p \geq 5$, then $D_E = D$. Suppose $p = 3$ and there exists $x \in Z(D_E) \setminus D$. Then $x \in E \setminus K$, $x \in C_E(D)$ and $D_E = \langle D, x \rangle$ is abelian. If $Z(D_E) \leq D$, then take $z \in Z(D_E)$ with $|z| = 3$, so that $D_E \leq C_E(z) = LA$. A proof similar to that of Case 1.2 with some obvious modifications shows that D_E is abelian.

Suppose $z \in D$ with $|z| = 9$ and $zZ(K) \in Z(D/Z(K))$. By [20, Table 4.7.3A], $9 \mid (q - \epsilon)$ and $C_E(z) = \mathrm{Spin}_{10}^{\epsilon}(q) \circ (q - \epsilon).(6^* \times 2_{\epsilon}^*)$ or $(\mathrm{SL}_2(q) \times \mathrm{SL}_5^{\epsilon}(q)) \circ (q - \epsilon).6^*$. In this case, $C_{E/Z(K)}(zZ(K))$ is also given by [20, Table 4.7.3A], and we have $C_{K/Z(K)}(zZ(K)) = C_K(z)/Z(K)$. Thus $D/Z(K) \leq C_K(z)/Z(K)$ and $D \leq C_K(z)$. In particular, $z \in Z(D)$ and we may suppose (z, B_z) is major. Hence $D = D(B_z)$ is abelian. It follows that Property 7.1 (d) holds for B.

Case 5. Let $K := E_7(q)$ with q even and $B \in \mathrm{Blk}(K)$. Then either L_i is classical, or L_i is exceptional given in Cases 3 or 4. If L_i is classical, then apply Propositions 8.1, 8.2, 8.3 and 8.5. If L_i is exceptional, then apply the results given in Cases 3 and 4. Either Property 7.1 $(a*)$ holds for B or Property 7.1 (d) holds for B (with $H := K := E_7(q)$).

Let q be odd, $K = 2.E_7(q) \leq E := 2.E_7(q).2$, $B \in \mathrm{Blk}(K)$ and $B_E \in \mathrm{Blk}(E)$ covering B. Either Property 7.1 $(a*)$ holds for some B-subgroups $(P, g) \leq (R, b)$ or Property 7.1 (d) holds for B.

Again let $D := D(B)$ and $m^* := \gcd(m, q - \epsilon)$.

Suppose Property 7.1 $(a*)$ holds for some B_i-subgroups $(P_i, g_i) \leq (R_i, b_i)$. A proof similar to that of Case 3.2 with some obvious modifications shows that there exist B-subgroups $(P, g) \leq (R, b)$ and y satisfying the first part of Property 7.1 $(a*)$ (with $H = K$). Suppose B_i satisfies either Property 7.1 (b) or (c) for some i. Then $p = 3$ and C is given by Table 2. A proof similar to that of Case 3.1.3 shows that Property 7.1 (a*) holds for some B-subgroups $(P, g) \leq (R, b)$. Suppose $(P, g_E) \leq (R, b_E)$ are B_E-subgroups such that g_E covering g and b_E covering b. Then $D(g) = D(g_E)$ and $D(b) = D(b_E)$ for some $D(g_E)$ and $D(b_E)$. But $D(g) = D(b)$ is abelian, so $D(g_E) = D(b_E)$ is abelian. A proof similar to that of Case 4.1 shows that (9.5) holds and so Property 7.1 $(a*)$ (with $H = E$) holds for B-subgroups $(P, g) \leq (R, b)$.

Since $E/K = 2$ and p is odd, it follows that $D(B_E)$ is abelian whenever $D(B)$ is abelian.

Case 6. Suppose $K := E_8(q)$. Either Property 7.1 (a) holds for some B-subgroups $(P, g) \leq (R, b)$ or Property 7.1 (d) holds for B.

In this case (z, B_z) is a major subsection of B and either L_i is classical, or L_i is exceptional given in Cases 3, 4 or 5. If L_i is classical, then apply Theorem 8.6. If L_i is exceptional, then apply the results in Cases 3, 4 or 5. Thus if each $D(B_i)$ is abelian, then $D(B_{J_i})$ is abelian, and so $D = D(B_z) = D(B_J) \cap C$ is abelian. Suppose $D(B_i)$ is non-abelian for some i, say $i = 1$.

If $p \geq 7$, then B_1 satisfies Property 7.1 $(a*)$, z is of parabolic type and the proof is similar to that of Case 3.2.1.

Suppose $p = 5$, so that B_1 satisfies Property 7.1 $(a*)$ and C is given by [20, Table 4.7.3B]. Thus $\ell = 1$ or 2. If z is parabolic, then a proof similar to that of Case 3.2.1 shows that we may suppose (9.3) holds.

If z is equal-rank, then

$$C = \langle L_1 \circ L_2, 5{:}5 \rangle, \quad L_1 = L_2 = \mathrm{SL}_5^\epsilon(q),$$

so that $L = L_1 \circ L_2$. Here $\epsilon = \pm 1$ such that $q \equiv \epsilon \pmod 5$. A proof similar to that of Case 3.1.3 shows that we may suppose $L = L_1 \times L_2$. By Case 2, B satisfies Property 7.1 $(a*)$.

Suppose $p = 3$, so that C is given by [20, Table 4.7.3A] and $\ell = 1$ or 2.

If some B_i satisfies either Property 7.1 (b) or (c), then C is given by Table 2. In particular, $\ell = 2$ and $D(B_i) = 3_+^{1+2}$.

If $\ell = 1$, then B_1 satisfies Property 7.1 $(a*)$ and we may suppose (9.3) holds. By Case 2, B satisfies Property 7.1 $(a*)$.

Suppose $\ell = 2$, so that $L = L_\epsilon \circ E_6(q)_u$ and $C = \langle L, 3{:}3 \rangle$, where $L_\epsilon = \mathrm{SL}_3^\epsilon(q)$ with $q \equiv \epsilon \pmod 3$.

If $L_1 = E_6(q)_u$ and B_1 satisfies Property 7.1 $(a*)$ for $(P_1, g_1) \le (R_1, b_1)$. Let $P_2 = R_2 = A(D(B_2))$, and let (R_2, b_2) be a B_2-subgroup and set $(P_2, g_2) = (R_2, b_2)$. By Case 4 and the ramark of Case 3.1.2, $C_{J_i}(R_i)/Z_i = C_{J_i/Z_i}(R_i/Z_i)$ for $i = 1, 2$. By Lemma 9.1, we may suppose $L = L_1 \times L_2$ and a similar to that of Case 2 shows that B satisfies Property 7.1 $(a*)$.

Suppose $L_1 = L_\epsilon$ and B_2 satisfies Property 7.1 (d), so that $D(B_2)$ is abelian. In this case B_1 satisfies Property 7.1 (b) with $D(B_1) = 3_+^{1+2}$ or B_1 satisfies Property 7.1 (a*). Note that $J = J_1 \circ J_2$ over $Z(L) = 3$ and $J_1 = H_\epsilon$ given in Case 3.1.2.

Let $(P_\epsilon, g) \le (R_\epsilon, b)$ be defined in Case 3.1.2. Then there exists $y \in C_{H_\epsilon}(P_\epsilon) \cap N_{H_\epsilon}(R_\epsilon) \setminus C_{H_\epsilon}(R_\epsilon)$ satisfies Property 7.1 $(a*)$.

Let $P_2 = R_2 = A(D(B_{J_2})) = D(B_2)$ and let $(P_2, b_2) = (R_2, b_2)$ be the Sylow B_{J_2}-subgroup. Since $H_\epsilon = J_1$, it follows that $P_\epsilon \circ P_2 \le R_\epsilon \circ R_2 \le J$. Set

$$P = (P_\epsilon \circ P_2) \cap C, \quad \text{and} \quad R = (R_\epsilon \circ R_2) \cap C.$$

Then $P \le R \le D$ and P, R are abelian. Since $Z(L) = 3$, it follows that $b \circ b_2$ is a block $C_J(R_\epsilon \circ R_2)$ and $g \circ g_2 \in \mathrm{Blk}(C_J(P_\epsilon \circ P_2))$. Since $J/C = 3$, it follows that $b \circ b_2$ covers a unique block b_R of $C_C(R)$ and similarly, $g \circ g_2$ covers a unique block g_P of $C_C(P)$. Since B_J is the unique block covering B_z, it follows that $(P, b_P) \le (R, b_R)$ are B_z-subgroups. Since $H_\epsilon/L_\epsilon = 3$, it follows that $y \in L_\epsilon = L_1$. Now B satisfies Property 7.1 (a) (not $(a*)$) for B-subgroups $(P, b_P) \le (R, b_R)$ (with $H = K$). □

Lemma 9.3 *Let G be a quasisimple group such that $G/Z(G)$ is alternating or of Lie type and G is an exceptional cover. Let p be an odd prime. Then every p-block of G with nonabelian defect groups has a subpair with at least two irreducible Brauer characters.*

Proof: We must consider the cases $G/Z(G) \cong PSL_2(4)$, $PSL_2(9)$, A_7, $PSL_3(2)$, $PSL_3(4)$, $PSU_4(2)$, $PSU_4(3)$, $PSU_6(2)$, ${}^2B_2(8)$, $O_7(2)$, $O_7(3)$, $O_8^+(2)$, $G_2(3)$, $G_2(4)$, $F_4(2)$ and ${}^2E_6(2)$. We may use [18] to confirm all but the cases $F_4(2)$ for $p = 3$, and ${}^2E_6(2)$ for $p = 3, 5, 7$ (noting that the three double covers of $O_8^+(2)$ have the same block structure - see [14]), as in each case the block itself has at least two irreducible Brauer characters. The result holds for $F_4(2)$ for $p = 3$ by [22]. Note that ${}^2E_6(2)$ has abelian

Sylow 5- and 7-subgroups, so we are left with $p = 3$ and $G/Z(G) \cong {}^2E_6(2)$. In this case we do not know the Brauer characters of G, so we are forced into a slightly involved argument to make use of the current literature. Note that it suffices to consider the case $|Z(G)| = 4$. Our group G has three conjugacy classes of elements of order three, $3A$, $3B$ and $3C$. For each such $x \in G$, we have $C_G(x)/Z(G) \cong C_{G/Z(G)}(xZ(G))$. Consider a block B covering the block c of $Z(G)$ containing the irreducible character λ, say. We may assume that c is faithful. By examination of the character table in [14], only two irreducible characters lying over λ vanish on $3A$, $3B$ but not $3C$ (χ_{184} and χ_{202} in the notation of [14]). Since a 3-block of positive defect must possess at least three irreducible characters, it follows by a theorem of Green that B must have a defect group D containing elements of $3A$ or $3B$. Suppose $x \in Z(D)$ has order three. Write $Q = \langle x \rangle$. Note that $DC_G(D) \leq C_G(Q)$, so there is a B-subgroup (Q, b_Q) with defect group D. If $x \in 3A$, then $C_{G/Z(G)}(xZ(G)) \cong Q \times PSU_6(2)$. We have seen that every block with nonabelian defect groups of a double cover of $PSU_6(2)$ has at least two irreducible Brauer characters, so it follows that $l(b_Q) = 1$ if D is nonabelian. If $x \in 3B$, then $C_{G/Z(G)}(xZ(G)) \cong Q \times O_8^+(2).3$, and the same argument applies as for $3A$. □

Theorem 9.4 *Let G be a quasisimple group and B a nilpotent p-block of G with defect group D, where p is odd. Then D is abelian.*

PROOF: If $G/Z(G)$ is an alternating group, then the result follows by Theorem 3.3 and the remarks following it. For $G/Z(G)$ sporadic see Proposition 4.6. If $G/Z(G)$ is a classical group and G is a non-exceptional cover, see Propositions 8.1, 8.2, 8.3 and 8.5. For $G/Z(G)$ an exceptional group of Lie type and G is a non-exceptional cover, see Theorem 9.2. For the exceptional covers, see Lemma 9.3. □

10 Puig's conjecture

We complete the proof of Puig's conjecture for quasisimple groups for odd primes, present some general results and deduce some corollaries.

Theorem 10.1 *Let G be a finite quasisimple group and let B be a p-block of G with p odd. Then B is nilpotent if and only if $l(b_Q) = 1$ for each p-subgroup Q and each block b_Q of $C_G(Q)$ with $(b_Q)^G = B$.*

PROOF: The necessary condition for nilpotency follows from [13, 1.2]. By Corollary 7.5 and Propositions 8.1, 8.2, 8.3 and 8.5 the result holds for the classical groups. By Theorem 9.2 it holds for the exceptional groups of Lie type. The result holds for the double covers of the alternating groups by Corollary 3.6, and when $G/Z(G)$ is sporadic by Proposition 4.7. For the exceptional covers of the alternating groups and of the finite simple groups of Lie type, see Lemma 9.3. □

Lemma 10.2 *Let $N \lhd G$ such that G/N is cyclic and of order prime to p, and let $B \in \mathrm{Blk}(G)$ cover $b \in \mathrm{Blk}(N)$. Suppose there are abelian R and P and b-subgroups $(P, b_P) \leq (R, b_R)$ such that b_P and b_R have abelian defect groups and there is $x \in C_N(P)$ of order prime to $[G{:}N]$ such that $x \in N_N(R, b_R) \setminus C_N(R)$. Then there are B-subgroups $(P, B_P) \leq (R, B_R)$ such that B_R and B_P have abelian defect groups and $x \in N_G(R, B_R) \setminus C_G(R)$.*

PROOF: By [15, 4.1] there is $B_R \in \mathrm{Blk}(C_G(R))$ such that $(B_R)^G = B$ and B_R covers b_R. We claim that the number of such B_R divides $[G{:}N]$. Now $C_G(R)/C_N(R)$ is cyclic of order dividing $[G{:}N]$. The blocks of $C_N(R)$ and of $C_G(R)$ are in 1-1 correspondence with their canonical characters. Let θ_R be the canonical character for b_R. Since $C_G(R)/C_N(R)$ is cyclic, θ_R extends to an irreducible character of $C_G(R)$, and since $[C_G(R) : C_N(R)]$ is not divisible by p, the extensions are precisely the canonical characters of the blocks of $C_G(R)$ covering b_R. By Clifford theory, the group of irreducible characters of $C_G(R)/C_N(R)$ acts transitively on the blocks $B_1, \ldots, B_n$ of $C_G(R)$ covering b_R by inflation and multiplication, and also transitively on the set $\{B_i^G : 1 \leq i \leq n\}$. Consequently the number of blocks of $C_G(R)$ covering b_R with Brauer correspondent B divides $[C_G(R) : C_N(R)]$, and the claim follows.

For each i, we have $(B_i^x)^G = (B_i^G)^x = B^x = B$, and B_i^x overs $b_i^x = b_i$. Hence x permutes $\{B_i : B_i^G = B\}$. Since this set has order prime to the order of x, it follows that x must fix some such B_i. Call it B_R. Letting $B_P = (B_R)^{C_G(P)}$, we are done. □

As an almost immediate corollary we have:

Corollary 10.3 *Let G be a finite group such that there is $N \lhd G$ with $[G : N]$ odd and G/N is a p-regular cyclic group, where N is quasisimple and $p > 3$ is a prime. Let B be a p-block of G. Then B is nilpotent if and only if $l(b_Q) = 1$ for every B-subgroup (Q, b_Q).*

PROOF: Since the alternating and sporadic groups have outer automorphism groups of order at most two, it follows that it suffices to consider the groups of Lie type. Suppose first that N is not an exceptional covering group. Since $p > 3$, every block of N satisfies one of Property 7.1 (a) or (d), and the result follows by Lemma 10.2 and Corollary 7.5.

Suppose that N is an exceptional cover. Then the outer automorphism group is a 2-group except when $N/Z(N) \cong PSL_3(4)$, $PSU_6(2)$ or ${}^2E_6(2)$, in which case it has order three, and consists of diagonal automorphisms. In each case $Z(N)$ is a Klein-four group. However, in each of these cases the non-inner automorphisms transitively permute the blocks whose kernel does not contain $Z(N)$, and the result follows in these cases too, since B is nilpotent if and only if b is, and B-subgroups (Q, B_Q) covering b-subgroups (Q, b_Q) satisfy $l(B_Q) = l(b_Q)$. □

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