Sets, Numbers and Functions: Lecture 9

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New sets from old

Let $A$ and $B$ be sets.

Define $A \cap B = \{ x : x \in A \text{ and } x \in B \}$, the *intersection* of $A$ and $B$.

Note that $A \cap B = B \cap A$.

Define $A \cup B = \{ x : x \in A \text{ or } x \in B \}$, the *union* of $A$ and $B$.

Note that $A \cup B = B \cup A$.

**Examples**

Let $A = \{2, 1, \pi\}$, $B = \{\pi, 4, \{1, 4\}, 1\}$ and $C = \{5, 7, \{1\}, \{1, 4\}\}$. Then:

(i) $A \cap B = \{1, \pi\}$

(ii) $A \cup B = \{1, 2, 4, \pi, \{1, 4\}\}$

(iii) $A \cap C = \emptyset$.

(iv) $B \cap C = \{\{1, 4\}\}$. 
Lemma

Let $A$, $B$ and $C$ be sets. Then

(i) $A \cap B \subseteq A$ and $A \cap B \subseteq B$;
(ii) $A \subseteq A \cup B$ and $B \subseteq A \cup B$;
(iii) $A \cap A = A$ and $A \cap \emptyset = \emptyset$;
(iv) $A \cup A = A$ and $A \cup \emptyset = A$;
(v) $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$;
(vi) if $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$.

Proof.

(i)-(v) are immediate from the definitions.

(vi) Suppose that $A \subseteq B$. To show that $A \cap B = A$, then we must show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

That $A \cap B \subseteq A$ follows from (i).

Suppose that $x \in A$. Since $A \subseteq B$, we have $x \in B$. So $x \in A$ and $x \in B$. So $x \in A \cap B$. Hence $A \subseteq A \cap B$. We have shown that $A = A \cap B$.

$A \cup B = B$ is similar and left as an exercise.
Remark:

\[ A \cap (B \cap C) = (A \cap B) \cap C \] means that we may write \( A \cap B \cap C \) without ambiguity. Similarly for \( A \cup B \cup C \).
Complements and differences

Let $A$ and $B$ be sets.

Define $A \setminus B = \{ x : x \in A \text{ and } x \not\in B \}$, the difference of $A$ and $B$ (or “$A$ take away $B$”).

Often written $A - B$.

**Examples:**

\[
\{1, 3, 4\} \setminus \{3, 4, 5, 6\} = \{1\}
\]

\[
\{3, 4, 5, 6\} \setminus \{1, 3, 4\} = \{5, 6\}.
\]

**Remarks**

Let $A$ and $B$ be sets.

(i) $(A \setminus B) \cap (B \setminus A) = \emptyset$

(ii) $A \setminus \emptyset = A$

(iii) $A \setminus A = \emptyset$. 
Suppose our discussion is about some sets which are all contained in a given large set \( U \) (called a universal set - this is not uniquely defined).

E.g., if all our sets consist of integers, then \( \mathbb{Z} \) might be the universal set.

If \( A \) is a set, and \( A \subseteq U \), write

\[
A^c = U \setminus A,
\]

the complement of \( A \).

**Remarks**

(i) \((A^c)^c = A\).

(ii) \( U \) is only defined in a given context. We’re *not* defining a \( U \) which contains all sets. The notation for a complement is simply for convenience of notation.

**Example**

Suppose \( U = \mathbb{Z} \). If \( A \) is the set of even integers, then

\[
A^c = U \setminus A = \{x : x \in \mathbb{Z} \text{ and } 2 \nmid x\}
\]

is the set of odd integers.
The power set

Let $A$ be a set. The power set $\mathcal{P}(A)$ of $A$ is the set whose elements are (all of) the subsets of $A$.

**Examples**

(i) If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, A, \{a\}, \{b\}\}$.

(ii) If $A = \{a\}$, then $\mathcal{P}(A) = \{\emptyset, A\}$.

(iii) If $A = \{a, b, c\}$, then $\mathcal{P}(A) = \{\emptyset, A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

**Proposition**

*If $A$ has precisely $n$ elements, then $\mathcal{P}(A)$ has $2^n$ elements.*

**Proof.**

Exercise (use induction).
Let $A$ and $B$ be sets.

The *cartesian product* of $A$ and $B$, written $A \times B$, is the set consisting of ordered pairs whose first part is an element of $A$ and whose second part is an element of $B$.

So $A \times B = \{(a, b) : a \in A, b \in B\}$.

Note that $(a_1, b_1) = (a_2, b_2) \iff a_1 = a_2$ and $b_1 = b_2$.

**Examples**

(i) Let $A = \{2, 3\}$ and $B = \{2, 4, 5\}$. Then

$A \times B = \{(2, 2), (2, 4), (2, 5), (3, 2), (3, 4), (3, 5)\}$

$A \times A = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$.

(ii) Let $A = B = \mathbb{R}$. Then

$$A \times B = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\},$$

coordinates in the Euclidean plane (often written $\mathbb{R}^2$).
(iii) Write $(\mathbb{R} \times \{1\}) \cap (\{3\} \times \mathbb{R})$ in the form $A \times B$ for sets $A$ and $B$.

On board

(iv) Write $(\mathbb{R} \times \{0\}) \setminus (\{1\} \times \mathbb{R})$ in the form $A \times B$ for sets $A$ and $B$.

(v) Let $A$ be any set. Then $A \times \emptyset = \emptyset$. 
In general, for $n \in \mathbb{N}$, write

$$A^n = A \times \cdots \times A \quad \underbrace{}_{n \text{ times}} = \{(a_1, a_2, \ldots, a_n) : a_1 \in A, a_2 \in A, \ldots, a_n \in A\},$$

ordered $n$-tuples.